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Dynamics of Mass Variable System

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Contents

Abstract

In this dissertation the dynamics of the body and the system of bodies with time variable mass and time variable moment of inertia are treated. Namely, there are a lot of machines, mechanisms and system in practical use which parts are mass variable or have the variable moments of inertia. Let us mention some of them: centrifuges, sieves for sorting particles, transportation mechanisms, lifting mechanisms, cranes, automatic weight measuring instruments, and rotors used in the textile, cable, paper industry, etc. The aim of modern industry is to increase productivity and to fully automate, resulting in various new challenges in the dynamics of systems and machines with nonlinearities and mass and also moment of inertia variation. The both types of mass and moment of inertia variation are considered: the discontinual and the continual. The basic lows in dynamics are extended to the case when the mass is varying in time. The principle of momentum and of angular momentum are applied to obtain the velocity and angular velocity of the body after discontinual mass variation. The same results are applied analytically by introducing the procedures of analytical mechanics. The dynamics of mass addition is treated as the plastic impact, and of the separation as the inverse process to plastic impact. For the case of the continual variation of the mass and of the moment of inertia in time, beside the reactive force, the reactive torque is introduced. The case of the free motion of the mass variable body is investigated. The Lagrange's equations of motion are derived. As the special motion of the mass variable body, the vibration is considered. The main attention is given to approximate solving of the strong non-linear differential equations of motion. The influence of the reactive force on the vibration properties of the body is analyzed. This dissertation based on the dynamics of the particle with time variable mass and the basic laws of dynamics, in this dissertation the theoretical consideration of the dynamics of the body and system of bodies with time variable mass and moment of inertia are given. The dissertation is divided into following Chapters:

After the Introduction, various mechanisms and machines with variable mass and moment of inertia are shown. The construction and also the working properties of the machines and mechanisms are described.

In Chapter 2, the linear momentum and the angular momentum of the body with variable mass and moment of inertia are considered. The linear and angular momentum of the body before and after mass modification due to adding or separating of the mass are determined. These values are the fundamental ones for dynamic analysis of the bodies with discontinual or continual time variable properties.

In Chapter 3, the obtained linear and angular momentum relations are applied for calculation of the velocity and angular velocity of the body when the body separation or augmentation is discontinual. The principles of momentum and angular momentum are used as the basic ones. The special case of the in-plane separation is considered. Depending on the type of motion of the separated body, the kinematic properties of the remainder body are discussed. An example the dynamics of a crumbled rotor is discussed according to the fact of separation of a body from the initial one.

In the Chapter 4, the dynamics of the discontinual mass variation is treated by analytically, by using the principles of analytical mechanics. An analytical procedure for the velocity and angular velocity determination of a body in the process of mass variation is developed. The main attention is given to the case when no external forces and torques act. The process of body addition is treated as the plastic impact and the separation as an inverse process of the plastic impact of bodies. An example of separation of a rotating pendulum into two bodies is treated.

In the Chapter 5, we express the free motion of the body with continual time variation of the mass and moment of inertia. Due to mass and also moment of

inertia variation, beside the reactive force, the reactive torque acts. In this section the main attention is directed toward investigation of the influence of these two physical actions. As a special type of motion the in-plane motion of the body is considered. The obtained theory is applied for analyzing of the plane motion of the rotor on which the band is winding up.

In the Chapter 6, the Lagrange's equations of the motion for the body with continual variation of mass and moment of inertia is derived. The generalized forces due to the reactive force and the reactive torque are defined. The obtained Lagrange's equations represent the analytical description of the free motion of the body with variable mass.

The Chapter 7, the vibrations as a special type of the motion of the body with variable mass. Based on the general equations of motion given in the Chapter 6, the mathematical model for the oscillatory motion is formed. The main attention is directed to approximate solution procedures for solving the strong nonlinear differential equations with slow-time variable parameters describing the vibrations of various kinds of oscillators. Various types of oscillators are treated: one-degree-of-freedom oscillators, two-degree-of-freedom one-mass oscillators (rotors) and also the oscillators which contain two masses and have two-degrees-of-freedom. Specially the influence of the reactive force and of the reactive torque on the vibration properties of the system with variable mass and moment of inertia are investigated.

In the dissertation the conclusions and remarks for the future Investigations are given in Chapter 8.

The dissertation ends with the Reference list.

1 Introduction

The problem of the motion of mass variable systems is evident since the 17th century. Galileo discovered the anomaly in the Moon motion which he believed is the function of the system mass variation. Lately, Laplace theoretically explained the phenomena of the secular acceleration of the Moon. Dufour, 1886, explained that the mass of the earth varies continuously due to the falling shooting-stars and also due to combustion or spending in the atmosphere. He found that the dust of shooting-stars which fall on the surface of France in one year can cover a volume of 0.1 m^3. Oppalzer, 1884, was the first to analyze the reason for secular acceleration of the Moon as the result of Earth and Moon mass increase. Namely, during a hundred year a 2.8 mm dust layer is formed on the Earth. Gylden, 1884, extended the previous investigations in celestial mechanics by analyzing of the relative motion of two variable mass systems under influence of the Newton force. Meshchersky, 1893, continued the investigation and found that the body with variable mass would move along a spiral, tending toward zero, or it would increase the distance to the other mass variable system.

Cayley in his works (Cayley, 1857; 1858) was the first to consider the influence of the continual mass variation on the motion of the body. The class of the dynamic problem he studied was the 'continuous reactive problem', i.e., the problem when continuously the infinitesimal small mass is added to a system which causes the velocity of the system continuously to be changed for a definite value. The two most widely discussed examples were (see Cayley, 1859): one, the chain is on the table and is dropping vertically down from the table, and the second, the chain is moving straightforward on a horizontal plane without friction under the influence of a mass M which is fixed at the end of a chain which is rolling around a drum and changing the length during motion.

At the end of the nineteenth century and at the beginning of the twentieth Meshchersky, 1897, laid the foundations of the modern dynamics of a particle with variable mass. After that publication numerous investigations have been done and the dynamics of variable mass systems is developed. In the 'variable mass systems' particles are expelled and /or captured during motion.

Two kinds of systems are identified:

- 'continuously' particle-ejecting systems, where the mass variation is a continual function of the time, position (see Grudtsyn, 1972) or velocity of the particle, and
- 'discretely' particle-ejecting systems, where the mass variation is a discontinual function (for example, automatic weapons that fire rounds, one at a time).

For the case of discontinual mass variation Meshchersky, 1952, calculated velocity of the particle after mass variation. The finite discontinual mass variation in a very short time was not of special interest for a long time and was not intensively discussed. Meshchersky was the first to consider the velocity change of a translatory moving body during step-like mass variation. Mass which is separated or added has a finite amount and the mass variation of the body is discontinual. The theory is mostly applied for solving of the Keplerian twobody problem (see Luk'yanov, 2005), and also the three and four-body problems (Cveticanin, 2007).

The motion of the continuously mass variable systems is much more investigated due to its application in rocket theory (Meirovitch, 1970; Cormelisse *et al*, 1979; Tran & Eke), astronomy (Kayuk & Denisenko, 2004), for charged particle motion in a magnetic field with decreasing mass and charge (Howard, 2007), in robotics (McPhee & Djerassi, 1991; Djerassi, 1998) in machinery (Cveticanin, 1984; 1988; 1989; 1991; 1993_1; 1993_2; 1995; 1998_1; 2001) etc. The motion is described with differential equations with variable parameters (Kayuk & Tivalov, 1987; Wang & Eke, 1995; Eke & Mao, 2002; Pesce, 2003). The effect of expulsion and /or capture of particles on the motion of the continuously mass variable system is evident as changes in the integration variables of the governing dynamic equations. In spite of that the variable mass systems may be conservative (Leubner & Krumm, 1990; Cveticanin, 1994) and with non-holonomic properties (Ge & Cheng, 1982; Ge, 1984). For the case when the mass is continually varying in time, the influence of the reactive force on the motion (see Apykhtin & Jakovlev, 1980; Azizov, 1986; Cveticanin, 1992; 1993_3; 2004) and also on the stability (Ignat'yev, 1991; Cveticanin, 1996_1; 1996_2) were investigated. The reactive force is mathematically the product of the mass variation function and the relative velocity of mass separated or added to the particle. Usually, two special cases were considered: first, the relative velocity is zero and second, the absolute velocity of separated or added mass is zero. If the relative velocity is zero, i.e., the absolute velocity of the separated or added particle is equal to the velocity of the basic particle, then the reactive force is also zero. Levi-Civita, 1928, investigated the motion of the particle for the case when the absolute velocity of the separated or added mass is zero and the reactive force exists. The most comprehensive consideration of the dynamics of the body with variable mass is given in the books of Bessonov, 1967, Conelisse *et al.,* 1979, and Cveticanin, 1998_2.

In this chapter various types of machines and mechanisms with mass variable elements are described. The expression "variable mass element" or "body with variable mass" as used in the context of this dissertation, refers to mechanical systems that lose and/or gain mass while in motion. Examples of such devices abound in contemporary engineering literature. They include complex systems such as aircraft, rockets, automobiles, moving robots picking up or lifting objects, as well as simpler systems such as transportation machines, mining machines, excavators, vibrating machines used as conveyors, separators, machines for segregation, rolling mills, metallurgical machines, casting machines, agricultural machines, centrifuges, measuring mechanisms, etc. In these mechanisms, the mass of elements and the position of their centre of mass and moment of inertia vary during addition or removal of the material. In this chapter various types of machines and mechanisms with mass variable elements are described. Their constructive properties and the working procedure are explained.

A pouring machine, shown in Fig.1, represents a mechanism with variable mass. The hook of the crane grips the vessel at point B, and rotates it around A, pouring out the molten metal for continuous casting. The basic requirement for the equipment is that pouring of metal from the vessel has to be uniform, and it is regulated by the velocity of the hook.

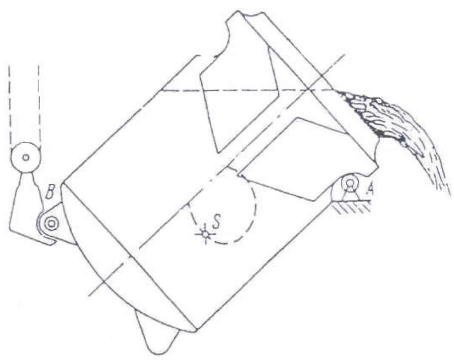


Fig.1. Mechanism for tipping a vessel

To estimate the process of lifting of the vessel and to give good control it is necessary to take into consideration the mass variation of the vessel, because it is the basic cause of uniform flow of metal. The position of the centre of mass of the vessel and metal varies. As a simplification this system can be considered as a rotating bar AB with variable mass and variable position of the centre of mass inside the element. Let us consider some other mechanisms. The pouring of concrete from the vessel in Fig.2 is achieved by a mechanism which contains an electric motor 1, which by a system of gears 4 and 6 and screws 2 and 3 transmits the motion to 4. As 4 is fixed to vessel 5 its motion about 6 turns the vessel. To analyze the elements and the process of vessel turning, the variable mass of the vessel has to be taken into account. Only in this way can the correct position of the turning axle and the torque for vessel rotating be obtained.

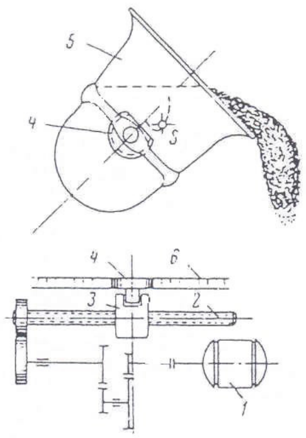


Fig.2.Mechanism for rotating a vessel

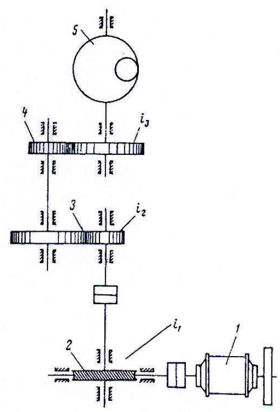


Fig.3. Mechanism for converter turning

Converters are used for steel production. For melting the steel, the converter-vessel system is usually lifted and turned during the introduction some additives and pouring out the finished steel. An electric motor 1 with a system of gear-wheels 2, 3 and 4 rotates the converter 5 (Fig.3). Exact dynamic analysis taking into account the variation of the mass of metal in the converter is necessary for automatic control of the process.A system of a rotor type of wagon turner is shown in Fig.4. The wagon with material 1 is in the rotor 2. The rotor rotates on cylinders 3 and is driven with a mechanism 4 and rope 5. The wagon is supported by positioners 6 and 7. By turning the wagon by an angle smaller than 180 degrees discharging is completed. The efficiency of this mechanism is very high.

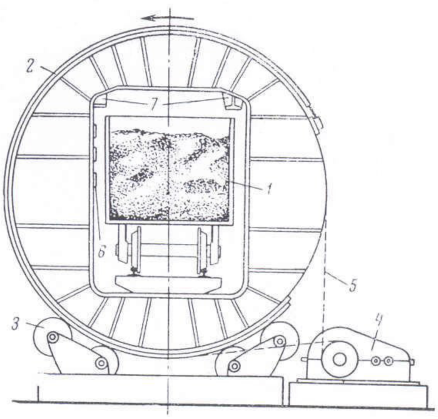


Fig.4. Mechanism for wagon turning

The mechanism for discharging wagon which is widely used in the metallurgical industry for minerals is of the rotating type (Fig.5). The wagon 1 is on the carrier 2, and is supported on the wall 3 and elements 4. The rope 5 is turning on the wheel and tips the carrier, discharging the material from the wagon. Torques are balanced with a system of counterweights. The mechanism productivity is high as the discharging is rapid. For dynamic analysis, the mass variation has to be taken into consideration.

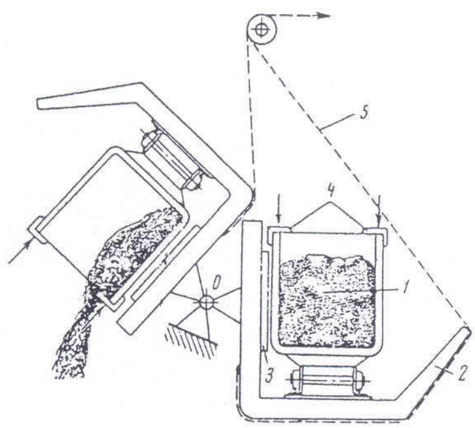


Fig.5. Mechanism for discharging a wagon

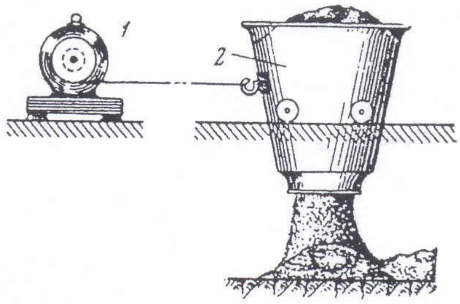


Fig.6. Wagon with variable mass

A wagon 2 which moves laterally is shown in Fig.6. It is driven by a motor 1. Another type of laterally moved wagon is shown in Fig.7. Movement in one direction increases the mass and in the other it is decreased. The mechanism is a bar mechanism, and the motion may be reversible. In both mechanisms the mass varies.

A rolling mill is shown in Fig.8. Tin rolls down off the drum 1, travels through the rollers 2 and rolls up on the drum 3. The system is fixed by supporting rollers 4. Cold rolling is caused at a velocity 10 to 15 m/sec and more. During rolling up this tin and rolling it off the radius and the moment of inertia of the drum varies. One of the most important requirements is constant velocity of rolling up, and special automatic regulation and controls have to be introduced. To increase the efficiency of these machines, the dynamics of motion of this system with variable mass have to be analyzed.

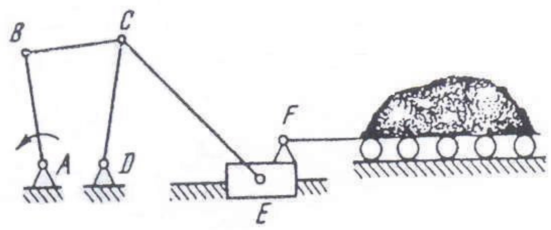


Fig.7. Transportation mechanism

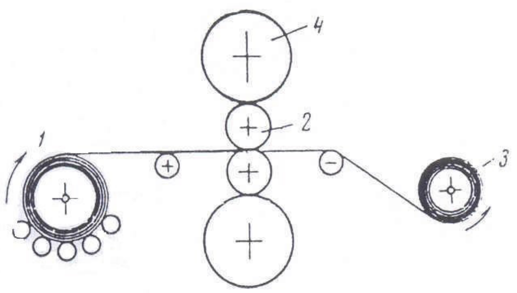


Fig.8. Rolling machine scheme

Another mechanism with variable moment of inertia is shown in Fig.9.

It is a centrifugal lifting basket regulator of motion. It contains shaft 1, system of gears, and a system of mercury filled tubes. The velocity of the shaft and of the tubes regulates the velocity of the basket. If the velocity of tubes is zero, the level of mercury in all tubes is the same. If the velocity increases, the mercury level attains a parabolic shape and the level in the middle tube decreases. If it gets to a very small level, the connection with the motor is broken and the shaft 1 slows down. If the velocity of basket motion is smaller than permitted, the regulator sets up the motor and transfers the motion to the shaft 1. As the position of the mercury depends on velocity, the moment of inertia varies, as a function of angular velocity, J(w). To give the correct instructions for regulator dynamic behavior it is necessary to know the change of moment of inertia of the mechanism.

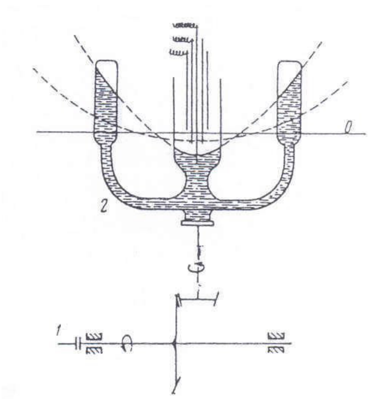


Fig.9. Centrifugal regulator

A simple planar model of an excavator is shown in Fig.10. It consists of five rigid bodies: body 1 which is in contact with the ground, body 2 mounted on 1, body 3 connected to 2 by means of pivot, body 4 which is an excavator arm connected with 3 by means of a joint, and body 5 which is the excavator scoop and can rotate around the body 4.

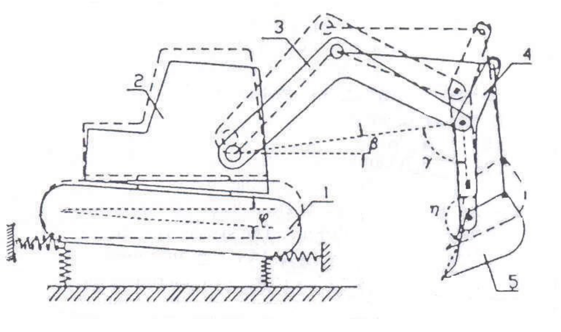


Fig.10. Excavator model

The excavator scoop has a varying mass. The operating movements can be realized by means of the jib 3, arm 4 or the scoop 5 movements. A crane represents a mechanism for load transportation.

A typical sectional model of a crane (Fig.11) consists of four rigid bodies: body 1 which is in elastic contact with the ground, body 2 mounted on 1, body 3 connected with 2 by means of joint and a component 4 connected with 3 by a flexible cord. Component 4 has a time variable mass, because it includes the load. Variation of mass and its distribution along the elements of a mechanism have a significant influence on belt type automatic dosing devices.

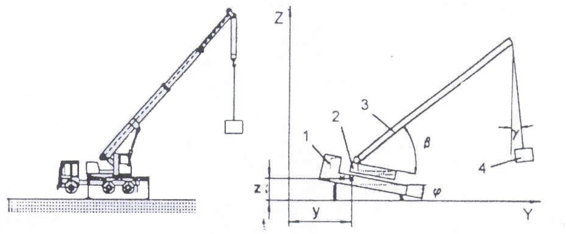


Fig.11. Sectional model of crane mechanism

The aim of the mechanism is to achieve constant mass flow of material. The mechanism (Fig.12) contains basket 1, belt conveyor 2 and bar mechanism 3. The bar is a sensitive element which directs the opening and shutting of the dosing device. The process of separating material from the conveyor may result in large vibrations around O. These vibrations are undesirable and they are due to mass variation.

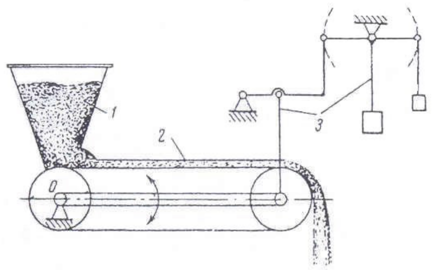


Fig.12. Belt type automatic dosing device

A mechanism for automatic measurement of liquid is shown in Fig.13. It contains bar 1, which is pivoted at 2 and is under the influence of weight 3, and vessel 4 which is to be filled with liquid. The quantity of liquid is regulated by a special dosing device 5 which automatically regulates the liquid in the vessel by stopping the liquid getting in. It is connected to bar 1.

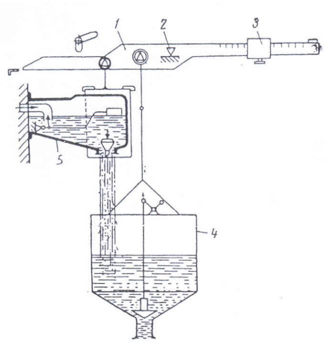


Fig.13. Mechanism for automatic measurement of liquid

A special group of mechanisms are the rotors with variable mass. Rotors with variable mass are the fundamental working elements of many machines in process, cable, textile and paper industry, as well as in transportation, etc. The term "rotor with variable mass" will refer to all parts which are mounted on the rotating shaft and whose mass is varying. The shaft which is a fundamental element of the rotor is designed to support machine parts rotating with it and to transmit bending moments and torques and also longitudinal forces.

As can be seen, mechanisms with variable mass are widely used in various industrial fields: process, transportation, civil engineering, regulators etc. Most of them can be reduced to a simple dynamic model with mass variation which is a function of time (for example, vibration mechanisms, dosing devices, etc). For correct dynamic analysis of the mechanical system it is necessary to take this property of the system into consideration.

2 Linear and angular momentums for the mass variable body

In this Chapter the linear and angular momentums for the bodies in the process of separation or addition of the bodies are determined. These values are necessary to express the general principles of dynamics of system of bodies with variable mass and for mathematical modeling of separation or adding of bodies.

Let us consider the discontinual mass variation caused by the body separation or augmentation. The initial body has the mass M and the moment of inertia $\mathbf{I}_S$ with respect to the mass centre S whose position vector due to the fixed point O is $\mathbf{r}_S$ (see Fig.14.).

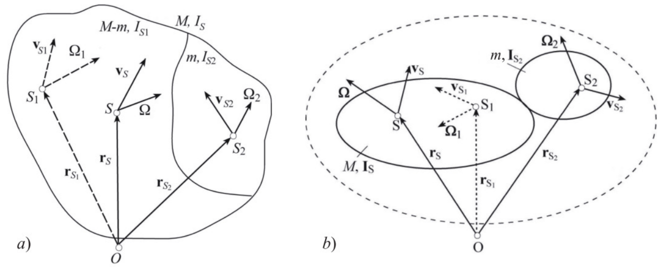


Fig.14. Position vectors, velocities and angular velocities of the system: a) body separation, b) body augmentation.

The linear velocity of the mass centre is v_S and the angular velocity of the initial body around the mass centre S is $\mathbf{\Omega}$. The position vector of the mass centre S to the fixed point O is $\mathbf{r}_S$. The separating or adding body has mass m and the mass centre S_2 . Moment of inertia of that body is $\mathbf{I}_{S2}$ with respectto the mass centre S_2.The absolute velocity of the mass center S_2 is $\mathbf{v}_{S2}$ while the angular velocity of the body with respect to point of rotation S_2 is $\mathbf{\Omega}_2$. The position vector of the mass centre S_2 according to the fixed point O at the moment of adding or separating is $\mathbf{r}_{S2}$. If the separation of the body occurs, the remainder body has the mass $(M-m)$ and if the augmentation occurs the mass is $(M+m)$. The moment of inertia of the final body is $\mathbf{I}_{S1}$ due to the mass centre S_1 whose position to the fixed point O is given with the vector $\mathbf{r}_{S1}$. The unknown linear and angular velocity of the body after mass variation are $\mathbf{v}_{S1}$ and $\mathbf{\Omega}_1$. In the Table 1. the introduced Nomenclature is shown.

The linear momentum of the initial body with mass M and velocity of mass center $\mathbf{v}_S$ is

$$\mathbf{K} = M\mathbf{v}_S. \tag{1}$$

If the separated or added mass m has the velocity $\mathbf{v}_{S2}$, its linear momentum is

$$\mathbf{K}_2 = m\mathbf{v}_{S2}. \tag{2}$$

After process of separation or addition the final mass is $M \mp m$, where the minus sign is for mass separation and plus sign for mass addition.

Table 1. Nomenclature.

	Initial body	Final body	Separated/ Added body
Index	-	1	2
Mass center	S	S_1	S_2
Position of mass center		$\mathbf{SS}_1$	$\mathbf{SS}_2$
Position vector of mass center	$\mathbf{r}_S$	$\mathbf{r}_{S1}$	$\mathbf{r}_{S2}$
Mass	M	$M \mp m$	$\mp m$
Absolute velocity of mass center	$\mathbf{v}_S$	$\mathbf{v}_{S1}$	$\mathbf{v}_{S2}$
Relative velocity of mass center		$\mathbf{v}_{r1}$	$\mathbf{v}_r$
Absolute angular velocity	$\mathbf{\Omega}$	$\mathbf{\Omega}_1$	$\mathbf{\Omega}_2$
Relative angular velocity		$\mathbf{\Omega}_{r1}$	$\mathbf{\Omega}_r$
Moment of inertia tensor	$\mathbf{I}_S$	$\mathbf{I}_{S1}$	$\mathbf{I}_{S2}$
Linear momentum	$\mathbf{K}$	$\mathbf{K}_1$	$\mathbf{K}_2$
Angular momentum relating to O	$\mathbf{L}_O$	$\mathbf{L}_{O1}$	$\mathbf{L}_{O2}$
Angular momentum relating to mass center	$\mathbf{L}_S$	$\mathbf{L}_{S1}$	$\mathbf{L}_{S2}$

Remark 1 *In general in this Chapter in any relation, the minus sign is for change of some quantity caused by mass separatiom and the plus sign is for mass addition.*

The unknown velocity of the mass centre S_1 is $\mathbf{v}_{S1}$ and the corresponding linear momentum is

$$\mathbf{K}_1 = (M \mp m)\mathbf{v}_{S1}. \tag{3}$$

Introducing the assumption that the bodies during the mass separation or addition belong to an unique system (see Fig.14), we obtain the linear momentums $\mathbf{K}_b$ before and $\mathbf{K}_a$ after mass variation:

for mass separation

$$\mathbf{K}_b = \mathbf{K} = M\mathbf{v}_S, \qquad \mathbf{K}_a = \mathbf{K}_1 + \mathbf{K}_2 = m\mathbf{v}_{S2} + (M - m)\mathbf{v}_{S1}, \tag{4}$$

for mass augmentation

$$\mathbf{K}_b = \mathbf{K} + \mathbf{K}_2 = M\mathbf{v}_S + m\mathbf{v}_{S2}, \qquad \mathbf{K}_a = \mathbf{K}_1 = (M + m)\mathbf{v}_{S1}, \tag{5}$$

The difference between the linear momentums before and after mass variation is

$$\Delta\mathbf{K} = M(\mathbf{v}_{S1} - \mathbf{v}_S) \pm m(\mathbf{v}_{S2} - \mathbf{v}_{S1}). \tag{6}$$

The angular momentum of the initial body before mass variation with respect to the fixed point O (Fig.14) is

$$\mathbf{L}_O = \mathbf{r}_S \times M\mathbf{v}_S + \mathbf{L}_S. \tag{7}$$

After mass variation the angular of the final body relating to the fixed point O is

$$\mathbf{L}_{O1} = \mathbf{r}_{S1} \times (M \mp m)\mathbf{v}_{S1} + \mathbf{L}_{S1}. \tag{8}$$

The angular momentum of the separated or added body is

$$\mathbf{L}_{O2} = \mathbf{r}_{S2} \times m\mathbf{v}_{S2} + \mathbf{L}_{S2}. \tag{9}$$

Dependently on the type of mass variation the angular momentums before $\mathbf{L}_{Ob}$ and after $\mathbf{L}_{Oa}$ mass variation are:

for mass separation

$$\begin{aligned} \mathbf{L}_{Ob} &= \mathbf{L}_O = \mathbf{r}_S \times M\mathbf{v}_S + \mathbf{L}_S, \\ \mathbf{L}_{Oa} &= \mathbf{L}_{O1} + \mathbf{L}_{O2} = \mathbf{r}_{S1} \times (M-m)\mathbf{v}_{S1} + \mathbf{L}_{S1} + \mathbf{r}_{S2} \times m\mathbf{v}_{S2} + \mathbf{L}_{S2}, \end{aligned} \tag{10}$$

for mass addition

$$\begin{aligned} \mathbf{L}_{Ob} &= \mathbf{L}_O + \mathbf{L}_{O2} = \mathbf{r}_S \times M\mathbf{v}_S + \mathbf{L}_S + \mathbf{r}_{S2} \times m\mathbf{v}_{S2} + \mathbf{L}_{S2}, \\ \mathbf{L}_{Oa} &= \mathbf{L}_{O1} = \mathbf{r}_{S1} \times (M+m)\mathbf{v}_{S1} + \mathbf{L}_{S1}. \end{aligned} \tag{11}$$

Based on (10) and (11), the difference between the angular momentums is

$$\Delta\mathbf{L}_O = \mathbf{L}_{S1} \pm \mathbf{L}_{S2} - \mathbf{L}_S + \mathbf{r}_{S1} \times (M \mp m)\mathbf{v}_{S1} \pm \mathbf{r}_{S2} \times m\mathbf{v}_{S2} - \mathbf{r}_S \times M\mathbf{v}_S. \tag{12}$$

The relation (6) and (12) are the basic ones for dynamic analysis of the mass variation problems.

For the position of the system mass center S

$$\mathbf{r}_S = \frac{M \mp m}{M}\mathbf{r}_{S1} \pm \frac{m}{M}\mathbf{r}_{S2}, \tag{13}$$

and position vectors $\mathbf{r}_{S1}$ and $\mathbf{r}_{S2}$ (Fig.14)

$$\mathbf{r}_{S1} = \mathbf{r}_S + \mathbf{SS}_1, \qquad \mathbf{r}_{S2} = \mathbf{r}_S + \mathbf{SS}_2, \tag{14}$$

we obtain

$$(M \mp m)\mathbf{SS}_1 = \mp m\mathbf{SS}_2, \tag{15}$$

Substituting (14) into (12) we obtain

$$\Delta L_O = \mathbf{L}_{S1} \pm \mathbf{L}_{S2} - \mathbf{L}_S + \mathbf{SS}_1 \times (M \mp m)\mathbf{v}_{S1} \pm \mathbf{SS}_2 \times m\mathbf{v}_{S2}. \tag{16}$$

Introducing the relation (6) into (16) it is

$$\Delta\mathbf{L}_O = \mathbf{L}_{S1} \pm \mathbf{L}_{S2} - \mathbf{L}_S + \mathbf{r}_S \times \Delta K \mp \mathbf{SS}_2 \times m(\mathbf{v}_{S1} - \mathbf{v}_{S2}). \tag{17}$$

Due to (15) the relation (16) transforms into

$$\Delta L_O = \mathbf{L}_{S1} - \mathbf{L}_S \pm \mathbf{L}_{S2} \mp \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m, \tag{18}$$

i.e.,

$$\Delta L_O = \mathbf{I}_{S1}\mathbf{\Omega}_1 - \mathbf{I}_S\mathbf{\Omega} \pm \mathbf{I}_{S2}\mathbf{\Omega}_2 \mp \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m, \tag{19}$$

where $\mathbf{L}_S = \mathbf{I}_S\mathbf{\Omega}$ is the known angular momentum of the initial body before mass variation, $\mathbf{L}_{S1} = \mathbf{I}_{S1}\mathbf{\Omega}_1$ depends on the angular velocity $\mathbf{\Omega}_1$ of the final body with respect to S_1 and is proportional to the moment of inertia $\mathbf{I}_{S1}$ for S_1 and $\mathbf{L}_{S2} = \mathbf{I}_{S2}\mathbf{\Omega}_2$ is the known angular momentum of the separated or added body with the angular velocity $\mathbf{\Omega}_2$ with respect to S_2 and moment of inertia $\mathbf{I}_{S2}$ for S_2.

3 Dynamics of the body with discontinual mass variation

In this Chapter the dynamics of the discontinual mass variaton of the bodies is considered. The Chapter has four sections. In the first and the second section the velocity and the angular velocity of the final body, formed after separation or addition of an additional body, are determined. In the third section, as the special case of the previous two, the in-plane motion of the system of bodies during addition and separation is analyzed. The obtained results are applied for solving of the real problem of separation of a part from the rotor whose disc is assumed to have an in-plane motion. The Chapter ends with Conclusion.

Dynamics of the discontinual mass variation requires some assumptions during the process of body separation or addition and are as follows (Cveticanin and Djukic, 2008):

1. The separated and the final body, and also the initial body and the added body, form a unique system during mass variation (see Fig.14);

2. The separated body leaves the system after the process of separation. The added body gets into the system before mass augmentation;

3. Separation or adding of the body is done in a very short time interval τ;

4. The considered bodies are rigid during mass separation;

5. Due to the assumption 1) we can regard the two parts of the body as a complex system, where the reaction forces and torques between these parts are internal within the system;

6. During the process of mass variation the external forces $\mathbf{F}_i$ and torques $\mathfrak{M}_j$, which act on the system, produce the impulses.

Let $\mathbf{F}_r$ be the resultant force of all external activ forces and constraint reactions, which are acting on the bodies. According to the principle of the momentum, the variation of the linear momentum (6) for the time interval from $\tau = \Delta t$ is equal to the impulse $\mathbf{I}^{Fr}$ of the resultant force $\mathbf{F}_r$

$$\Delta \mathbf{K} = \mathbf{F}_r \Delta t \equiv \mathbf{I}^{Fr}. \tag{20}$$

Substituting (6) into (20) it is

$$M(\mathbf{v}_{S1} - \mathbf{v}_S) \pm m(\mathbf{v}_{S2} - \mathbf{v}_{S1}) = \mathbf{I}^{Fr}. \tag{21}$$

According to the principle of the angular momentum, the variation of the angular momentum (19) in the time interval Δt is equal to the impulse $\mathbf{I}^M$ which is the sum of the impulse of the moment of resultant force for the point O, $\mathbf{M}_0^{Fr}$, and of the impulse of the resultant torque $\mathfrak{M}$, caused by activ torque and reaction torque, i.e.,

$$\Delta \mathbf{L}_O = (\mathbf{M}_0^{Fr} + \mathfrak{M})\Delta t = \mathbf{I}^M. \tag{22}$$

Substituting (16) into (22) we have

$$\mathbf{L}_{S1} \pm \mathbf{L}_{S2} - \mathbf{L}_S + \mathbf{SS}_1 \times (M \mp m)\mathbf{v}_{S1} \pm \mathbf{SS}_2 \times m\mathbf{v}_{S2} = \mathbf{I}^M. \tag{23a}$$

Usually, the impulses of the external forces and torques are quite small due to the short time τ. It is the reason, that the system is usually assumed to be without action of the external forces and torques. Then, the linear momentum of the system before and after mass variation remains invariable. The same conclusion is valid also for the angular momentum.

3.1 Velocity of the body after mass variation

According to (21) the velocity of the body after mass variation is

$$\mathbf{v}_{S1} = \frac{1}{(M \mp m)}(M\mathbf{v}_S \mp m\mathbf{v}_{S2} + \mathbf{I}^{Fr}). \tag{24}$$

Using the fact that the absolute velocity of the mass center S_2 of the separated or added body is the sum of the dragging velocity of S_2 and the relative velocity $\mathbf{v}_r$ of the point S_2 with respect to the point S. The dragging velocity of S_2 has two components: translatory $\mathbf{v}_S$ and velocity of rotation $\mathbf{\Omega} \times \mathbf{SS}_2$ of the point S_2 with respect to point S. The absolute velocity of the point S_2 for mass varaiation is

$$\mathbf{v}_{S2} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_2 + \mathbf{v}_r. \tag{25}$$

The absolute velocity of the mass center S_1 of the final body after mass variation has the form

$$\mathbf{v}_{S1} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_1 + \mathbf{v}_{r1}, \tag{26}$$

where $\mathbf{v}_S$ and velocity of rotation $\mathbf{\Omega} \times \mathbf{SS}_1$ are the translatory and rotational components of velocity of mass center S_1 and $\mathbf{v}_{r1}$ is the relative velocity of the point S_1. Substituting (25) and (26) into (24), the following relation is

$$(M \mp m)\mathbf{v}_{r1} \pm m\mathbf{v}_r = \mp m(\mathbf{\Omega} \times \mathbf{SS}_2) - \mathbf{\Omega} \times (M \mp m)\mathbf{SS}_1) + \mathbf{I}^{Fr}. \tag{27}$$

The relations (27) and (15) give the relative velocity of the mass center S_1 of the final body

$$(M \mp m)\mathbf{v}_{r1} = \mathbf{I}^{Fr} \mp m\mathbf{v}_r, \tag{28}$$

and the corresponding absolute velocity

$$\mathbf{v}_{S1} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_1 + \frac{\mathbf{I}^{Fr} \mp m\mathbf{v}_r}{(M \mp m)}. \tag{29}$$

Let us assume the system without external forces. In that case the linear momentum of the system is same before and after mass variation, hence $\Delta\mathbf{K} = 0$. According to the relation (29), we determine the velocity of mass center S_1 of the body after mass variation as

$$\mathbf{v}_{S1} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_1 \mp \frac{m\mathbf{v}_r}{(M \mp m)}. \tag{30}$$

For the case when the separation of the body occurs, the velocity of the body after mass transformation is due to (30)

$$\mathbf{v}_{S1} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_1 - \frac{m}{M - m}\mathbf{v}_r. \tag{31}$$

3.2 Angular velocity of the body after mass variation

Using the relations (22) and (19), it follows

$$\mathbf{I}_{S1}\mathbf{\Omega}_1 = \mathbf{I}^M + \mathbf{I}_S\mathbf{\Omega} \mp \mathbf{I}_{S2}\mathbf{\Omega}_2 \pm \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m. \tag{32}$$

The relation is suitable for calculation of the angular velocity $\mathbf{\Omega}_1$ of the final body after mass variation.

Using the assumption 1) that two bodies form one unique system and no external forces and torques act, it is stated that the angular momentums of the body before and after mass variation are invariable, i.e., $\Delta \mathbf{L}_O = 0$, and due to (32) the angular velocity $\boldsymbol{\Omega}_1$ is

$$\mathbf{I}_{S1}\boldsymbol{\Omega}_1 = \mathbf{I}_S\boldsymbol{\Omega} \mp \mathbf{I}_{S2}\boldsymbol{\Omega}_2 \pm \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m. \tag{33}$$

For the case when mass separation occurs, the angular velocity of the final body is

$$\mathbf{I}_{S1}\boldsymbol{\Omega}_1 = \mathbf{I}_S\boldsymbol{\Omega} - \mathbf{I}_{S2}\boldsymbol{\Omega}_2 + \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m, \tag{34}$$

where $\mathbf{v}_{S1}$ satisfies the Eq. (31).

The value of the angular velocity $\boldsymbol{\Omega}_1$ and the velocity of mass center after mass variation $\mathbf{v}_{S1}$ represent the initial values for the motion of the final body.

3.2.1 Remarks

Due to the assumptions and the relations (31) and (34), it can be concluded:

1. During the process of mass variation, which lasts for the infinitesimal time interval, $t \in [t_1, t_1 + \tau]$, the interaction of the separated and final body or initial and added bodies results in a finite change of the linear and the angular velocity of the body parts. The linear momentum and the velocity of the final body and also the angular momentum and the angular velocity of the final body receive finite increments during the infinitesimal time, that is, these quantities change in a jump-like manner.

2. According to the assumption 3), during the process of mass variation the position change of the bodies is negligible, i.e., the position vectors of mass centers and the angle position of bodies are not varying during mass variation.

3. Due to the aforementioned Remarks it is concluded that the body additon corresponds to the perfect plastic impact where the relative velocity of the adding body is zero.

4. Accordint to here obtained results it is obvious that the body separation is the inverse process to the perfectly plastic impact where the relative velocity of the separated body is zero. As for the plastic impact the restitution coefficient is zero, the same is evident for the body separation. It means that the motion does not depend on the geometric and dynamical properties of the separation surface.

5. In this Chapter more attention is given to the mass separation as the dynamics of mass augmentation can be treated as the plastic impact.

3.3 In-plane separation of the body

Consider the body which moves in-plane before and after body separation. The absolute velocity of the initial body, remainder (final) and separated body are defined by (24) and (28), with projections

$$\mathbf{v}_S = v_{Sx}\mathbf{i} + v_{Sy}\mathbf{j}, \qquad \mathbf{v}_{S1} = v_{S1x}\mathbf{i} + v_{S1y}\mathbf{j}, \qquad \mathbf{v}_{S2} = v_{S2x}\mathbf{i} + v_{S2y}\mathbf{j}, \tag{35}$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors in the plane of motion (Fig.15).

For the in-plane motion the angular moments of the initial body, separated body and remainder body are

$$\mathbf{L}_{S1} = I_{S1}\Omega_1\mathbf{k}, \qquad \mathbf{L}_{Sb} = I_S\Omega\mathbf{k}, \qquad \mathbf{L}_{S2} = I_{S2}\Omega_2\mathbf{k}, \tag{36}$$

where $\mathbf{k}$ is the unit vector orthogonal to the plane of motion. Substituting (36) into (34) we obtain the angular velocity of the remainder body as a function of the angular velocity of the separated body

$$I_{S1}\Omega_1\mathbf{k} = I_S\Omega\mathbf{k} - I_{S2}\Omega_2\mathbf{k} + \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m, \tag{37}$$

where

$$\mathbf{SS}_2 = SS_{2x}\mathbf{i} + SS_{2y}\mathbf{j}, \tag{38}$$

In relation (37) the absolute angular velocity of the remainder body is the function of the absolute angular velocity of the separated body.

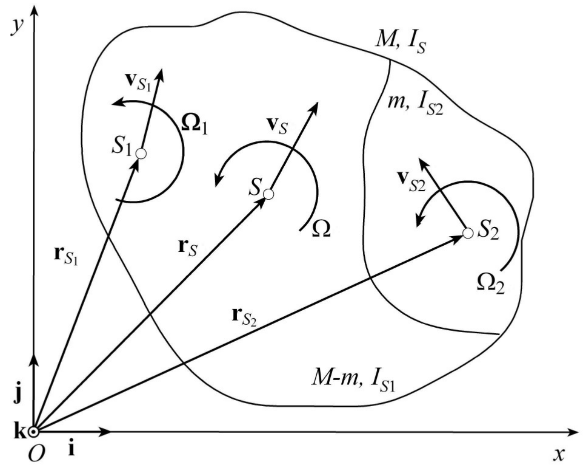


Fig.15. Position vectors of the plane body centers S, S_1 and S_2 with respect to a fixed point O.

Introducing (25) and (26) and also (36) into (32) leads to

$$\begin{aligned} I_S\Omega\mathbf{k} &= I_{S1}\Omega_1\mathbf{k} + \mathbf{SS}_1 \times (M-m)(\mathbf{v}_S + \mathbf{\Omega}\times\mathbf{SS}_1 + \mathbf{v}_{r1}) \\ &\quad + I_{S2}\Omega_2\mathbf{k} + \mathbf{SS}_2 \times m(\mathbf{v}_S + \mathbf{\Omega}\times\mathbf{SS}_2 + \mathbf{v}_r). \end{aligned} \tag{39}$$

Using the relation (15), the Stainer formulas for the moment of inertia for the axis in S parallel to the axis in S_1 and S_2

$$I_{S1} = I_1 + (SS_1)^2(M-m), \qquad I_{S2} = I_2 + (SS_2)^2 m, \tag{40}$$

and the relative angular velocities

$$\Omega_1 = \Omega + \Omega_{r1}, \qquad \Omega_2 = \Omega + \Omega_r, \tag{41}$$

the relation (39) yields

$$I_{S1}\Omega_{r1}\mathbf{k} = -\, I_{S2}\Omega_r\mathbf{k} - \mathbf{SS}_2 \times m(\mathbf{v}_r - \mathbf{v}_{r1}). \tag{42}$$

For the difference of the relative velocities

$$\mathbf{v}_r - \mathbf{v}_{r1} = (v_{rx} - v_{r1x})\mathbf{i} + (v_{ry} - v_{r1y})\mathbf{j}, \tag{43}$$

and (42), the relative angular velocity of the remainder body is obtained

$$\Omega_{r1} = \frac{m}{I_{S1}}[SS_{2y}(v_{rx} - v_{r1x}) - SS_{2x}(v_{ry} - v_{r1y})] - \frac{I_{S2}}{I_{S1}}\Omega_r. \tag{44}$$

The relations (28) and (44) define the relative velocity and angular velocity of the remainder body during in-plane body separation.

3.3.1 Some special cases

Depending on the velocity and the angular velocity of the separated body, the various cases of body motion are possible. In the following Tables some special cases of body separation are shown, together with the corresponding properties of the remainder body, calculated on the bases of (28) and (42).

Table 2. provides properties of the remainder body for the case when the initial body is in translation. Subcases for various absolute velocity $\mathbf{v}_{S2}$ of mass center and absolute angular velocity Ω_2 of the separated body are considered.

Table 2. Translation of the whole body

$\mathbf{v}_S \neq 0$ $\quad \Omega = 0$

No.	Separated body	Remainder body
1	$\mathbf{v}_{S2} \neq 0, \quad \Omega_2 \neq 0$	$\mathbf{v}_{r1} = -\frac{m}{M-m}\mathbf{v}_r,$ $I_{S1}\Omega_1\mathbf{k} = \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m$ $-I_{S2}\Omega_2\mathbf{k}.$
2	$\mathbf{v}_{S2} \neq 0, \quad \Omega_2 = 0$	$\mathbf{v}_{r1} = -\frac{m}{M-m}\mathbf{v}_r,$ $I_{S1}\Omega_1\mathbf{k} = \mathbf{SS}_2 \times (\mathbf{v}_{S1} - \mathbf{v}_{S2})m.$
3	$\mathbf{v}_{S2} = 0, \quad \Omega_2 \neq 0$	$\mathbf{v}_{S1} = \frac{M}{M-m}\mathbf{v}_S,$ $I_{S1}\Omega_1\mathbf{k} = \frac{Mm}{M-m}\mathbf{SS}_2 \times \mathbf{v}_S$ $-I_{S2}\Omega_2\mathbf{k}.$
4	$\mathbf{v}_{S2} = 0, \quad \Omega_2 = 0$	$\mathbf{v}_{S1} = \frac{M}{M-m}\mathbf{v}_S,$ $I_{S1}\Omega_1\mathbf{k} = \frac{Mm}{M-m}\mathbf{SS}_2 \times \mathbf{v}_S.$
5	$\mathbf{v}_{S2} \neq 0, \quad \Omega_2\mathbf{k} = -\frac{mM}{I_{S2}(M-m)}\mathbf{SS}_2 \times \mathbf{v}_r$	$\mathbf{v}_{S1} = \mathbf{v}_S - \frac{m}{M-m}\mathbf{v}_r,$ $\Omega_1 = 0.$
6	$\mathbf{v}_{S2} = \frac{M}{m}\mathbf{v}_S, \quad \Omega_2 \neq 0$	$\mathbf{v}_{S1} = 0,$ $I_{S1}\Omega_1\mathbf{k} = -M(\mathbf{SS}_2 \times \mathbf{v}_S)$ $-I_{S2}\Omega_2\mathbf{k}.$
7	$\mathbf{v}_{S2} = \frac{M}{m}\mathbf{v}_S, \quad \Omega_2\mathbf{k} = \frac{M}{I_{S2}}(\mathbf{v}_S \times \mathbf{SS}_2)$	$\mathbf{v}_{S1} = 0, \quad \Omega_1 = 0.$
8	$\mathbf{v}_{S2} = \mathbf{v}_S, (\mathbf{v}_r = 0) \quad \Omega_2 \neq 0$	$\mathbf{v}_{S1} = \mathbf{v}_S, \quad I_{S1}\Omega_{r1}\mathbf{k}$ $= -I_{S2}\Omega_r\mathbf{k}.$

In Table 3. the velocity and the angular velocity of the remainder body for the case when the initial body is rotating are shown. Subcases are considered with respect to various values of the relative velocity v_r of mass center and the relative angular velocity Ω_r of the separated body.

Table 3. Rotating of the whole body

$\mathrm{v}_S = 0$ $\quad \Omega \neq 0$

No.	Separated body	Remainder body
1	$\mathbf{v}_r \neq 0, \quad \Omega_r \neq 0$	$\mathbf{v}_{S1} = \boldsymbol{\Omega} \times \mathbf{SS}_1 - \frac{m}{M-m}\mathbf{v}_r,$ $\Omega_{r1}\mathbf{k} = \frac{M}{I_{S1}}(\mathbf{SS}_1 \times \mathbf{v}_r) - \frac{I_{S2}}{I_{S1}}\Omega_r\mathbf{k}.$
2	$\mathbf{v}_r = 0, \quad \Omega_r \neq 0$	$\mathbf{v}_{S1} = \boldsymbol{\Omega} \times \mathbf{SS}_1,$ $\Omega_{r1}\mathbf{k} = -\frac{I_{S2}}{I_{S1}}\Omega_r\mathbf{k}.$
3	$\mathbf{v}_r = 0, \quad \Omega_r = 0$	$\mathbf{v}_{S1} = \boldsymbol{\Omega} \times \mathbf{SS}_1,$ $\Omega_{r1} = 0, \quad \Omega_1 = \Omega.$
4	$\mathbf{v}_r \parallel \mathbf{SS}_1, \quad \Omega_r \neq 0$	$\mathbf{v}_{S1} = \boldsymbol{\Omega} \times \mathbf{SS}_1 - \frac{m}{M-m}\mathbf{v}_r,$ $\Omega_{r1}\mathbf{k} = -\frac{I_{S2}}{I_{S1}}\Omega_r\mathbf{k}.$
5	$\mathbf{v}_{S2} = \boldsymbol{\Omega} \times \mathbf{SS}_2 + \mathbf{v}_r,$ $\boldsymbol{\Omega}_r = -\boldsymbol{\Omega}, (\Omega_2 = 0)$	$\mathbf{v}_{S1} = -\frac{m}{M-m}\mathbf{v}_{S2},$ $I_{S1}\Omega_1\mathbf{k} = I_S\Omega\mathbf{k} - \frac{Mm}{M-m}(\mathbf{SS}_2 \times \mathbf{v}_{S2}).$

Table 3.			
6	$\mathbf{v}_r = \frac{M-m}{m}(\mathbf{\Omega} \times \mathbf{SS}_1)$,	$\Omega_r \neq 0$	$\mathbf{v}_{S1} = 0$, $\Omega_{r1} = \frac{M}{I_{S1}}\frac{M-m}{m}\Omega(SS_1)^2 + \frac{I_{S2}}{I_{S1}}\Omega_r$.
7	$\mathbf{v}_r \neq 0$, $I_{S2}\Omega_r\mathbf{k} = M(\mathbf{SS}_1 \times \mathbf{v}_r) + I_{S1}\Omega\mathbf{k}$		$\mathbf{v}_{S1} = \mathbf{\Omega} \times \mathbf{SS}_1 - \frac{m}{M-m}\mathbf{v}_r$, $\Omega_{r1} = 0$, $\Omega_1 = \Omega$.
8	$\mathbf{v}_r = \frac{M-m}{m}(\mathbf{\Omega} \times \mathbf{SS}_1)$, $\Omega_r = (\frac{I_{S1}}{I_{S2}} - \frac{M(M-m)}{m}\frac{SS_1^2}{I_{S2}})\Omega$		$\mathbf{v}_{S1} = 0$, $\Omega_{r1} = 0$, $\Omega_1 = \Omega$.

Table 4.	Plane motion of the whole body		
$\mathbf{v}_S \neq 0$	$\Omega \neq 0$		
No.	Separated body		Remainder body
1	$\mathbf{v}_r = \mathbf{SS}_2 \times \mathbf{\Omega}$, $(\mathbf{v}_{S2} = \mathbf{v}_S)$,	$\Omega_2 \neq 0$	$\mathbf{v}_{r1} = \mathbf{SS}_1 \times \mathbf{\Omega}$, $(\mathbf{v}_{S1} = \mathbf{v}_S)$, $\Omega_1 = \frac{I_S}{I_{S1}}\Omega - \frac{I_{S2}}{I_{S1}}\Omega_2$.
2	$\mathbf{v}_r = \mathbf{SS}_2 \times \mathbf{\Omega}$,	$\Omega_2 = (I_S/I_{S2})\Omega$	$\mathbf{v}_{S1} = \mathbf{v}_S$, $\Omega_1 = 0$.
3	$\mathbf{v}_r = \mathbf{SS}_2 \times \mathbf{\Omega}$,	$\Omega_2 = 0$	$\mathbf{v}_{S1} = \mathbf{v}_S$, $\Omega_1 = (I_S/I_{S1})\Omega$.
4	$\mathbf{v}_r = \mathbf{SS}_2 \times \mathbf{\Omega}$,	$\Omega_r = 0$	$\mathbf{v}_{S1} = \mathbf{v}_S$, $\Omega_1 = \Omega$, $\Omega_{r1} = 0$.
5	$\mathbf{v}_r = 0$,	$\Omega_r = 0$	$\mathbf{v}_{r1} = 0$, $\Omega_{r1} = 0$.

Table 4. provides properties of the remainder body for the case when the whole body has the plane motion.

Analyzing the results in the Tables 2-4 the following is concluded:

1. If the relative velocity and the relative angular velocity of body separation are zero, the relative velocity and the relative angular velocity of the remainder body are also zero, independently of the type of motion of the initial body (see Table 2 case 4, Table 2 case 3 and Table 4 case 5). The absolute velocity of mass center of the remainder body is equal to the dragging velocity of S_1 before body separation. The angular velocity of the remainder body is equal to the angular velocity of the initial body before separation.

2. If the motion of the initial body and of the separated body is translatory with velocity $\mathbf{v}_S$ (the relative velocity $\mathbf{v}_r$ is zero), the velocity of mass center of the remainder body is also $\mathbf{v}_S$. This result was previously obtained by I.V. Meshchersky, 1896, for the continual mass variation of the translatory moving particle. Namely, due to the fact that the relative velocity of mass separation is zero, the reactive force is also zero and the equation of motion is the same as for the body without mass change.

3. If the motion of the initial body is translatory with the velocity $\mathbf{v}_S$ and the absolute velocity and the absolute angular velocity of separation of the body are zero, the absolute velocity of mass center of the remainder body differs from the velocity of the initial body. The velocity depends on the mass which is separated: for the higher value of separated mass m i.e., for smaller value of the remainder mass, the velocity is higher. The solution of the Levi-Civita equation (Levi Civita, 1928)

$$\mathbf{v} = \frac{\mathbf{Q}}{M}, \tag{45}$$

shows that if mass decreases the velocity of motion increases. $\mathbf{Q}$ is a constant which depends on the velocity properties of the system.

4. If the motion of the initial body is translatory, the relative angular velocity and the absolute angular velocity of the remainder body are equal (see Table 2).

3.3.2 Example: Separation of a part of the rotor

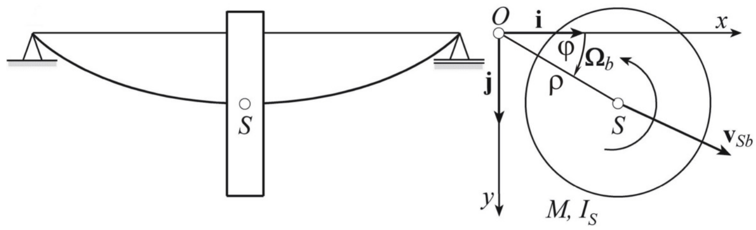


Fig.16. Model of the symmetrical rotor before body separation.

A symmetrically supported rotor, which is modelled as a shaft-disc system, is considered (see Fig.16). Mass of the disc is M. The mass center S is in the geometric center of the disc. Mass of the shaft is negligible in comparison to the mass of the disc. The moment of inertia of the disc is I_S for the axis z in mass center S. The rigidity of the shaft is c. The motion of the disc is in a plane Oxy. The differential equations which describe the motion of the rotor are

$$M\ddot{x} + cx = 0, \qquad M\ddot{y} + cy = 0, \qquad I_S\ddot{\psi} + k\dot{\psi} = \mathcal{M}, \tag{46}$$

where k is the damping coefficient and $\mathcal{M}$ is the torque.

The steady state solution for the third differential equation (46) is

$$\Omega_b = \frac{\mathcal{M}}{k}. \tag{47}$$

The angular velocity depends on the external moment and damping of the system.

By introducing the complex deflection $z_S = x + iy$ where x, y are the coordinates of the mass center S, $i = \sqrt{-1}$ is the imaginary unit and $\omega_w = \sqrt{c/M}$ is the frequency of vibration, the differential equation of motion of mass center is

$$\ddot{z}_S + \omega_w^2 z_S = 0. \tag{48}$$

For the initial conditions

$$x(0) = x_{S0}, \qquad y(0) = y_{S0}, \qquad \dot{x}_S(0) = v_{Sx0}, \qquad \dot{y}(0) = v_{Sy0}, \tag{49}$$

the deflection of mass center yields

$$z_S = (A_0 + iB_0)\exp(i\omega_w t) + (C_0 + iD_0)\exp(-i\omega_w t), \tag{50}$$

where

$$A_0 = \frac{1}{2}(x_{S0} + \frac{v_{Sy0}}{\omega_w}), \qquad B_0 = \frac{1}{2}(y_{S0} - \frac{v_{Sx0}}{\omega_w}),$$
$$C_0 = \frac{1}{2}(x_{S0} - \frac{v_{Sy0}}{\omega_w}), \qquad D_0 = \frac{1}{2}(y_{S0} + \frac{v_{Sx0}}{\omega_w}). \tag{51}$$

The rotor center oscillates around the initial position of mass center.

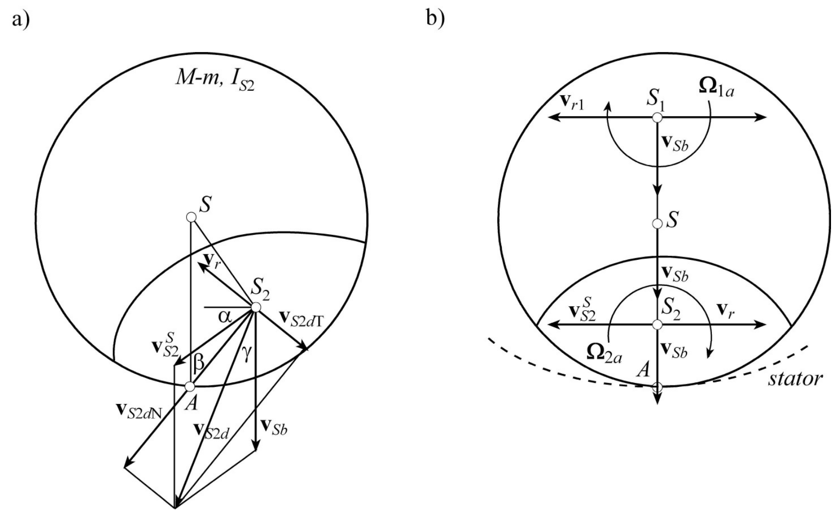


Fig.17. Velocity distribution during separation: a) The position of mass center is such that $\angle ASS_2 = \alpha$, b) SS_2 and AS are colinear.

If the rotor struck the fixed part of a stator in A (Fig.17a) at a moment t_1, a part of the rotor is separated. The mass of the separated part is m, with mass center S_2 and the moment of inertia I_{S2} related to the axis in S_2. The velocity of S_2 of the separated part is the sum of the dragging velocity $\mathbf{v}_{S2d}$ and the relative velocity $\mathbf{v}_r$

$$\mathbf{v}_{S2} = \mathbf{v}_{S2d} + \mathbf{v}_r,$$

where

$$v_{S2d} = \sqrt{v_{Sb}^2 + (v_{S2}^S)^2 + 2v_{Sb}v_{S2}^S \sin\alpha}, \qquad \sin\gamma = \frac{v_{S2}^S}{v_{S2d}} \cos\alpha, \tag{52}$$

with

$$v_{S2}^S = \Omega_b(SS_2),$$

and according to (50)

$$v_{Sb} = \sqrt{\dot{x}^2 + \dot{y}^2} = \omega_w \sqrt{(C_0 - A_0)^2 + (B_0 - D_0)^2 - 2(C_0 - A_0)(B_0 - D_0)\sin(2\omega_w t_1)},$$

α is the angle between SS_2 and SA. The dragging velocity

$$\mathbf{v}_{S2d} = \mathbf{v}_{S2dT} + \mathbf{v}_{S2dN}, \tag{53}$$

has two components: the normal one, in the AS_2 direction

$$v_{S2dN} = v_{S2d}\cos(\beta - \gamma), \tag{54}$$

and the tangential one, orthogonal to the AS_2

$$v_{S2dT} = v_{S2d}\sin(\beta - \gamma), \tag{55}$$

where

$$\tan\beta = \frac{SS_2}{R}\frac{\sin\alpha}{(1 - \cos\alpha)}. \tag{56}$$

In the moment of contact between the rotor and the stator, the relative velocity and relative angular velocity of the separated body are

$$\mathbf{v}_r = -\mathbf{v}_{S2dT}, \qquad \Omega_r = \frac{v_{S2dT}}{AS_2}. \tag{57}$$

Using the relations (28), (44) and (57) the relative velocity of the mass center of the remainder body and the relative angular velocity are calculated

$$\mathbf{v}_{r1} = \frac{m}{M-m}\mathbf{v}_{S2dT}, \qquad I_{S1}\Omega_{r1}\mathbf{k} = -I_{S2}\frac{v_{S2dT}}{AS_2}\mathbf{k} + \frac{m}{M-m}\mathbf{SS}_2 \times \mathbf{v}_{S2dT}. \tag{58}$$

The absolute velocity and angular velocity of the remainder rotor are

$$\begin{aligned} \mathbf{v}_{S1} &= \mathbf{v}_{Sb} + \mathbf{\Omega}_b \times \mathbf{SS}_1 + \frac{m}{M-m}\mathbf{v}_{S2dT}, \\ I_{S1}\Omega_{1a}\mathbf{k} &= I_S\Omega_b\mathbf{k} - I_{S2}\Omega_{2a}\mathbf{k} + \frac{m}{M-m}SS_2^2\Omega_b\mathbf{k} \\ &\quad + \frac{m}{M-m}\mathbf{SS}_2 \times \mathbf{v}_{S2dT}. \end{aligned} \tag{59}$$

For the special case when $\alpha = 0$ and $\beta = 0$, i.e., the mass center of the separated body is in the SA direction (Fig.4b) and also the velocity $\mathbf{v}_{Sb}$, the angle γ are determined

$$\sin\gamma = \Omega_b \frac{SS_2}{v_{S2d}}, \tag{60}$$

where

$$v_{S2d} = \sqrt{v_b^2 + (SS_2)^2\Omega_b^2}. \tag{61}$$

The separation of the body is with the relative velocity

$$\mathbf{v}_r = \mathbf{\Omega}_b \times \mathbf{SS}_2, \tag{62}$$

and the relative angular velocity

$$\Omega_r = \frac{SS_2}{AS_2}\Omega_b. \tag{63}$$

The absolute velocity and the angular velocity of the remainder body are

$$\mathbf{v}_{S1} = \mathbf{v}_{Sb}, \qquad \Omega_{1a} = \frac{I_S}{I_{S1}}\Omega_b - \frac{I_{S2}}{I_{S1}}\Omega_b\frac{R}{R-SS_2} = \Omega_b\left(1 - \frac{I_{S2}}{I_{S1}}\frac{SS_2}{R-SS_2}\right), \tag{64}$$

where R is the radius of the rotor. The angular velocity of the remainder body jumps to a lower value during separation: if the moment of inertia of the separated body is larger, the decrease of the angular velocity is higher.

For the special case when $I_{S1} = I_{S2}(SS_2/AS_2)$, the angular velocity of the remainder body is zero.

The velocity $\mathbf{v}_{S1}$ and angular velocity Ω_{1a} represent the initial velocity and angular velocity for motion of the remainder body after separation.

Transient motion of the remainder body after separation The motion of the remainder body after separation is described with the following differential equations

$$(M-m)\ddot{x} = X, \qquad (M-m)\ddot{y} = Y, \qquad I_{S1}\ddot{\psi} + k\dot{\psi} = \mathcal{M} + cxSS_1\sin\psi - cySS_1\cos\psi, \tag{65}$$

where X and Y are the projections of the elastic force in the shaft

$$X = -c(x + SS_1\cos\psi), \qquad Y = -c(y + SS_1\sin\psi). \tag{66}$$

Introducing the notation

$$\omega^2 = \frac{c}{M-m}, \qquad s = SS_1, \qquad k^* = \frac{k}{I_{S1}}, \qquad \mathcal{M}^* = \frac{\mathcal{M}}{I_{S1}}, \qquad p^2 = \frac{M-m}{I_{S1}}, \tag{67}$$

the differential equations (65) transform to

$$\begin{aligned}\ddot{x}+\omega^2 x &= -s\omega^2\cos\psi, \qquad \ddot{y}+\omega^2 y = -s\omega^2\sin\psi,\\ \ddot{\psi}+k^*\dot{\psi} &= \mathcal{M}^* + sp^2\omega^2 x\sin\psi - sp^2\omega^2 y\cos\psi.\end{aligned} \tag{68}$$

Due to shortness of the time, the position variation of the body is small ($x<1$, $y<1$, $\psi<1$). The linearized differential equations (68) are

$$\begin{aligned}\ddot{x}+\omega^2 x &= -s\omega^2,\\ \ddot{y}+\omega^2 y &= -s\omega^2\psi,\\ \ddot{\psi}+k^*\dot{\psi} &= \mathcal{M}^* - sy\omega^2 p^2.\end{aligned} \tag{69}$$

The first differential equation in the system (69) is independent and the solution for the initial conditions $x(0)=x(t_1)$ (see (50)) and $\dot{x}(0)=v_{S1x}$, where v_{S1x} is the projection in the x direction of the velocity of S_1 of the remainder body after separation, are obtained

$$x=(x_{S1}+s)\cos\omega t+\frac{v_{S1x}}{\omega}\sin\omega t - s, \tag{70}$$

For the initial conditions $y(0)=y_{S1}(t_1)$, $\dot{y}(0)=v_{S1y}$, $\psi(0)=\psi(t_1)=\psi_0$ and $\dot{\psi}(0)=\Omega_{1a}$, the displacement in the y direction and the angular velocity $\dot{\psi}$ are

$$\begin{aligned}y &= \frac{s}{k^*}(\Omega-\Omega_{1a}) - s\psi_0 - s(\Omega t) - \frac{s\omega^2}{(\omega^2+k^{*2})k^*}(\Omega-\Omega_{1a})\exp(-k^* t)\\ &\quad +[\frac{v_{S1y}}{\omega}+\frac{s\omega}{\omega^2+k^{*2}}(\Omega_{1a}+\Omega\frac{k^{*2}}{\omega^2})]\sin\omega t\\ &\quad +[(y_0+s\psi_0)-(\Omega-\Omega_{1a})\frac{sk^*}{\omega^2+k^{*2}}]\cos\omega t.\end{aligned} \tag{71}$$

and

$$\dot{\psi}=\Omega-(\Omega-\Omega_{1a})\exp(-k^* t). \tag{72}$$

For the initial velocity and the angular velocity (64), where $\Omega_b=\Omega$ and $SS_2 = R-SS_1$, the transient motion yields

$$\begin{aligned}y &= \frac{\Omega}{k^*}\frac{I_{S2}}{I_{S1}}(R-s) - s\psi_0 - s(\Omega t) - \frac{\omega^2\Omega}{(\omega^2+k^{*2})k^*}\frac{I_{S2}}{I_{S1}}(R-s)\exp(-k^* t)\\ &\quad +[\frac{v_{S1y}}{\omega}+\frac{s\omega\Omega}{\omega^2+k^{*2}}(1-\frac{I_{S2}}{I_{S1}}\frac{R-s}{s}+\frac{k^{*2}}{\omega^2})]\sin\omega t\\ &\quad +[y_0+s\psi_0-\Omega\frac{I_{S2}}{I_{S1}}\frac{k^*(R-s)}{\omega^2+k^{*2}}]\cos\omega t,\end{aligned} \tag{73}$$

and

$$\dot{\psi}=\Omega(1-\frac{I_{S2}}{I_{S1}}\frac{R-s}{s}\exp(-k^* t)). \tag{74}$$

During separation the angular velocity starts with Ω_{1a} and tends to the steady state angular velocity Ω, which depends on the moment and damping acting on the rotor: the higher the damping, the lower the angular velocity. If the torque is larger, the angular velocity is also higher.

Mass center of the remainder body moves with the velocity ($s\Omega$), which depends on the properties of the system and the distance between mass center and rotation center of the remainder body: the smaller the parameter s, the motion is slower. If the velocity is significant, the remainder body impacts the fixed stator, but if the velocity is smaller, the mass center tends to its steady state position.

Stability analysis of the steady state motion of the remainder body By introducing the polar coordinates $x = r\cos\varphi$ and $y = r\sin\varphi$ into the differential equations (68),

$$\begin{aligned}(\ddot{r} - r\dot{\varphi}^2) + \omega^2 r &= -s\omega^2\cos(\varphi-\psi),\\ r\ddot{\varphi} + 2\dot{r}\dot{\varphi} &= s\omega^2\sin(\varphi-\psi),\\ \ddot{\psi} + k^*\dot{\psi} &= \mathcal{M}^* - sr\omega^2p^2\sin(\varphi-\psi).\end{aligned} \tag{75}$$

Analyzing the solution it is obvious that the steady state motion corresponds to

$$\dot{\psi}_{SS} = \Omega, \qquad \psi_{SS} = \varphi_{SS} = \Omega t + \alpha, \qquad r_{SS} = -\frac{s\omega^2}{\omega^2 - \Omega^2}, \tag{76}$$

and

$$\dot{\psi}_{SS} = \Omega, \qquad \psi_{SS} = \varphi_{SS} - \pi = \Omega t + \alpha, \qquad r_{SS} = \frac{s\omega^2}{\omega^2 - \Omega^2}. \tag{77}$$

For the stability analysis, let us perturb the steady state motion

$$r = r_0 + \rho, \qquad \varphi = \varphi_0 + \xi, \qquad \psi = \psi_0 + \eta, \tag{78}$$

where ρ, ξ and η are small perturbations. Substituting (78) with (76) into (75) and after linearization, the differential equation of the perturbed values are

$$\begin{aligned}\ddot{\rho} + (\omega^2 - \Omega^2)\rho - 2\Omega r_0\dot{\xi} &= 0,\\ r_0\ddot{\xi} - s\omega^2\xi + 2\Omega\dot{\rho} + s\omega^2\eta &= 0,\\ \ddot{\eta} + k^*\dot{\eta} - s\omega^2p^2r_0\eta + s\omega^2p^2r_0\xi &= 0.\end{aligned} \tag{79}$$

The solution of the system (79) is assumed as

$$\rho = A\exp(\lambda t), \qquad \xi = B\exp(\lambda t), \qquad \eta = C\exp(\lambda t). \tag{80}$$

Substituting (80) into (79), the algebraic equation of the eigenvalues is obtained

$$\begin{aligned}0 = {} & [\lambda^2 + (\omega^2 - \Omega^2)]^2\left(\lambda^2 + k^*\lambda + \frac{s^2p^2\omega^4}{\omega^2 - \Omega^2}\right)\\ & -[\lambda^2 + (\omega^2 - \Omega^2)]s^2p^2\omega^4 + 4\Omega^2\left(\lambda^2 + k^*\lambda + \frac{s^2p^2\omega^4}{\omega^2 - \Omega^2}\right).\end{aligned} \tag{81}$$

Expansion restricted to $O(s^2)$ and using the Routh-Hurwitz criteria of stability, it can be concluded that the motion is stable for

$$\Omega^2 < \omega^2, \qquad i.e., \qquad \Omega < c/(M - m). \tag{82}$$

A numerical example is considered. For the values of the parameter of the remainder rotor $\omega^2 = 1$, $s = 0.5$, $p^2 = 4$ and the initial conditions $x(0) = 0$, $\dot{x}(0) = v_{S1x} = 0$, $y_{S1}(0) = 0$, $\dot{y}(0) = v_{S1y} = 0$, $\psi(0) = 0.5$, $\dot{\psi}(0) = 0$ the $r - t$ diagram is plotted in Fig.18. It represents the time history of the rotor center.

Three different values of parameter Ω are considered: 0.625, 1.0 and 2.5. From the Figure it is evident that for $\Omega = 2.5 > \omega$ the motion of the rotor is unstable and the distance of mass center to the initial position increases in time. For $\Omega = 0.625 < \omega$ the motion is stable and tends to the steady state position. It proves the conclusion obtained analytically (82). The analytically calculated steady state value (77) for mass center is $r = 0.82051$ and it agrees with the numerically obtained one.

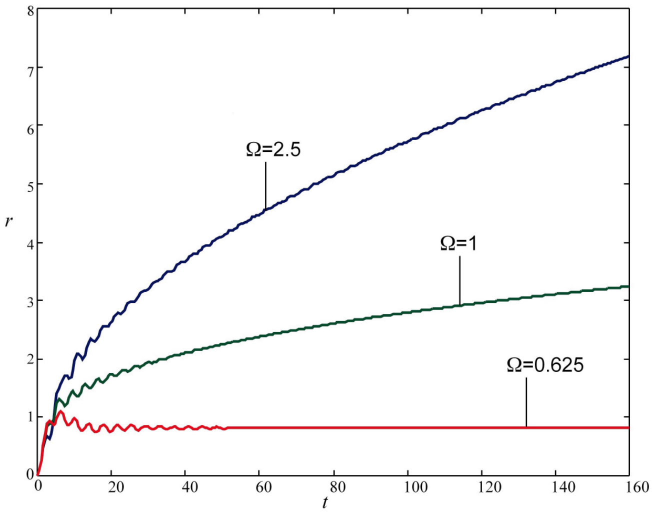


Fig.18. The time history $r-t$ diagrams obtained for various values of the parameter Ω.

3.4 Conclusion

Based on the general laws of dynamics, the procedure for obtaining the velocity and the angular velocity of the final body for discontinual mass variation is developed. It is concluded that the position and the angular position of the body during separation is approximately unchanged. The velocity and the angular velocity of the body has a jump-like variation during the process of body variation which is caused by mass and geometry variation of the body. The determined velocity of the mass center and the angular velocity of the final body represent the initial values for its further motion. If the relative velocity and the relative angular velocity of the separated or added body are zero, the relative velocity and the relative angular velocity of the final body are also zero, independently of the type of motion of the initial body.

4 Analytical procedures applied in dynamics of the body with discontinual mass variation

In this Chapter, applying the procedures of analytical dynamics the results obtained previously are rederived.

Let us start with the Lagrange-D'Alambert principle. Here, $\delta \mathbf{r}_S$ is virtual displacement of position of the mass center S and $.\delta \boldsymbol{\phi}$ is virtual change of the angle position $\boldsymbol{\phi}$ of the initial body

Multiplying the Eq. (21) with a virtual displacement $\delta \mathbf{r}_S$. and using the relations (14), we obtain

$$\delta \mathbf{r}_{S1} = \delta \mathbf{r}_S + \delta \boldsymbol{\phi} \times \mathbf{SS}_1, \qquad \delta \mathbf{r}_{S2} = \delta \mathbf{r}_S + \delta \boldsymbol{\phi} \times \mathbf{SS}_2, \tag{83}$$

and

$$\begin{aligned} & \pm m \mathbf{v}_{S2} \delta \mathbf{r}_{S2} + (M \mp m) \mathbf{v}_{S1} (\delta \mathbf{r}_{S1} - \delta \boldsymbol{\phi} \times \mathbf{SS}_1) \\ & \mp m \mathbf{v}_{S2} (\delta \boldsymbol{\phi} \times \mathbf{SS}_2) - M \mathbf{v}_S \delta \mathbf{r}_S \\ = \ & \mathbf{I}^{Fr} \delta \mathbf{r}_S, \end{aligned} \tag{84}$$

Multiplying the Eq. (23a) with $\delta \boldsymbol{\phi}$ and using the relations $\mathbf{L}_{S1} = \mathbf{I}_{S1} \boldsymbol{\Omega}_1$, $\mathbf{L}_{S2} = \mathbf{I}_{S2} \boldsymbol{\Omega}_2$ and $\mathbf{L}_S = \mathbf{I}_S \boldsymbol{\Omega}$, we have

$$\mathbf{I}_{S1} \boldsymbol{\Omega}_1 \delta \boldsymbol{\phi}_1 \pm \mathbf{I}_{S2} \boldsymbol{\Omega}_2 \delta \boldsymbol{\phi}_2 + \mathbf{SS}_1 \times (M \mp m) \mathbf{v}_{S1} \delta \boldsymbol{\phi} \pm \mathbf{SS}_2 \times m \mathbf{v}_{S2} \delta \boldsymbol{\phi} - \mathbf{I}_S \boldsymbol{\Omega} \delta \boldsymbol{\phi} = \mathbf{I}^M \delta \boldsymbol{\phi}, \tag{85}$$

where $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2 = \boldsymbol{\phi}$ and the angle variations are $\delta \boldsymbol{\phi}_1 = \delta \boldsymbol{\phi}_2 = \delta \boldsymbol{\phi}$. Adding the equations (84) and (85), we obtain

$$\begin{aligned} & \pm m \mathbf{v}_{S2} \delta \mathbf{r}_{S2} \pm \mathbf{I}_{S2} \boldsymbol{\Omega}_2 \delta \boldsymbol{\phi}_2 + (M \mp m) \mathbf{v}_{S1} \delta \mathbf{r}_{S1} \\ & + \mathbf{I}_{S1} \boldsymbol{\Omega}_1 \delta \boldsymbol{\phi}_1 - (M \mathbf{v}_S \delta \mathbf{r}_S - \mathbf{I}_S \boldsymbol{\Omega} \delta \boldsymbol{\phi}) \\ = \ & \mathbf{I}^{Fr} \delta \mathbf{r}_S + \mathbf{I}^M \delta \boldsymbol{\phi}. \end{aligned} \tag{86}$$

Introducing the generalized coordinates q_i where $i = 1, 2, ..., N$, and assuming that all quantities $\mathbf{r}_S$, $\mathbf{r}_{S1}, \mathbf{r}_{S2}$, $\boldsymbol{\phi}$,, $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$ are functions of the generalized coordinates we have

$$\delta \mathbf{r}_S = \sum_{i=1}^{N} \frac{\partial \mathbf{r}_S}{\partial q_i} \delta q_i, \qquad \delta \mathbf{r}_{S1} = \sum_{i=1}^{N} \frac{\partial \mathbf{r}_{S1}}{\partial q_i} \delta q_i, \qquad \delta \mathbf{r}_{S2} = \sum_{i=1}^{N} \frac{\partial \mathbf{r}_{S2}}{\partial q_i} \delta q_i, \tag{87}$$

$$\delta \boldsymbol{\phi} = \sum_{i=1}^{N} \frac{\partial \boldsymbol{\phi}}{\partial q_i} \delta q_i, \qquad \delta \boldsymbol{\phi}_1 = \sum_{i=1}^{N} \frac{\partial \boldsymbol{\phi}_1}{\partial q_i} \delta q_i, \qquad \delta \boldsymbol{\phi}_2 = \sum_{i=1}^{N} \frac{\partial \boldsymbol{\phi}_2}{\partial q_i} \delta q_i. \tag{88}$$

Using the relations (87), (88) and the equalities

$$\frac{\partial \mathbf{r}_S}{\partial q_i} = \frac{\partial \dot{\mathbf{r}}_S}{\partial \dot{q}_i} = \frac{\partial \mathbf{v}_S}{\partial \dot{q}_i}, \qquad \frac{\partial \mathbf{r}_{S1}}{\partial q_i} = \frac{\partial \dot{\mathbf{r}}_{S1}}{\partial \dot{q}_i} = \frac{\partial \mathbf{v}_{S1}}{\partial \dot{q}_i}, \qquad \frac{\partial \mathbf{r}_{S2}}{\partial q_i} = \frac{\partial \dot{\mathbf{r}}_{S2}}{\partial \dot{q}_i} = \frac{\partial \mathbf{v}_{S2}}{\partial \dot{q}_i}, \tag{89}$$

$$\frac{\partial \boldsymbol{\phi}}{\partial q_i} = \frac{\partial \dot{\boldsymbol{\phi}}}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{\Omega}}{\partial \dot{q}_i}, \qquad \frac{\partial \boldsymbol{\phi}_1}{\partial q_i} = \frac{\partial \dot{\boldsymbol{\phi}}_1}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{\Omega}_1}{\partial \dot{q}_i}, \qquad \frac{\partial \boldsymbol{\phi}_2}{\partial q_i} = \frac{\partial \dot{\boldsymbol{\phi}}_2}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{\Omega}_2}{\partial \dot{q}_i}, \tag{90}$$

the Eq. (86) becomes

$$\begin{aligned}\sum_{i=1}^{N}(M\mathbf{v}_S\frac{\partial \mathbf{v}_S}{\partial \dot{q}_i}+\mathbf{I}_S\boldsymbol{\Omega}\frac{\partial \boldsymbol{\Omega}}{\partial \dot{q}_i})\delta q_i-\sum_{i=1}^{N}[& \pm(m\mathbf{v}_{S2}\frac{\partial \mathbf{v}_{S2}}{\partial \dot{q}_i}+\mathbf{I}_{S2}\boldsymbol{\Omega}_2\frac{\partial \boldsymbol{\Omega}_2}{\partial \dot{q}_i}) \\ & +(M\mp m)\mathbf{v}_{S1}\frac{\partial \mathbf{v}_{S1}}{\partial \dot{q}_i}+\mathbf{I}_{S1}\boldsymbol{\Omega}_1\frac{\partial \boldsymbol{\Omega}_1}{\partial \dot{q}_i}]\delta q_i \\ = & -\sum_{i=1}^{N}\{\frac{\partial \mathbf{r}_S}{\partial q_i}(\mathbf{F}_r\Delta t)+[(\mathbf{M}_0^{Fr}+\mathfrak{M})\Delta t]\frac{\partial \phi}{\partial q_i}\}\delta q_i\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{N}(M\mathbf{v}_S\frac{\partial \mathbf{v}_S}{\partial \dot{q}_i}+\mathbf{I}_S\boldsymbol{\Omega}\frac{\partial \boldsymbol{\Omega}}{\partial \dot{q}_i})\delta q_i &= \sum_{i=1}^{N}(m\mathbf{v}_{S2}\frac{\partial \mathbf{v}_{S2}}{\partial \dot{q}_i}+\mathbf{I}_{S2}\boldsymbol{\Omega}_2\frac{\partial \boldsymbol{\Omega}_2}{\partial \dot{q}_i}) \\ & +(M-m)\mathbf{v}_{S1}\frac{\partial \mathbf{v}_{S1}}{\partial \dot{q}_i}+\mathbf{I}_{S1}\boldsymbol{\Omega}_1\frac{\partial \boldsymbol{\Omega}_1}{\partial \dot{q}_i})\delta q_i.\end{aligned} \tag{91}$$

i.e.,

$$\begin{aligned} & \sum_{i=1}^{N}\frac{\partial}{\partial \dot{q}_i}[\pm(\frac{1}{2}m\mathbf{v}_{S2}\mathbf{v}_{S2}+\frac{1}{2}\mathbf{I}_{S2}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2)+(\frac{1}{2}(M\mp m)\mathbf{v}_{S1}\mathbf{v}_{S1} \\ & +\frac{1}{2}\mathbf{I}_{S1}\boldsymbol{\Omega}_1\boldsymbol{\Omega}_1)]-\sum_{i=1}^{N}\frac{\partial}{\partial \dot{q}_i}(\frac{1}{2}M\mathbf{v}_S\mathbf{v}_S+\frac{1}{2}\mathbf{I}_S\boldsymbol{\Omega}\boldsymbol{\Omega})\delta q_i \\ = & \sum_{i=1}^{N}(\mathbf{I}^{Fr}\frac{\partial \mathbf{r}_S}{\partial q_i}+\mathbf{I}^{M}\frac{\partial \phi}{\partial q_i})\delta q_i.\end{aligned} \tag{92}$$

The first group of terms on the left side of the equation represent the kinetic energy of the system after, whereas the second group of terms is equal to the kinetic energy before mass variation

$$T_1 = \frac{1}{2}M\mathbf{v}_S\mathbf{v}_S+\frac{1}{2}\mathbf{I}_S\boldsymbol{\Omega}\boldsymbol{\Omega}, \tag{93}$$

$$T_2 = \pm(\frac{1}{2}m\mathbf{v}_{S2}\mathbf{v}_{S2}+\frac{1}{2}\mathbf{I}_{S2}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2)+(\frac{1}{2}(M\mp m)\mathbf{v}_{S1}\mathbf{v}_{S1}+\frac{1}{2}\mathbf{I}_{S1}\boldsymbol{\Omega}_1\boldsymbol{\Omega}_1), \tag{94}$$

while the terms on the right side of (92) give the generalized impulse

$$Q_i^I=\mathbf{I}^{Fr}\frac{\partial \mathbf{r}_S}{\partial q_i}+\mathbf{I}^{M}\frac{\partial \phi}{\partial q_i}. \tag{95}$$

Substituting the notations for the kinetic energy (93) and (94) and also of the generalized impulse (95) into (93) and separating the equations with the same variation of the generalized coordinates, the following system of equations is obtained

$$\frac{\partial (T_2-T_1)}{\partial \dot{q}_i}=Q_i^I, \qquad i=1,2,...,N.$$

If the body mass variation is without external impulses, the system of equations is modified to (see Cveticanin, 2009_1)

$$\frac{\partial (T_1-T_2)}{\partial \dot{q}_i}=0, \qquad i=1,2,...,N. \tag{96}$$

Using these equations, one can calculate the velocity and angular velocity of the system after mass variation if the velocities and angular velocities before mass variation are known. In practical applications of these equations the kinetic energy functions before and after separation have to be differentiable and continual functions.

Let us consider the case of mass separation when the velocity and angular velocity of the body before mass separation and also the velocity and angular velocity of the separated mass are given. For the Eq. (96) it follows: The partial derivative in generalized velocity of the difference of the kinetic energy of the body before and the sum of kinetic energies of the separated and remainder bodies after separation is equal to zero.

Let us apply this results to analyze the mass separation for the case of the free motion of the body. The motion has six degrees of freedom.

The kinetic energy of the initial body with fee motion in the space is

$$T_1 = \frac{1}{2}M(\dot{x}_S^2+\dot{y}_S^2+\dot{z}_S^2)+\frac{1}{2}(I_{xx}\Omega_x^2+I_{yy}\Omega_y^2+I_{zz}\Omega_z^2+2I_{xy}\Omega_x\Omega_y+2I_{xz}\Omega_x\Omega_z+2I_{zy}\Omega_z\Omega_y), \tag{97}$$

where I_{xx}, I_{yy}, I_{zz} are axial moments of inertia; I_{xy}, I_{xz}, I_{zy} centrifugal moments of inertia; $\dot{x}_S$, $\dot{y}_S$ and $\dot{z}_S$ projections of the velocity $\mathbf{v}_S$ of mass center; Ω_x, Ω_y and Ω_z projections of the angular velocity $\mathbf{\Omega}$.

For the velocity of the mass center of the separated body

$$\mathbf{v}_{S2} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_2 + \mathbf{u}, \tag{98}$$

and the angular velocity

$$\mathbf{\Omega}_2 = \mathbf{\Omega} + \mathbf{\Omega}^*, \tag{99}$$

the kinetic energy of the free motion of separated body is

$$\begin{aligned} T_{S2} \quad = \quad & \frac{1}{2}m[\dot{x}_S^2 + \dot{y}_S^2 + \dot{z}_S^2 + u_x^2 + u_y^2 + u_z^2 + 2\dot{x}_S u_x + 2\dot{y}_S u_y + 2\dot{z}_S u_z + \\ & (\Omega_y z_{SS2} - \Omega_z y_{SS2})^2 + (\Omega_z x_{SS2} - \Omega_x z_{SS2})^2 + (\Omega_x y_{SS2} - \Omega_y x_{SS2})^2 + \\ & 2(\Omega_y z_{SS2} - \Omega_z y_{SS2})u_x + 2(\Omega_z x_{SS2} - \Omega_x z_{SS2})u_y + 2(\Omega_x y_{SS2} - \Omega_y x_{SS2})u_z + \\ & 2(\Omega_y z_{SS2} - \Omega_z y_{SS2})\dot{x}_S + 2(\Omega_z x_{SS2} - \Omega_x z_{SS2})\dot{y}_S + 2(\Omega_x y_{SS2} - \Omega_y x_{SS2})\dot{z}_S] + \\ & \frac{1}{2}[I_{xx2}(\Omega_x + \Omega_x^*)^2 + I_{yy2}(\Omega_y + \Omega_y^*)^2 + I_{zz2}(\Omega_z + \Omega_z^*)^2 + \\ & 2I_{xy2}(\Omega_x + \Omega_x^*)(\Omega_y + \Omega_y^*) + \\ & 2I_{xz2}(\Omega_x + \Omega_x^*)(\Omega_z + \Omega_z^*) + 2I_{zy2}(\Omega_z + \Omega_z^*)(\Omega_y + \Omega_y^*)], \end{aligned} \tag{100}$$

where $\mathbf{u}$ is the relative velocity of the separation with projections u_x, u_y and u_z; $\mathbf{\Omega}^*$ is the relative angular velocity of separation with projections Ω_x^*, Ω_y^* and Ω_z^*.

For the velocity of the remainder system

$$\mathbf{v}_{S1} = \mathbf{v}_S + \mathbf{\Omega} \times \mathbf{SS}_1 + \mathbf{v}^*, \tag{101}$$

and the angular velocity

$$\mathbf{\Omega}_1 = \mathbf{\Omega} + \mathbf{\Omega}_1^*, \tag{102}$$

the kinetic energy of the final (remainder) body in free motion is

$$\begin{aligned} T_{S1} \quad = \quad & \frac{1}{2}(M-m)(\dot{x}_{S1}^2 + \dot{y}_{S1}^2 + \dot{z}_{S1}^2) + \frac{1}{2}(I_{xx1}\Omega_{x1}^2 + I_{yy1}\Omega_{y1}^2 + \\ & I_{zz1}\Omega_{z1}^2 + 2I_{xy1}\Omega_{x1}\Omega_{y1} + 2I_{xz1}\Omega_{x1}\Omega_{z1} + 2I_{zy1}\Omega_{z1}\Omega_{y1}), \end{aligned} \tag{103}$$

where $I_{xx1}, I_{yy1}, I_{zz1}$ are the axial moments of inertia; $I_{xy1}, I_{xz1}, I_{zy1}$ are the centrifugal moments of inertia; $\mathbf{v}^*$ and $\mathbf{\Omega}_1$ are the unknown velocity and angular velocity of the remainder body.

The total kinetic energy of the system after the separation is

$$T_2 = T_{S1} + T_{S2}. \tag{104}$$

Taking into account (96) and using the relations (97) and (100), the following system of equations is obtained

$$\begin{aligned}
(M-m)\dot{x}_S &= (M-m)\dot{x}_{S1} + m(u_x + \Omega_y z_{S2} - \Omega_z y_{S2}),\\
(M-m)\dot{y}_S &= (M-m)\dot{y}_{S1} + m(u_y + \Omega_z x_{S2} - \Omega_x z_{S2}),\\
(M-m)\dot{z}_S &= (M-m)\dot{z}_{S1} + m(u_z + \Omega_x y_{S2} - \Omega_y x_{S2}),
\end{aligned}$$

$$\begin{aligned}
0 &= -(I_{xx}\Omega_x + I_{xy}\Omega_y + I_{xz}\Omega_z) + I_{xx1}\Omega_{1x} + I_{xy1}\Omega_{1y} + I_{xz1}\Omega_{1z}\\
&\quad + I_{xx2}(\Omega_x + \Omega_x^*) + I_{xy2}(\Omega_y + \Omega_y^*) + I_{xz2}(\Omega_z + \Omega_z^*) + m[-(\Omega_z x_{S2} - \Omega_x z_{S2})z_{S2}\\
&\quad + (\Omega_x y_{S2} - \Omega_y x_{S2})y_{S2} - z_{S2}(u_y + \dot{y}_S) + y_{S2}(u_z + \dot{z}_S)] + (M-m)[-(\Omega_z x_{S1}\\
&\quad - \Omega_x z_{S1})z_{S1} + (\Omega_x y_{S1} - \Omega_y x_{S1})y_{S1} - z_{S1}(v_y + \dot{y}_S) + y_{S1}(v_z + \dot{z}_S)],\\
0 &= -(I_{yy}\Omega_y + I_{xy}\Omega_x + I_{zy}\Omega_z) + I_{yy1}\Omega_{1y} + I_{xy1}\Omega_{1x} + I_{zy1}\Omega_{1z}\\
&\quad + I_{yy2}(\Omega_y + \Omega_y^*) + I_{xy2}(\Omega_x + \Omega_x^*) + I_{zy2}(\Omega_z + \Omega_z^*) + m[(\Omega_y z_{S2} - \Omega_z y_{S2})z_{S2}\\
&\quad - (\Omega_x y_{S2} - \Omega_y x_{S2})x_{S2} + z_{S2}(u_x + \dot{x}_S) - x_{S2}(u_z + \dot{z}_S)] + (M-m)[(\Omega_y z_{S1}\\
&\quad - \Omega_z y_{S1})z_{S1} - (\Omega_x y_{S1} - \Omega_y x_{S1})x_{S1} + z_{S1}(v_x + \dot{x}_S) - x_{S1}(v_z + \dot{z}_S),\\
0 &= -(I_{zz}\Omega_z + I_{xz}\Omega_x + I_{zy}\Omega_y) + I_{zz1}\Omega_{1z} + I_{xz1}\Omega_{1x} + I_{zy1}\Omega_{1y} + I_{zz2}(\Omega_z + \Omega_z^*)\\
&\quad + I_{xz2}(\Omega_x + \Omega_x^*) + I_{zy2}(\Omega_y + \Omega_y^*) + m[-y_{S2}(\Omega_y z_{S2} - \Omega_z y_{S2}) + x_{S2}(\Omega_z x_{S2}\\
&\quad - \Omega_x z_{S2}) - y_{S2}(u_x + \dot{x}_S) + x_{S2}(u_y + \dot{y}_S)] + (M-m)[-y_{S1}(\Omega_y z_{S1} - \Omega_z y_{S1})\\
&\quad + x_{S1}(\Omega_z x_{S1} - \Omega_x z_{S1}) - y_{S1}(v_x + \dot{x}_S) + x_{S1}(v_y + \dot{y}_S)].
\end{aligned} \tag{105}$$

Using the relations

$$\begin{aligned}
\Omega_{2x} &= \Omega_x + \Omega_x^*, \qquad \Omega_{2y} = \Omega_y + \Omega_{2y}^*, \qquad \Omega_{2z} = \Omega_z + \Omega_z^*,\\
\dot{x}_{S1} &= \dot{x}_S + v_x + (\Omega_y z_{S1} - \Omega_z y_{S1}), \qquad \dot{y}_{S1} = \dot{y}_S + v_y + (\Omega_z x_{S1} - \Omega_x z_{S1}),\\
\dot{z}_{S1} &= \dot{z}_S + v_z + (\Omega_x y_{S1} - \Omega_y x_{S1}), \qquad \dot{x}_{S2} = \dot{x}_S + u_x + (\Omega_y z_{S2} - \Omega_z y_{S2}),\\
\dot{y}_{S2} &= \dot{y}_S + u_y + (\Omega_z x_{S2} - \Omega_x z_{S2}), \qquad \dot{z}_{S2} = \dot{z}_S + u_z + (\Omega_x y_{S2} - \Omega_y x_{S2}),
\end{aligned} \tag{106}$$

the Eqs. (105) are transformed into

$$\begin{aligned}
M\dot{x}_S &= m\dot{x}_{S2} + (M-m)\dot{x}_{S1},\\
M\dot{y}_S &= m\dot{y}_{S2} + (M-m)\dot{y}_{S1},\\
M\dot{z}_S &= m\dot{z}_{S2} + (M-m)\dot{z}_{S1},\\
I_{xx}\Omega_x + I_{xy}\Omega_y + I_{xz}\Omega_z &= I_{xx1}\Omega_{1x} + I_{xy1}\Omega_{1y} + I_{xz1}\Omega_{1z} + I_{xx2}\Omega_{2x} + I_{xy2}\Omega_{2y}\\
&\quad + I_{xz2}\Omega_{2z} + (M-m)(y_{S1}\dot{z}_{S1} - z_{S1}\dot{y}_{S1})\\
&\quad + m(y_{S2}\dot{z}_{S2} - z_{S2}\dot{y}_{S2}), \qquad (107)\\
I_{xy}\Omega_x + I_{yy}\Omega_y + I_{yz}\Omega_z &= I_{xy1}\Omega_{1x} + I_{yy1}\Omega_{1y} + I_{yz1}\Omega_{1z} + I_{xy2}\Omega_{2x} + I_{yy2}\Omega_{2y}\\
&\quad + I_{yz2}\Omega_{2z} + (M-m)(z_{S1}\dot{x}_{S1} - x_{S1}\dot{z}_{S1})\\
&\quad + m(z_{S2}\dot{x}_{S2} - x_{S2}\dot{z}_{S2}), \qquad (108)\\
I_{xz}\Omega_x + I_{yz}\Omega_y + I_{zz}\Omega_z &= I_{xz1}\Omega_{1x} + I_{yz1}\Omega_{1y} + I_{zz1}\Omega_{1z} + I_{xz2}\Omega_{2x} + I_{yz2}\Omega_{2y}\\
&\quad + I_{zz2}\Omega_{2z} + (M-m)(x_{S1}\dot{y}_{S1} - y_{S1}\dot{x}_{S1}) \qquad (109)\\
&\quad + m(x_{S2}\dot{y}_{S2} - y_{S2}\dot{x}_{S2}). \qquad (110)
\end{aligned}$$

Introducing the projections of the velocities, angular velocities and position vectors

$$\begin{aligned}
\mathbf{v}_S &= \dot{x}_S\mathbf{i}+\dot{y}_S\mathbf{j}+\dot{z}_S\mathbf{k}, \qquad \mathbf{v}_{S1}=\dot{x}_{S1}\mathbf{i}+\dot{y}_{S1}\mathbf{j}+\dot{z}_{S1}\mathbf{k},\\
\mathbf{v}_{S2} &= \dot{x}_{S2}\mathbf{i}+\dot{y}_{S2}\mathbf{j}+\dot{z}_{S2}\mathbf{k},\\
\boldsymbol{\Omega} &= \Omega_x\mathbf{i}+\Omega_y\mathbf{j}+\Omega_z\mathbf{k}, \qquad \boldsymbol{\Omega}_1=\Omega_{1x}\mathbf{i}+\Omega_{1y}\mathbf{j}+\Omega_{1z}\mathbf{k},\\
\boldsymbol{\Omega}_2 &= \Omega_{2x}\mathbf{i}+\Omega_{2y}\mathbf{j}+\Omega_{2z}\mathbf{k},\\
\mathbf{SS}_1 &= x_{S1}\mathbf{i}+y_{S1}\mathbf{j}+z_{S1}\mathbf{k}, \qquad \mathbf{SS}_2=x_{S2}\mathbf{i}+y_{S2}\mathbf{j}+z_{S2}\mathbf{k},
\end{aligned} \tag{111}$$

as well as the inertia tensors

$$\begin{aligned}
\mathbf{I}_S &= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz}\\ I_{yx} & I_{yy} & I_{yz}\\ I_{zx} & I_{zy} & I_{zz}\end{bmatrix}, \qquad \mathbf{I}_{S1}=\begin{bmatrix} I_{xx1} & I_{xy1} & I_{xz1}\\ I_{yx1} & I_{yy1} & I_{yz1}\\ I_{zx1} & I_{zy1} & I_{zz1}\end{bmatrix}, \qquad (112)\\
\mathbf{I}_{S2} &= \begin{bmatrix} I_{xx2} & I_{xy2} & I_{xz2}\\ I_{yx2} & I_{yy2} & I_{yz2}\\ I_{zx2} & I_{zy2} & I_{zz2}\end{bmatrix},
\end{aligned}$$

into Eq. (21) and Eq. (23a), we obtain the above mentioned Eq. (109). The solutions of (96) are equal to those obtained from (21) and Eq. (23a) without external forces and torques. The main advantage of the suggested analytical procedure is its simplicity for practical use in comparison to the classical method based on the general principles of dynamics which have the vectorial form (Cveticanin, 2009_1).

4.1 Increase of the kinetic energy

Analyzing the relation (96) it can be concluded that the kinetic energy of the body before separation T_1 and the sum of the kinetic energies of the remainder and separated bodies T_2 differs. The difference between the kinetic energies T_1 and T_2 is the result of transformation of the deformation energy of the body into kinetic energies of the separated and remainder bodies during separation.

Theorem 2 *In the perfectly plastic separation of a body the increase of the kinetic energy is equal to the sum of the kinetic energies corresponding to the relative velocities and angular velocities of the remainder and separated bodies*

$$\Delta T=[\frac{1}{2}(M-m)(\mathbf{v}^*)^2+\frac{1}{2}\mathbf{I}_{S1}(\boldsymbol{\Omega}_1^*)^2]+[\frac{1}{2}m\mathbf{u}^2+\frac{1}{2}\mathbf{I}_{S2}(\boldsymbol{\Omega}^*)^2]. \tag{113}$$

Proof. *Using the relations (93) and (94) the difference between the kinetic energy before separation T_1 and after separation T_2 is*

$$\begin{aligned}
T_2-T_1 &= [\frac{1}{2}(M-m)\mathbf{v}_{S1}\mathbf{v}_{S1}+\frac{1}{2}\mathbf{I}_{S1}\boldsymbol{\Omega}_1\boldsymbol{\Omega}_1 \qquad (114)\\
&\quad +(\frac{1}{2}m\mathbf{v}_{S2}\mathbf{v}_{S2}+\frac{1}{2}\mathbf{I}_{S2}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2)]-\frac{1}{2}M\mathbf{v}_S\mathbf{v}_S+\frac{1}{2}\mathbf{I}_S\boldsymbol{\Omega}\boldsymbol{\Omega},
\end{aligned}$$

i.e.

$$\begin{aligned}
T_2-T_1 &= [\frac{1}{2}(M-m)\mathbf{v}_{S1}\mathbf{v}_{S1}-M\mathbf{v}_S\mathbf{v}_S+\frac{1}{2}m\mathbf{v}_{S2}\mathbf{v}_{S2}]+\frac{1}{2}M\mathbf{v}_S\mathbf{v}_S\\
&\quad +[\frac{1}{2}\mathbf{I}_{S1}\boldsymbol{\Omega}_1\boldsymbol{\Omega}_1-\mathbf{I}_S\boldsymbol{\Omega}\boldsymbol{\Omega}+\frac{1}{2}\mathbf{I}_{S2}\boldsymbol{\Omega}_2\boldsymbol{\Omega}_2]+\frac{1}{2}\mathbf{I}_S\boldsymbol{\Omega}\boldsymbol{\Omega}.
\end{aligned} \tag{115}$$

Substituting the equalities (25) and (26) for $\mathbf{v}_r = \mathbf{v}_{r1} = 0$*, the relation (115) is transformed into*

$$\begin{aligned} T_2 - T_1 &= \frac{1}{2}(M-m)(\mathbf{v}_{S1}-\mathbf{v}_S)(\mathbf{v}_{S1}-\mathbf{v}_S)+\frac{1}{2}m(\mathbf{v}_{S2}-\mathbf{v}_S)(\mathbf{v}_{S2}-\mathbf{v}_S) \\ &-(M-m)(\mathbf{SS}_1\times\mathbf{v}_{S1})\boldsymbol{\Omega}-m(\mathbf{SS}_2\times\mathbf{v}_{S2})\boldsymbol{\Omega} \\ &+\frac{1}{2}\mathbf{I}_{S1}(\boldsymbol{\Omega}_1-\boldsymbol{\Omega})(\boldsymbol{\Omega}_1-\boldsymbol{\Omega})+\frac{1}{2}\mathbf{I}_{S2}(\boldsymbol{\Omega}_2-\boldsymbol{\Omega})(\boldsymbol{\Omega}_2-\boldsymbol{\Omega}). \end{aligned} \tag{116}$$

Introducing the relations (101) and (98) and also (102) and (99) for the velocities $\mathbf{v}_{S1}$ *and* $\mathbf{v}_{S2}$ *and angular velocities* $\boldsymbol{\Omega}_1$ *and* $\boldsymbol{\Omega}_2$ *into (116) we obtain*

$$\begin{aligned} T_2 - T_1 &= \frac{1}{2}(M-m)(\mathbf{SS}_1\times\boldsymbol{\Omega}+\mathbf{v}^*)(\mathbf{SS}_1\times\boldsymbol{\Omega}+\mathbf{v}^*) \\ &+\frac{1}{2}m(\mathbf{SS}_2\times\boldsymbol{\Omega}+\mathbf{u})(\mathbf{SS}_2\times\boldsymbol{\Omega}+\mathbf{u}) \\ &-(M-m)(\mathbf{SS}_1\times(\mathbf{v}_S+\mathbf{SS}_1\times\boldsymbol{\Omega}+\mathbf{v}^*))\boldsymbol{\Omega} \\ &-m(\mathbf{SS}_2\times(\mathbf{v}_S+\mathbf{SS}_2\times\boldsymbol{\Omega}+\mathbf{u}))\boldsymbol{\Omega} \\ &+\frac{1}{2}\mathbf{I}_{S1}(\boldsymbol{\Omega}_1^*)(\boldsymbol{\Omega}_1^*)+\frac{1}{2}\mathbf{I}_{S2}(\boldsymbol{\Omega}^*)(\boldsymbol{\Omega}^*). \end{aligned} \tag{117}$$

Since the time of separation is negligibly short and the displacements of mass centers and the angle positions of the bodies during the separation are also negligibly small we assume that the positions of mass centers and angle positions of the bodies remain constant during the separation. Using this assumption that S *is the mass center of the body and* S_1 *and* S_2 *are the mass centers of the remainder and separated body and after some calculation the relation (117) is simplified to*

$$\begin{aligned} \Delta T &= T_2 - T_1 = \frac{1}{2}(M-m)\mathbf{v}^*\mathbf{v}^* + \frac{1}{2}m\mathbf{uu} \\ &+\frac{1}{2}\mathbf{I}_{S1}\boldsymbol{\Omega}_1^*\boldsymbol{\Omega}_1^* + \frac{1}{2}\mathbf{I}_{S2}\boldsymbol{\Omega}^*\boldsymbol{\Omega}^*. \end{aligned} \tag{118}$$

The theorem is proved. ■

Remark 3 *For the perfectly plastic direct central impact of two perfectly inelastic bodies moving translatory Carnot proved the following theorem (see Starzhinskii, 1982): There is the loss of kinetic energy which is equal to the kinetic energy corresponding to the loss of velocities of the two bodies in impact. Carnot's theorem talks about the loss of the kinetic energy during impact, and the suggested theorem (113) about the increase of the kinetic energy during separation. The perfectly plastic impact of two perfectly inelastic bodies is opposite to the perfectly plastic separation of the body into separated and remainder bodies.*

4.2 Example: Separation of a pendulum

Let us consider a pendulum with mass M and length L rotating around a fixed point O (Fig.19). The aim is to obtain the angular velocity of the remainder pendulum after the separation of a part whose mass is m and length l.

For the angular velocity of rotation $\dot{\varphi}$, the kinetic energy of the motion is

$$T_1 = \frac{1}{2}\frac{ML^2}{3}\dot{\varphi}^2. \tag{119}$$

The part of the pendulum is separating with the velocity $\mathbf{u}$ and the angular velocity Ω and it is moving in the xy plane. Its kinetic energy is

$$T_{S2} = \frac{1}{2}mv_{S2}^2 + \frac{1}{2}\frac{ml^2}{12}\dot{\varphi}_2^2, \tag{120}$$

where

$$\dot{\varphi}_2 = \dot{\varphi} + \Omega, \qquad \mathbf{v}_{S2} = \mathbf{v}_{S2t} + \mathbf{u}, \qquad v_{S2t} = \dot{\varphi}(OS_2) = \dot{\varphi}(L - \frac{l}{2}). \tag{121}$$

Substituting (121) into (120), we obtain

$$T_{S2} = \frac{1}{2}m[u^2 + (\dot{\varphi}+\Omega)^2(L-\frac{l}{2})^2 - 2(\dot{\varphi}+\Omega)(u_x y_{S2} - u_y x_{S2})] + \frac{1}{2}\frac{ml^2}{12}(\dot{\varphi}+\Omega)^2, \tag{122}$$

where

$$x_{S2} = (L - \frac{l}{2})\sin\varphi, \qquad y_{S2} = (L - \frac{l}{2})\cos\varphi. \tag{123}$$

The kinetic energy of the remainder pendulum is

$$T_{S1} = \frac{1}{2}\frac{(M-m)(L-l)^2}{3}\dot{\varphi}_1^2, \tag{124}$$

where $\dot{\varphi}_1$ is an unknown angular velocity. The derivation of the kinetic energies (119), (122) and (124) for the generalized velocity $\dot{\varphi}$ is done according to (96). Then, the angular velocity of the remainder pendulum is determined

$$\dot{\varphi}_1 = \frac{1}{(M-m)(L-l)^2}[ML^2\dot{\varphi} - \frac{ml^2}{4}(\dot{\varphi}+\Omega) - 3m(\dot{\varphi}+\Omega)(L-\frac{l}{2})^2 + 3m(u_x y_{S2} - u_y x_{S2})]. \tag{125}$$

In order to prove the obtained result (125), the method based on the angular momentum of the system is considered. The angular momentum of the pendulum for the fixed point O is

$$L_0 = L_S + Mr_S v_S = \frac{1}{3}ML^2\dot{\varphi}, \tag{126}$$

where $L_S = I_S\dot{\varphi}$ is the angular momentum of the pendulum with the moment of inertia $I_S = \frac{1}{12}ML^2$, the position r_S and the velocity of the mass center v_S

$$r_S = \frac{L}{2}, v_S = \frac{L}{2}\dot{\varphi}. \tag{127}$$

The angular momentum of the separated body is

$$L_2\mathbf{k} = L_{S2}\mathbf{k} + mr_{S2}v_{S2}\mathbf{k} + \mathbf{r}_{S2}\times m\mathbf{u} = [\frac{1}{12}ml^2\dot{\varphi}_2 + m(L-\frac{l}{2})(\dot{\varphi}+\Omega) + m(x_{S2}u_y - y_{S2}u_x)]\mathbf{k}, \tag{128}$$

for

$$\begin{aligned} L_{S2} &= I_{S2}\dot{\varphi}_2 = \frac{1}{12}ml^2(\dot{\varphi}+\Omega), \\ mr_{S2}v_{S2} &= m(L-\frac{l}{2})(\dot{\varphi}+\Omega), \qquad \mathbf{r}_{S2}\times m\mathbf{u} = m\mathbf{k}(x_{S2}u_y - y_{S2}u_x), \end{aligned} \tag{129}$$

where $\mathbf{k}$ is the unit vector perpendicular to the xy plane. The angular momentum of the remainder pendulum is

$$L_0^r = L_{S1} + (M-m)r_{S1}v_{S1} = \frac{1}{3}(M-m)(L-l)^2\dot{\varphi}_1, \tag{130}$$

where

$$L_{S1} = I_{S1}\dot{\varphi}_1 = \frac{1}{12}(M-m)(L-l)^2\dot{\varphi}_1, \qquad r_{S1}v_{S1} = \frac{(L-l)^2}{4}\dot{\varphi}_1. \tag{131}$$

Using the assumption that the angular momenta of the system before and after mass separation are equal to each other, the following equation is obtained

$$\frac{1}{3}ML^2\dot{\varphi} = \frac{(M-m)(L-l)^2}{3}\dot{\varphi}_1 + \frac{ml^2}{12}(\dot{\varphi}+\Omega) + m[(\dot{\varphi}+\Omega)(L-\frac{l}{2})^2 - (u_x y_{S2} - u_y x_{S2})]. \tag{132}$$

Comparing the eq.(125) with (132), it is obvious that they are in agreement.

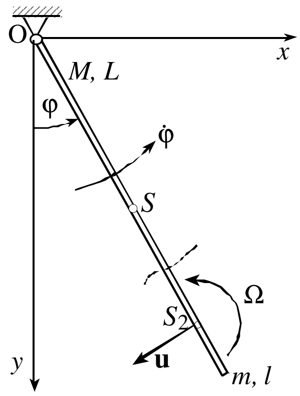


Fig.19. Model of the separated pendulum.

4.3 Conclusion

The proposed analytical procedure for obtaining the velocity and angular velocity of the remainder body after the mass separation is based on the principles of momentum and angular momentum of the body before the separation and the system of bodies after separation. By using the derivatives in the velocity of the kinetic energy of the whole body and of the system of bodies after separation, the required quantities are found. These quantities depend on the velocity and angular velocity of the separated mass, mass of the body before separation and also on the mass of the separated body. In addition, these values depend on the moment of inertia of the whole body, moment of inertia of the separated body and on the position of the separated part. The procedure given in this paper is of a general type and applicable to solving all the cases where discontinual mass variation occurs. The developed analytical method is much more suitable for engineering applications than the direct use of the general vectorial principles of dynamics. The results obtained with the developed analytical method are equal to those obtained by using the classical procedure.

The kinetic energy of the system increases during separation in general. The increase of the kinetic energy of the separated and remainder bodies during the perfectly plastic separation is equal to the kinetic energies of relative motion of the two bodies.

As it is stated in the previous Chapter, the dynamics of body separation represents an inverse process of the perfectly plastic impact of the two perfectly inelastic bodies. During the separation, which lasts for a very short time interval τ, the body undergoes a relaxation which causes the energy of deformation (potential energy) to be transformed into kinetic energies of the separated and the remainder bodies. This additional kinetic energy causes the relative motion of the separated and remainder bodies. The separation forces and torques which act between the separated and remainder bodies give the separation impulses and moments of separation impulses. We regard the separated and remainder body as one complex system. Then the separation impulses and moments between these bodies are internal within that system. The external forces and torques are assumed to be negligible in comparison to those caused by body separation. It is this fact that the law of conservation of the momentum and angular momentum of the system can be applied. The linear momentum of the body before the separation and the sum of the linear momenta of two parts after the separation remain invariable. Also, the angular momentum of the body before the separation and the sum of angular momenta of two parts after the separation remain invariable.

5 Dynamics of the body with continual mass variation

Based on the mathematical expressions of the principle of the momentum and of the angular momentum, in this Chapter the differential equations of motion of the continaul mass variation of a body are obtained.

Namely, according to (6) and (20) and the principle of the momentum, it is

$$\Delta\mathbf{K} \equiv M\Delta\mathbf{v}_{S1} \pm m(\mathbf{v}_{S2} - \mathbf{v}_{S1}) = \mathbf{F}_r\Delta t, \tag{133}$$

where the velocity variation is

$$\Delta\mathbf{v}_{S1} = \mathbf{v}_{S1} - \mathbf{v}_S. \tag{134}$$

Substituting the relation (17) into (22), the principle of the angular momentum has the form

$$\Delta\mathbf{L}_O \equiv \mathbf{r}_S \times \Delta K \mp \mathbf{SS}_2 \times m(\mathbf{v}_{S1} - \mathbf{v}_{S2}) + \Delta\mathbf{L}_S \pm \mathbf{L}_{S2} = (\mathbf{M}_0^{Fr} + \mathfrak{M})\Delta t, \tag{135}$$

where

$$\Delta\mathbf{L}_S = \mathbf{L}_{S1} - \mathbf{L}_S, \tag{136}$$

and

$$\mathbf{L}_{S2} = \mathbf{I}_{S2}\mathbf{\Omega}_2. \tag{137}$$

Introducing the notation for the adding or separating mass and moment of inertia and its absolute velocity as

$$m = \Delta M, \qquad \mathbf{I}_{S2} = \Delta\mathbf{I}, \qquad \mathbf{v}_{S2} = \mathbf{u}, \tag{138}$$

and also

$$\mathbf{v}_{S1} = \mathbf{v}, \tag{139}$$

the relations (133) and (135) are transformed

$$M\Delta\mathbf{v} = \mathbf{F}_r\Delta t \mp \Delta M(\mathbf{u} - \mathbf{v}), \tag{140}$$

$$\Delta\mathbf{L}_S = (\mathbf{M}_0^{Fr} + \mathfrak{M})\Delta t - \mathbf{r}_S \times \Delta K \pm \mathbf{SS}_2 \times \Delta M(\mathbf{v} - \mathbf{u}) \mp \Delta\mathbf{I}\mathbf{\Omega}_2. \tag{141}$$

By introducing the moment of external forces $\mathbf{M}_S^{Fr}$ for the mass center of the body S the connection between the two resultant moments $\mathbf{M}_O^{Fr}$ and $\mathbf{M}_S^{Fr}$ for the two points O and S is

$$\mathbf{M}_O^{Fr} = \mathbf{M}_S^{Fr} + \mathbf{r}_S \times \mathbf{F}_r. \tag{142}$$

Multiplying the Eq. (20) with the position vector $\mathbf{r}_S$ it is

$$\mathbf{r}_S \times \Delta K = \mathbf{r}_S \times \mathbf{F}_r. \tag{143}$$

Substituting (142) and (143) into (141) yields

$$\Delta\mathbf{L}_S = (\mathbf{M}_S^{Fr} + \mathfrak{M})\Delta t \pm \mathbf{SS}_2 \times \Delta M(\mathbf{v} - \mathbf{u}) \mp \Delta\mathbf{I}\mathbf{\Omega}_2. \tag{144}$$

Dividing the Eqs. (140) and (144) with the infinitesimal time Δt, it is

$$M\frac{\Delta\mathbf{v}}{\Delta t} = \mathbf{F}_r \mp \frac{\Delta M}{\Delta t}(\mathbf{u} - \mathbf{v}), \tag{145}$$

$$\frac{\Delta\mathbf{L}_S}{\Delta t} = (\mathbf{M}_S^{Fr} + \mathfrak{M}) \pm \mathbf{SS}_2 \times \frac{\Delta M}{\Delta t}(\mathbf{v} - \mathbf{u}) \mp \frac{\Delta\mathbf{I}}{\Delta t}\mathbf{\Omega}_2. \tag{146}$$

For the limit condition, when the infinitesimal time tends to zero, the relations (145) and (146) transform into

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_r + (\mp\left|\frac{dM}{dt}\right|)(\mathbf{u}-\mathbf{v}), \tag{147}$$

$$\frac{d\mathbf{L}_S}{dt} = (\mathbf{M}_S^{Fr} + \mathfrak{M}) + \mathbf{SS}_2 \times (\mp\left|\frac{dM}{dt}\right|)(\mathbf{u}-\mathbf{v}) + (\mp\left|\frac{d\mathbf{I}}{dt}\right|)\mathbf{\Omega}_2, \tag{148}$$

i.e.,

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_r + \frac{dM}{dt}(\mathbf{u}-\mathbf{v}), \tag{149}$$

$$\frac{d\mathbf{L}_S}{dt} = (\mathbf{M}_S^{Fr} + \mathfrak{M}) + \mathbf{SS}_2 \times \frac{dM}{dt}(\mathbf{u}-\mathbf{v}) + \frac{d\mathbf{I}}{dt}\mathbf{\Omega}_2. \tag{150}$$

Remark 4 *It must be emphasized that the sign 'minus' or 'plus' ($\mp$) in front of the elementary mass dM and moment of inertia $d\mathbf{I}$ have to be eliminated in the equations which describe the motion of the continually mass variable bodies. Namely, the sign of the first time derivative of the mass (dM/dt) and of the moment of inertia ($d\mathbf{I}/dt$) are negative, if the mass and the moment of inertia are decreasing in time (mass separation), and positive, if the mass and the moment of inertia are increasing in time (mass addition). The sign of these first derivatives is automatically obtained during the calculation and it availables the elimination of the mentioned signs in the formulas in front of the time derivatives of the mass and moment of inertia.*

For $\mathbf{L}_S = \mathbf{I\Omega}$ where $\mathbf{I} \equiv \mathbf{I}_S$ is the tensor of moment of inertia, and after some simple modification the differential equations of motion of the body with continual mass variation follow as

$$\frac{d}{dt}(M\mathbf{v}) = \mathbf{F}_r + \frac{dM}{dt}\mathbf{u}, \tag{151}$$

$$\frac{d}{dt}(\mathbf{I\Omega}) = (\mathbf{M}_S^{Fr} + \mathfrak{M}) + \mathbf{SS}_2 \times \frac{dM}{dt}(\mathbf{u}-\mathbf{v}) + \frac{d\mathbf{I}}{dt}\mathbf{\Omega}_2. \tag{152}$$

For (see for example Goldstein, 1980)

$$\frac{d}{dt}(\mathbf{I\Omega}) = \mathbf{\Omega}\frac{d\mathbf{I}}{dt} + \mathbf{I}\frac{d\mathbf{\Omega}}{dt} + \mathbf{\Omega} \times \mathbf{I\Omega}, \tag{153}$$

the differential equations of motion transform into

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_r + \frac{dM}{dt}(\mathbf{u}-\mathbf{v}), \tag{154}$$

$$\mathbf{I}\frac{d\mathbf{\Omega}}{dt} + \mathbf{\Omega} \times \mathbf{I\Omega} = (\mathbf{M}_S^{Fr} + \mathfrak{M}) + \mathbf{SS}_2 \times \frac{dM}{dt}(\mathbf{u}-\mathbf{v}) + \frac{d\mathbf{I}}{dt}(\mathbf{\Omega}_2 - \mathbf{\Omega}). \tag{155}$$

The last terms in the Eqs. (154) and (155) represent the reactive force

$$\mathbf{\Phi} = \frac{dM}{dt}(\mathbf{u}-\mathbf{v}), \tag{156}$$

and the reactive torque

$$\mathbb{R} = \frac{d\mathbf{I}_S}{dt}(\mathbf{\Omega}_2 - \mathbf{\Omega}), \tag{157}$$

which exist due to variation of mass and moment of inertia of the body. The reactive force $\mathbf{\Phi}$ gives the moment due to point S and it is

$$\mathbf{M}_S^{\mathbf{\Phi}} = \mathbf{SS}_2 \times \mathbf{\Phi}. \tag{158}$$

Substituting (156) - (158) into (154) and (155), we have

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_r + \mathbf{\Phi}, \tag{159}$$

$$\mathbf{I}\frac{d\mathbf{\Omega}}{dt} + \mathbf{\Omega} \times \mathbf{I\Omega} = \mathbf{M}_S^{Fr} + \mathfrak{M} + \mathbf{M}_S^{\mathbf{\Phi}} + \mathbb{R}. \tag{160}$$

The first equation defines the translational motion and the second the rotation around the mass center S.

For practical reasons it is convenient to rewrite the vector differential equations (159) and (160) into the scalar ones. Introducing the fixed coordinate system $Oxyz$, the components u, v, w of the velocity $\mathbf{v}$, the components u_2, v_2, w_2 of the velocity $\mathbf{u}$ and F_x, F_y, F_z the components of the resultant $\mathbf{F}_r$ the vector differential equation of translational motion is given with three scalar equations

$$M\frac{du}{dt} = F_x + \Phi_x, \qquad M\frac{dv}{dt} = F_y + \Phi_y, \qquad M\frac{dw}{dt} = F_z + \Phi_z, \tag{161}$$

The terms on the right side of (161)

$$\Phi_x = \frac{dM}{dt}(u_2 - u), \qquad \Phi_y = \frac{dM}{dt}(v_2 - v), \qquad \Phi_z = \frac{dM}{dt}(w_2 - w), \tag{162}$$

are called the projections of the reactive force. The reactive force is the consequence of mass variation. This result is published in the paper Cveticanin and Kovacic, 2007.

For the reference system $S\xi\eta\zeta$ fixed to the body with the origin in the center of mass of the body the inertial tensor $\mathbf{I}$ has nine components, but only six of them are independent: $I_{\xi\xi}, I_{\eta\eta}, I_{\varsigma\varsigma}$, are the moments of inertia and $I_{\xi\eta}, I_{\xi\varsigma}, I_{\eta\varsigma}$and also $I_{\eta\xi}, I_{\varsigma\xi}, I_{\varsigma\eta}$ are the products of inertia. If the axes are principal and products of inertia are zero simultaneously the inertial tensor $\mathbf{I}$ has only three principal moments of inertia $I_{\xi\xi}, I_{\eta\eta}, I_{\varsigma\varsigma}$. The angular velocity $\mathbf{\Omega}$ has three components p, q, r in this frame. If p_2, q_2, r_2 are the components of the angular velocity $\mathbf{\Omega}_2$, $M_\xi^{\mathbf{\Phi}}$, $M_\eta^{\mathbf{\Phi}}$ and $M_\varsigma^{\mathbf{\Phi}}$ are the body-axis components of $\mathbf{M}_S^{\mathbf{\Phi}}$, M_ξ, M_η and M_ζ are the body-axis components of $\mathbf{M}_S^{Fr}$ and $\mathfrak{M}_\xi, \mathfrak{M}_\eta$ and $\mathfrak{M}_\zeta$ are the projections of the torque the vector $\mathfrak{M}$, the equation for rotational motion (160) is given with three scalar equations

$$\begin{aligned}
I_{\xi\xi}\frac{dp}{dt} + (I_{\varsigma\varsigma} - I_{\eta\eta})qr &= M_\xi + \mathfrak{M}_\xi + M_\xi^{\mathbf{\Phi}} + \Re_\xi,\\
I_{\eta\eta}\frac{dq}{dt} + (I_{\xi\xi} - I_{\varsigma\varsigma})pr &= M_\eta + \mathfrak{M}_\eta + M_\eta^{\mathbf{\Phi}} + \Re_\eta,\\
I_{\varsigma\varsigma}\frac{dr}{dt} + (I_{\eta\eta} - I_{\xi\xi})pq &= M_\varsigma + \mathfrak{M}_\zeta + M_\varsigma^{\mathbf{\Phi}} + \Re_\zeta.
\end{aligned} \tag{163}$$

The terms on the right side of (163)

$$\Re_\xi = \frac{dI_{\xi\xi}}{dt}(p_2 - p), \qquad \Re_\eta = \frac{dI_{\eta\eta}}{dt}(q_2 - q), \qquad \Re_\zeta = \frac{dI_{\varsigma\varsigma}}{dt}(r_2 - r), \tag{164}$$

are called the projections of the reactive torque. The reactive torque is the consequence of variation of moment of inertia of the body.

The Eqs. (161) and (163) are the six scalar differential equations of motion of the body with variable mass. For the both cases, when mass is separating or adding, the differential equations of motion have the same form (161) and (163), but the signs of separating mass dM/dt and separating moment of inertia dI/dt are negative, while the signs of adding mass dM/dt and adding moment of inertia dI/dt are positive.

5.1 Discussion of the differential equations of motion

1) Comparing the relations (159) and (160) with those for the free motion of a body with constant mass (see for example Starzhinskii, 1982)

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_r, \qquad \mathbf{I}\frac{d\mathbf{\Omega}}{dt} + \mathbf{\Omega}\times\mathbf{I}\mathbf{\Omega} = \mathbf{M}_S^{Fr} + \mathfrak{M}, \tag{165}$$

it is evident that due to variation of the mass and moment of inertia some additional terms exist which represent the reactive force $\mathbf{\Phi}$, its moment $\mathbf{M}_S^{\mathbf{\Phi}}$ and the reactive torque $\mathbb{R}$.

2) For the case when the relative velocity and angular velocity of mass and moment of inertia variation is zero, the differential equations of motion have the form (165), but M and $\mathbf{I}$ are time variable. Namely, due to the fact that $\mathbf{u} = \mathbf{v}$ and $\mathbf{\Omega}_2 = \mathbf{\Omega}$, the reactive force and torque and also the corresponding moment are zero, i.e.,

$$\mathbf{\Phi} = 0, \qquad \mathbb{R} = 0, \qquad \mathbf{M}_S^{\mathbf{\Phi}} = 0. \tag{166}$$

3) For the case when the absolute velocity $\mathbf{u} = 0$ and the angular velocity $\mathbf{\Omega}_2 = 0$ of added or separated mass are zero, the differential equations of motion transform into

$$\frac{d}{dt}(M\mathbf{v}) = \mathbf{F}_r, \tag{167}$$

$$\frac{d}{dt}(\mathbf{I}\mathbf{\Omega}) = \mathbf{M}_S^{Fr} + \mathfrak{M} + \mathbf{M}_S^{\mathbf{\Phi}^*}, \tag{168}$$

where the modified reactive force and its moment are

$$\mathbf{\Phi}^* = \mathbf{v}\frac{dM}{dt}, \qquad \mathbf{M}_S^{\mathbf{\Phi}^*} = \mathbf{S}\mathbf{S}_2 \times \mathbf{\Phi}^*. \tag{169}$$

The sign of the moment of the reactive force is negative.

Cornelisse *et al.*, 1979, introduce the additional assumption: the moment of the reactive force is sufficiently small in comparison with other values in the system and can be omitted. The differential equations of motion (167) and (168) simplify into

$$\frac{d}{dt}(M\mathbf{v}) = \mathbf{F}_r, \qquad \frac{d}{dt}(\mathbf{I}\mathbf{\Omega}) = \mathbf{M}_S^{Fr} + \mathfrak{M}. \tag{170}$$

In the paper of Meshchersky (1896) this special case is considered for the rotating body around a fixed axle

$$\frac{d}{dt}(I_\zeta r) = M_\zeta^{Fr} + \mathfrak{M}_\zeta. \tag{171}$$

where I_ζ is the time variable moment of inertia, r is the angular velocity, M_ζ^{Fr} is the moment of the resultant force due to the axle ς and $\mathfrak{M}_\zeta$ is the torque rotating the body around the axle ς.

4) Mathematical model of the mass variable body with translatory motion is equal to that of the particle with variable mass and it is

$$M\frac{d\mathbf{v}}{dt} = \mathbf{F}_r + \mathbf{\Phi}. \tag{172}$$

This relation was for the first time introduced by Meshchersky. Based on this equation the modern rocket dynamics is developed.

5) If the mass variable body rotates around a fixed axle, the system of differential equations of motion (161) and (163) simplify into only one

$$I_\varsigma\frac{dr}{dt} = M_\varsigma + \mathfrak{M}_\zeta + \Re_\zeta, \tag{173a}$$

The reactive torque $\Re_\zeta$ depends on the variation of the moment of inertia for the rotation axle and the relative angular velocity which is the difference between the angular velocity of body rotation and the angular velocity of the separated or added mass. If the relative angular velocity of the mass variation is zero, the differential equation of body rotation is

$$I_\varsigma \ddot{\varphi} = M_\varsigma + \mathfrak{M}_\zeta. \tag{174}$$

where $\dot{\varphi} = r$.and the moment of inertia is time dependent.

6) For the straightforward motion of a mass variable body the differential equation is according to (172)

$$M\frac{dv}{dt} = F_r + \Phi, \tag{175}$$

where

$$\Phi = \frac{dM}{dt}(u - v). \tag{176}$$

Comparing (173a) and (175) it can be concluded that the differential equations have the same form, where the reactive force Φ for the translatory motion corresponds to the reactive torque $\Re_\zeta$.

Remark 5 *Using the principle of solidification Bessonov, 1967, obtained the differential equations of free motion of the body. Bessonov does not take the reactive torque into consideration.*

5.2 Band is winding up on a drum

Let us consider the in-plane motion of a drum on which the band is winding up (Fig.20). The differential equations of the motion are according to (161) and (163)

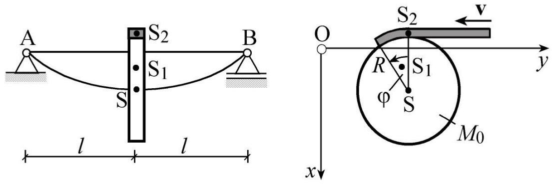
$$\begin{aligned}
\frac{d}{dt}(M\dot{x}_{S1}) &= F_x + \frac{dM}{dt}v_{xb}, \\
\frac{d}{dt}(M\dot{y}_{S1}) &= F_y + \frac{dM}{dt}v_{yb}, \\
\frac{d}{dt}(I_{S1}\dot{\varphi}) &= M_{S1} + M_{S1}^{\mathbf{\Phi}} + \frac{dI_{S1}}{dt}\Omega_b,
\end{aligned} \tag{177}$$

where

$$M_{S1}^{\mathbf{\Phi}} = \frac{dM}{dt}[(\mathbf{S}_1\mathbf{S}_2)_x(v_{yb} - \dot{y}_{S1}) - (\mathbf{S}_1\mathbf{S}_2)_y(v_{xb} - \dot{x}_{S1}), \tag{178}$$

$(\mathbf{S}_1\mathbf{S}_2)_x$ and $(\mathbf{S}_1\mathbf{S}_2)_y$ are projections of the position vector of the point of mass addition due to the mass centre S_1, Ω_b is the angular velocity of the winding band, v_{xb} and v_{yb} are the projections of the linear velocity of the winding band, M is the mass of the drum with band, I_{S1} is moment of inertia of the drum with band, $\dot{x}_{S1}$ and $\dot{y}_{S1}$ are the projections of the velocity of the mass center of the drum with band and $\dot{\varphi}$ is the angular velocity of the drum with band (Cveticanin & Kovacic, 2007).

Fig.20. The model of the rotor on which the band is winding up.



The technical requirement for winding up of the band is the absolute velocity of the band v_b to be constant. Only for that condition the rolling up of the band on the drum is accurate without crumpling of the band or its plucking. The band is moving translatory with velocity v horizontally, parallel to y-axle in Fig.20. The projections of the band velocity are

$$v_{xb} = 0, \qquad v_{yb} = v. \tag{179}$$

The rolling up of one band layer is discussed. The angle of rolling up of the band is in the interval from $\varphi = 0$ to $\varphi = 2\pi$.

5.2.1 The geometric and physical properties of the drum with band

If the mass of the drum with unrolled band is M_0 and the rolling mass is M_r

$$M_r = \mu\varphi, \tag{180}$$

where

$$\mu = Rhb\rho, \tag{181}$$

h is the thickness, b is the width and ρ is the density of band, the mass M variation is a linear function of the angle φ

$$M = M_0 + M_r = M_0 + \mu\varphi. \tag{182}$$

The position of mass centre of the drum with unrolled mass is

$$SS_1 = \frac{M_r}{M}(SS'), \tag{183}$$

where SS' is the distance of the mass centre of the unrolled mass on the drum

$$SS' = R\frac{\sin(\varphi/2)}{\varphi/2}. \tag{184}$$

According to the relations (182), (183) and (184) the distance between the mass centre of the whole system and the rotation centre is obtained

$$SS_1 = \frac{2\mu}{M_0 + \mu\varphi}\sin(\frac{\varphi}{2}). \tag{185}$$

The moment of inertia of the drum with the unrolled mass is J_0 and the moment of inertia of the band which is rolling up is

$$J_r = \int_0^{\varphi} R^2 dM_r = \int_0^{\varphi} R^3 hb\rho d\varphi = j\varphi, \tag{186}$$

where $j = R^3 hb\rho = \mu R^2$ is the unit moment of inertia. The total moment of inertia is obtained by superposition the both moments of inertia

$$J_S = J_0 + j\varphi. \tag{187}$$

Applying the Steiner theorem the moment of inertia for the parallel axis settled in the mass centre is obtained

$$J_{S1} = J_S - M(SS_1)^2. \tag{188}$$

5.2.2 Forces acting on the system

During winding up of the band the following forces act: the elastic force of the shaft, the damping torque, the reactive force and the reactive torque.

The elastic force of the shaft is projected in the fixed coordinate system

$$F_x = -cx_S = -c(x_{S1} - SS_1 \cos\frac{\varphi}{2}), \qquad F_y = -cy_S = -c(y_{S1} - SS_1 \sin\frac{\varphi}{2}), \quad (189)$$

where c is the rigidity of the shaft.

According to (177) and (179) the projections of the reactive force Φ and the reactive torque $\Im$ are obtained

$$\Phi_x = \frac{dM}{dt}(-\dot{x}_{S1}), \qquad \Phi_y = \frac{dM}{dt}(v - \dot{y}_{S1}), \qquad \Re = \frac{dI_{S1}}{dt}(-\dot{\varphi}).$$

If the rotational damping torque acts

$$M_D = -D\dot{\varphi}, \quad (190)$$

where D is the damping coefficient, and the moment of the reactive force according to S_1 is considered, the differential equations of the plane motion is obtained

$$\begin{aligned} M\ddot{x}_{S1} + cx_{S1} &= c(SS_1)\cos\frac{\varphi}{2} + \frac{dM}{dt}(-\dot{x}_{S1}), \\ M\ddot{y}_{S1} + cy_{S1} &= c(SS_1)\sin\frac{\varphi}{2} + \frac{dM}{dt}(v - \dot{y}_{S1}), \\ J_{S1}\ddot{\varphi} + D\dot{\varphi} &= \frac{dJ_{S1}}{dt}(-\dot{\varphi}) + x_{S1}c(SS_1)\sin\frac{\varphi}{2} - y_{S1}c(SS_1)\cos\frac{\varphi}{2} \\ &+ \frac{dM}{dt}(v - \dot{y}_{S1})(\mathbf{S}_1\mathbf{S}_2)_x - \frac{dM}{dt}(-\dot{x}_{S1})(\mathbf{S}_1\mathbf{S}_2)_y. \end{aligned} \quad (191)$$

It is worth to be mentioned that due to winding up of the band the symmetry of the disc is disturbed and the products of inertia J_{xz} and J_{xz} are not zero. As the thickenss of the band which is winding up is quite small, the mentioned products of inertia are also small and can be neglected.

For the case when the mass centre and the geometric centre of the disc are quite close and the eccentricity is $SS_1 \approx 0$, we have the coordinates $x_{S1} \approx x_S$ and $y_{S1} \approx y_S$. Omitting the terms with SS_1 in the equation (191) and using the relations (182), (187) and (188) the differential equations of the plane motion of the disc, with variable mass and without the unbalance force, are obtained

$$M\ddot{x}_S + cx_S = -\mu\dot{\varphi}\dot{x}_S, \qquad M\ddot{y}_S + cy_S = \mu\dot{\varphi}(v - \dot{y}_S), \quad (192)$$

$$J_S\ddot{\varphi} + D\dot{\varphi} = -j\dot{\varphi}^2 + \mu\dot{\varphi}(v - \dot{y}_S)(\mathbf{SS}_2)_x + \mu\dot{\varphi}\dot{x}_S(\mathbf{SS}_2)_y. \quad (193)$$

The system of differential equations (192) and (193) is nonlinear.

5.2.3 The shaft is rigid

If the shaft of the drum with winding up band is rigid the motion of the system transforms to a rotation around the rigid axle ($x_S = y_S = 0$)

$$(J_0 + j\varphi)\ddot{\varphi} + D\dot{\varphi} = -j\dot{\varphi}^2 + \mu\dot{\varphi}vR. \quad (194)$$

Introducing the new variable $u(\varphi) = \dot{\varphi}$ the differential equation (194) is transformed to the Bernoulli equation

$$(J_0 + j\varphi)\frac{du}{d\varphi} + ju = (\mu vR - D), \quad (195)$$

whose solution for the initial condition $\dot{\varphi}(0) = \Omega_b$ has the form

$$\dot{\varphi} = \frac{J_0\Omega_b + (\mu v R - D)\varphi}{J_0 + j\varphi}. \tag{196}$$

The relation (196) describes the variation of the angular velocity of the drum when the absolute velocity v of the winding band is constant. The angular velocity of the drum decreases during winding up of a layer.

Integrating the differential equation (196) for the initial angle $\varphi(0) = 0$ the time history of angle variation is obtained

$$\varphi + (\frac{J_0}{j} - \frac{J_0\Omega_b}{\mu v R - D}) \ln \left| 1 + \varphi \frac{\mu v R - D}{J_0\Omega_b} \right| = \frac{\mu v R - D}{j} t. \tag{197}$$

This form of solution is not convenient for discussion. Introducing the new variable

$$r = 1 + \frac{\mu v R - D}{J_0\Omega_b}\varphi, \tag{198}$$

the equation (197) is

$$r + \frac{D}{j\Omega_b} \ln r = 1 + \frac{(\mu v R - D)^2}{J_0 j\Omega_b} t. \tag{199}$$

Let us introduce the new function

$$w = -\ln r - \frac{f_1}{k_1}, \tag{200}$$

where

$$f_1 = 1 + \frac{(\mu v R - D)^2}{J_0 j \Omega_b} t, \quad k_1 = -\frac{\mu v R - D - j\Omega_b}{j\Omega_b}. \tag{201}$$

After substituting (200) into (197) and some transformation the obtained result is

$$w \exp(w) = x, \tag{202}$$

where

$$x = \frac{1}{k_1} \exp(-\frac{f_1}{k_1}). \tag{203}$$

The solution $w(x)$ of (202) is the Lambert's w function (see Weisstein, 2007)

$$w(x) \equiv lambertw(\frac{1}{k_1} \exp(-\frac{f_1}{k_1})). \tag{204}$$

Substituting into (200) the solution for r is obtained

$$r = -k_1(lambertw(-\frac{1}{k_1} \exp(-\frac{f_1}{k_1}))) \equiv -k_1 w(x), \tag{205}$$

which gives the implicit solution for (197)

$$\varphi = J_0(\frac{1}{j} - \frac{\Omega_b}{\mu v R - D})w - \frac{J_0\Omega_b}{\mu v R - D}. \tag{206}$$

For the case when damping is neglected and assuming that $v = \Omega_b R$ the relation (197) is simplified and the angle time function is linear

$$\varphi = \Omega_b t. \tag{207}$$

5.2.4 The shaft is elastic

Let us transform the differential equations (192) introducing the variables

$$x_S = x(\varphi), \qquad \dot{x}_S = \frac{dx}{d\varphi}\dot{\varphi}, \qquad \ddot{x}_S = \frac{d^2x}{d\varphi^2}\dot{\varphi}^2 + \frac{dx}{d\varphi}\ddot{\varphi}, \tag{208}$$

$$y_S = y(\varphi), \qquad \dot{y}_S = \frac{dy}{d\varphi}\dot{\varphi}, \qquad \ddot{y}_S = \frac{d^2y}{d\varphi^2}\dot{\varphi}^2 + \frac{dy}{d\varphi}\ddot{\varphi}, \tag{209}$$

The obtained system of differential equations of plane motion is

$$(M_0 + \mu\varphi)\dot{\varphi}^2\frac{d^2x}{d\varphi^2} + \frac{dx}{d\varphi}[\mu\dot{\varphi}^2 + (M_0 + \mu\varphi)\ddot{\varphi}] + cx = 0, \tag{210}$$

$$(M_0 + \mu\varphi)\dot{\varphi}^2\frac{d^2y}{d\varphi^2} + \frac{dy}{d\varphi}[\mu\dot{\varphi}^2 + (M_0 + \mu\varphi)\ddot{\varphi}] + cy = \mu\dot{\varphi}v, \tag{211}$$

$$J_S\ddot{\varphi} + (D - \mu vR)\dot{\varphi} + j\dot{\varphi}^2 = -\mu\dot{\varphi}^2R\frac{dy}{d\varphi}. \tag{212}$$

Substituting (196) into (210) and (211), assuming that μ/M_0, j/J_0, μ/J_0 and $D/J_0\Omega_b$ are small parameters, the simplified differential equations are formed

$$\frac{d^2x}{d\varphi^2} + 2\delta\frac{dx}{d\varphi} + \omega^2(\varphi)x = 0, \tag{213}$$

$$\frac{d^2y}{d\varphi^2} + 2\delta\frac{dy}{d\varphi} + \omega^2(\varphi)y = \frac{\mu}{M_0}R, \tag{214}$$

$$\ddot{\varphi} + \frac{D - \mu Rv}{J_S}\dot{\varphi} + \frac{j}{J_S}\dot{\varphi}^2 = -\frac{\mu}{J_0}R\dot{\varphi}^2\frac{dy}{d\varphi}, \tag{215}$$

where

$$2\delta = 2\frac{\mu}{M_0} - \frac{j}{J_0} - \frac{D}{J_0\Omega_b}, \qquad \omega^2(\varphi) \equiv \omega^2 = k^2(1 - A\varphi),$$

$$k^2 = \frac{\omega_1^2}{\Omega_b^2}, \qquad \omega_1^2 = \frac{c}{M_0}, \qquad A = 3\frac{\mu}{M_0} - 2\frac{j}{J_0} - \frac{D}{J_0\Omega_b}.$$

To obtain the approximate analytic solutions of (213) - (215) the Bogolubov-Mitropolski method is modified for the non-homogenous rheo-linear differential equations.

Omitting the terms on the right side of the equation (215) as small values the approximate solution of (215) corresponds to the case of rigid shaft (196). Substituting (196) into (214) the solution of the differential equation (214) is assumed as

$$y = a(\varphi)\exp(-\delta\varphi)\cos\Psi(\varphi) + \frac{1}{\omega^2(\varphi)}\frac{\mu}{M_0}R \equiv a\exp(-\delta\varphi)\cos\Psi + \frac{1}{\omega^2(\varphi)}\frac{\mu}{M_0}R, \tag{216}$$

where

$$\Psi(\varphi) = \int \omega(\varphi)d\varphi + \alpha(\varphi), \tag{217}$$

and the first derivative of the function y

$$\frac{dy}{d\varphi} = (-\delta a\cos\Psi - a\omega(\varphi)\sin\Psi)\exp(-\delta\varphi) - (\frac{\mu}{M_0})^2\frac{R}{k^2(1 - A\varphi)^2}, \tag{218}$$

with

$$\frac{da}{d\varphi}\cos\Psi - a\frac{d\alpha}{d\varphi}\sin\Psi = 0. \tag{219}$$

Eliminating the second order small term in (218) the relation transforms to

$$\frac{dy}{d\varphi} \approx (-\delta a \cos\Psi - a\omega(\varphi)\sin\Psi)\exp(-\delta\varphi). \tag{220}$$

Using the relations (216) - (220) the differential equation (214) is transformed into a system of two first order differential equations

$$\frac{da}{d\varphi} = -\frac{a}{\omega}\frac{d\omega}{d\varphi}\sin^2\Psi, \qquad \frac{d\alpha}{d\varphi} = \frac{1}{2\omega}\frac{d\omega}{d\varphi}\sin 2\Psi. \tag{221}$$

It is at this point the averaging procedure $\frac{1}{2\pi}\int\limits_0^{2\pi}(\bullet)d\Psi$ is introduced and the relations (221) are simplified into

$$\frac{da}{d\varphi} = -\frac{a}{2\omega}\frac{d\omega}{d\varphi}, \qquad \frac{d\alpha}{d\varphi} = 0. \tag{222}$$

For the initial conditions

$$\varphi = 0, \qquad a = a_0, \qquad \alpha = \alpha_0, \tag{223}$$

the solution of the equation (214) in the first approximation is

$$y_S = y(\varphi) = \frac{a_0}{\sqrt[4]{1 - A\varphi}}\exp(-\delta\varphi)\cos(k\sqrt{(1 - A\varphi)} + \alpha_0) + \frac{R}{k^2}\frac{\mu}{M_0}. \tag{224}$$

According to the suggested procedure the solution of the equation (213) is

$$x_S = x(\varphi) = \frac{b_0}{\sqrt[4]{1 - A\varphi}}\exp(-\delta\varphi)\cos(k\sqrt{(1 - A\varphi)} + \beta_0), \tag{225}$$

where b_0 and β_0 are initial amplitude and phase. The parameter values have to satisfy the relation

$$3\frac{\mu}{M_0} - 2\frac{j}{J_0} - \frac{D}{J_0\Omega_b} < \frac{1}{2\pi}. \tag{226}$$

The motion of the rotor center depends on the ratio between the small parameters μ/M_0, j/J_0 and $D/J_0\Omega_b$. For small value of the rotational damping and higher velocity of the rolling band the vibrations decrease.

Using the obtained solution (224) the correction for the angle velocity (196) can be denoted. Due to the fact that $\dot{y}_S$ tends to zero for technical reasons the relation (196) is guessed to be accurate enough.

5.3 Conclusion

During the process of continual mass variation, the mass and moment of inertia of the rigid body vary due to adding or separating of mass in the short infinitesimal time interval: mass but also the form and the volume of the body are continually varying in time. It causes the body mass center position variation and also the change of the moment of inertia and the products of moment of inertia. Due to mass and moment of inertia variation the reactive force and reactive torque act. Namely, the absolute velocity and angular velocity of addition or separation differs in general from the velocity of mass center and angular velocity of the initial body and it causes the impact to occur. As the mass variation is continual the impact is substituted with a "reactive force" and "reactive torque" which continually act on the body. The force and torque depend on the absolute velocity of mass centre and angular velocity of the separated or added body.

For the rolling up of the band on the drum mass and moment of inertia of the drum with band is varying. Mass and moment of inertia depend on the angle position of the wounded band. Due to geometry variation of the drum with band the mass center position inside the system is varying, too. This variation seems to be small and is neglected in our consideration. During winding up of the band on the drum the impact occurs due to difference of velocity and angular velocity of the band and drum. It causes the vibrations of the mass centre of drum. The vibrations of drum mass centre depend on the amount on the band winding up on the drum: the higher the amount of band on the drum the smaller the vibrations. The damping property of drum also has an influence on the vibrations: the higher the damping the smaller the vibrations.

The band is winding up with constant velocity. This requires the angle velocity of drum to vary. The angle velocity variation is the function of the moment of inertia of the band which is winding up and also of the damping properties of the system: for higher damping the angle velocity decreases faster than for the smaller damping; the larger the moment of inertia of the winding up band the slower the decrease of the angular velocity. This result is of technical importance for regulating of the rotation of the drum.

6 Lagrange's equations of the body with continual mass variation

As it is always in analytical approach to the problem, in this Chapter the famous Lagrange's equations of the second kind for the body with continual mass variation are derived. Let us rewrite the Eqs. (151) and (152) into the form

$$\frac{d}{dt}(M\mathbf{v}) = \mathbf{F}_r + \mathbf{\Phi}_a, \tag{227}$$

$$\frac{d}{dt}(\mathbf{I\Omega}) = \mathbf{M}_S^{Fr} + \mathfrak{M} + \mathbf{M}_S^{\Phi} + \mathbb{R}_a. \tag{228}$$

where

$$\mathbf{\Phi}_a = \frac{dM}{dt}\mathbf{u}, \qquad \mathbb{R}_a = \frac{d\mathbf{I}}{dt}\mathbf{\Omega}_2, \tag{229}$$

and the reactive force $\mathbf{\Phi}$ and the moment of the reactive force $\mathbf{M}_S^{\Phi}$ are given with Eqs. (156) and (158), respectively. Multiplying the Eq. (227) with the virtual displacement $\delta\mathbf{r}$, and the Eq. (228) with the virtual angle $\delta\mathbf{\Psi}$ and by adding them, it follows

$$(\mathbf{F}_r+\mathbf{\Phi}_a)\delta\mathbf{r} + (\mathbf{M}_S^{Fr}+\mathfrak{M}+\mathbf{M}_S^{\Phi}+\mathbb{R}_a)\delta\mathbf{\Psi} - M\ddot{\mathbf{r}}\delta\mathbf{r} - \dot{M}\dot{\mathbf{r}}\delta\mathbf{r} - \mathbf{I}\ddot{\mathbf{\Psi}}\delta\mathbf{\Psi} - \dot{\mathbf{I}}\dot{\mathbf{\Psi}}\delta\mathbf{\Psi} = \mathbf{0}, \tag{230}$$

where $\dot{\mathbf{\Psi}} = \mathbf{\Omega}$ and $(\cdot)^{\cdot} \equiv d(\cdot)/dt$, $(\cdot)^{\cdot\cdot} \equiv d^2(\cdot)/dt^2$.The relation (230) describes the D'Alambert-Lagrange principle for the body with continual mass variation: The total virtual work of all active forces and torques (including the non-ideal constraint reactions), of the reactive force and torque, of the moment of the reactive force and of the inertial force and torque is equal to zero for any virtual displacement and virtual angle of the body. Mathematically, it is

$$\delta A^I + \delta A^{\phi a} + \delta A^{Ra} + \delta A + \delta A^{M\phi} = 0, \tag{231}$$

where

$$\delta A^I = -(M\ddot{\mathbf{r}}+\dot{M}\dot{\mathbf{r}}\delta\mathbf{r})\delta\mathbf{r} - (\mathbf{I}\ddot{\mathbf{\Psi}} + \dot{\mathbf{I}}\dot{\mathbf{\Psi}})\delta\mathbf{\Psi}, \qquad \delta A^{\phi a} = \mathbf{\Phi}_a\delta\mathbf{r},$$
$$\delta A^{Ra} = \mathbb{R}_a\delta\mathbf{\Psi}, \qquad \delta A^{M\phi} = \mathbf{M}_S^{\Phi}\delta\mathbf{\Psi}, \qquad \delta A = \mathbf{F}_r\delta\mathbf{r} + (\mathbf{M}_S^{Fr} + \mathfrak{M})\delta\mathbf{\Psi} \tag{232}$$

Let us introduce $i = 1, 2, ..., 6$ independent generalized coordinates q_i. The virtual displacement and the virtual angle are determined by formulas

$$\delta \mathbf{r} = \sum_{i=1}^{6} \frac{\partial \mathbf{r}}{\partial q_i} \delta q_i, \qquad \delta \mathbf{\Psi} = \sum_{i=1}^{6} \frac{\partial \mathbf{\Psi}}{\partial q_i} \delta q_i, \tag{233}$$

where δq_i is the variation of the generalized coordinate q_i. Substituting (233) into (231) and after some modification we have

$$\sum_{i=1}^{6} (Z_i + Q_i^{\phi a} + Q_i^{Ra} + Q_i + Q_i^{M\phi}) \delta q_i = 0, \tag{234}$$

where the generalized inertial force Z_i, generalized force of the part of the reactive force $Q_i^{\phi a}$ and reactive torque Q_i^{Ra}, generalized force of the active forces and torques and reactions of non-ideal constraints Q_i and the generalized force of the moment of the reactive force $Q_i^{M\phi}$ are calculated according to following formulas

$$\begin{aligned} Z_i &= -[\frac{d}{dt}(M\mathbf{v})\frac{\partial \mathbf{r}}{\partial q_i} + \frac{d}{dt}(\mathbf{I\Omega})\frac{\partial \mathbf{\Psi}}{\partial q_i}], \qquad Q_i^{\phi a} = \mathbf{\Phi}_a \frac{\partial \mathbf{r}}{\partial q_i} \qquad Q_i^{Ra} = \mathbb{R}_a \frac{\partial \mathbf{\Psi}}{\partial q_i}, \\ Q_i &= \mathbf{F}_r \frac{\partial \mathbf{r}}{\partial q_i} + (\mathbf{M}_S^{Fr} + \mathfrak{M}) \frac{\partial \mathbf{\Psi}}{\partial q_i}, \qquad Q_i^{M\phi} = \mathbf{M}_S^{\Phi} \frac{\partial \mathbf{\Psi}}{\partial q_i}. \end{aligned} \tag{235}$$

The generalized inertial force Z_i is rewritten as

$$Z_i = -\frac{d}{dt}[(M\mathbf{v})\frac{\partial \mathbf{r}}{\partial q_i} + (\mathbf{I\Omega})\frac{\partial \mathbf{\Psi}}{\partial q_i}] + [(M\mathbf{v})\frac{d}{dt}\frac{\partial \mathbf{r}}{\partial q_i} + (\mathbf{I\Omega})\frac{d}{dt}\frac{\partial \mathbf{\Psi}}{\partial q_i}]. \tag{236}$$

The position vector $\mathbf{r}$ and the angle vector $\mathbf{\Psi}$ depend on the generalized coordinates q_i and time t

$$\mathbf{r} = \mathbf{r}(q_i, t), \qquad \mathbf{\Psi} = \mathbf{\Psi}(q_i, t). \tag{237}$$

As the generalized coordinates also depend on time, the velocity and angular velocity have the form

$$\mathbf{v} \equiv \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \mathbf{r}}{\partial q_i} \dot{q}_i, \qquad \mathbf{\Omega} \equiv \frac{d\mathbf{\Psi}}{dt} = \frac{\partial \mathbf{\Psi}}{\partial t} + \sum_{i=1}^{6} \frac{\partial \mathbf{\Psi}}{\partial q_i} \dot{q}_i. \tag{238}$$

Now we take the partial derivatives with respect to $\dot{q}_i$

$$\frac{\partial \mathbf{r}}{\partial q_i} = \frac{\partial \dot{\mathbf{r}}}{\partial \dot{q}_i} = \frac{\partial \mathbf{v}}{\partial \dot{q}_i}, \qquad \frac{\partial \mathbf{\Psi}}{\partial q_i} = \frac{\partial \dot{\mathbf{\Psi}}}{\partial \dot{q}_i} = \frac{\partial \mathbf{\Omega}}{\partial \dot{q}_i}. \tag{239}$$

On the other hand, taking the partial derivatives of both the sides of equalities (238) with respect to q_i we obtain

$$\frac{\partial \mathbf{v}}{\partial q_i} = \frac{\partial^2 \mathbf{r}}{\partial t \partial q_i} + \sum_{j=1}^{6} \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_j} \dot{q}_j, \qquad \frac{\partial \mathbf{\Omega}}{\partial q_i} = \frac{\partial^2 \mathbf{\Psi}}{\partial t \partial q_i} + \sum_{j=1}^{6} \frac{\partial^2 \mathbf{\Psi}}{\partial q_i \partial q_j} \dot{q}_j. \tag{240}$$

Directly, the time derivative of $(\partial \mathbf{r}/\partial q_i)$ and $(\partial \mathbf{\Psi}/\partial q_i)$ is

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right) = \frac{\partial^2 \mathbf{r}}{\partial t \partial q_i} + \sum_{j=1}^{6} \frac{\partial^2 \mathbf{r}}{\partial q_i \partial q_j} \dot{q}_j, \qquad \frac{d}{dt}\left(\frac{\partial \mathbf{\Psi}}{\partial q_i}\right) = \frac{\partial^2 \mathbf{\Psi}}{\partial t \partial q_i} + \sum_{j=1}^{6} \frac{\partial^2 \mathbf{\Psi}}{\partial q_i \partial q_j} \dot{q}_j. \tag{241}$$

The left sides of the Eqs. (240) and (241) are equal, and consequently,

$$\frac{\partial \mathbf{v}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathbf{r}}{\partial q_i}\right), \qquad \frac{\partial \mathbf{\Omega}}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial \mathbf{\Psi}}{\partial q_i}\right). \tag{242}$$

Applying (239) and (242), the generalized inertial force (236) is

$$Z_i = -\frac{d}{dt}[(M\mathbf{v})\frac{\partial \mathbf{v}}{\partial \dot{q}_i} + (\mathbf{I\Omega})\frac{\partial \mathbf{\Omega}}{\partial \dot{q}_i}] + [(M\mathbf{v})\frac{\partial \mathbf{v}}{\partial q_i} + (\mathbf{I\Omega})\frac{\partial \mathbf{\Omega}}{\partial q_i}]. \tag{243}$$

After some modification, Eq. (243) transforms into

$$Z_i = -\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} + \frac{\partial T}{\partial q_i}, \tag{244}$$

where T is the kinetic energy

$$T = \frac{1}{2}M\mathbf{v}\mathbf{v} + \frac{1}{2}\mathbf{I\Omega\Omega}. \tag{245}$$

Substituting (244) into (234) the general equation of dynamics for mass variation is

$$\sum_{i=1}^{6}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} - \left(Q_i + Q_i^{\phi a} + Q_i^{Ra} + Q_i^{M\phi}\right)\right]\delta q_i = 0 \tag{246}$$

Since the coordinates q_i are independent so are the variations δq_i and therefore condition (246) implies

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i + Q_i^{\phi a} + Q_i^{Ra} + Q_i^{M\phi}, \qquad i = 1, 2, ..., 6. \tag{247}$$

The Lagrange's equations of motion of the variable mass body is given by Bessonov, , by using the method of solidification. Bessonov's equations are not general, as he assumed that the absolute angular velocity of the added body is zero and the generalized force Q_i^{Ra} is omitted.

7 Vibration of the body with continual mass variation

As the special type of motion in this Chapter the oscillation of the body with time variable mass is considered. The motion is bounded, periodical and with monotone change of the direction of motion around the equilibrium position or it is added to the steady state motion of the body. In this Chapter the one and also two degrees of freedom oscillators with variable mass are investigated. The new procedure for solving the differential equation of vibration is developed. The obtained results are applied for analyzing of the motion of the Van der Pol oscillator with variable mass but also of the rotor, as the one-mass oscillator with two degrees of freedom, and the two-mass system with two-degrees of freedom and mass variation.

7.1 One-degree-of-freedom oscillator with strong nonlinearity

Based on the Eq. (161) with (162) and introducing the generalized coordinate x, the mathematical model of the one-degree-of-freedom oscillator with time variable mass is

$$M\ddot{x} = F_x + \frac{dM}{dt}(u_x - \dot{x}), \tag{248}$$

where F_x is the resultant force and u_x is the absolute velocity of the adding or separating particle in x direction. In general, the resultant force is a function of the displacement x, velocity $\dot{x}$ and time t

$$M\ddot{x} = F_x(x, \dot{x}, t) + \frac{dM}{dt}(u_x - \dot{x}). \tag{249}$$

If an elastic force of odd parity acts, i.e.,

$$F_e(-x) = F_e(-x), \tag{250}$$

the differential equation (249) is as follows

$$M\ddot{x} + F_e(x) = F_x(x, \dot{x}, t) + \frac{dM}{dt}(u_x - \dot{x}). \tag{251}$$

Eq. (251) describes the vibration of the time variable one-degree-of-freedom system.

Let us consider the oscillator where:

1. Mass variation is slow and depends on the 'slow time' $\tau = \varepsilon t$ where $\varepsilon << 1$ is a small parameter

$$M = m(\tau). \tag{252}$$

2. The elastic force depends on the nonlinear deflection $x\,|x|^{\alpha-1}$, where the order of nonlinearity $\alpha \in \mathbb{R}_+$ is the positive rational number written as a termination decimal or as an exact fraction, $\alpha \in \mathbb{Q}_+ = \left\{\frac{m}{n} > 0 : m \in \mathbb{Z}, n \in \mathbb{Z}, n \neq 0\right\}$ and $\mathbb{Z}$ is integer.

3. The absolute velocity of the adding or separating mass is zero, i.e., $u_x = 0$.

4. The additional force which acts on the oscillator is small and is the function of the deflection x and velocity $\dot{x} : F_x = \varepsilon f_1(x, \dot{x})$.

The mathematical model for such an oscillator is

$$m(\tau)\ddot{x} + k_\alpha x\,|x|^{\alpha-1} = -\varepsilon\frac{dm(\tau)}{d\tau}\dot{x} + \varepsilon f_1(x, \dot{x}), \tag{253}$$

i.e.

$$\ddot{x} + \omega^2(\tau)x\,|x|^{\alpha-1} = \varepsilon f(\tau, x, \dot{x}), \tag{254}$$

where

$$\varepsilon f(\tau, x, \dot{x}) = -\frac{\varepsilon}{m(\tau)}\frac{dm(\tau)}{d\tau}\dot{x} + \frac{\varepsilon}{m(\tau)}f_1(x, \dot{x}). \tag{255}$$

and

$$\omega(\tau) = \sqrt{\frac{k_\alpha}{m(\tau)}}. \tag{256}$$

7.2 Criteria for the generating solution

For $\varepsilon = 0$ the generating equation of (254) is

$$\ddot{x} + \omega_0^2 x\,|x|^{\alpha-1} = 0, \tag{257}$$

with initial conditions

$$x(0) = x_0, \qquad \dot{x}(0) = 0, \tag{258}$$

where $\omega_0^2 \equiv \omega^2(0)$ =const.Integrating (257) and using the initial conditions, the first integral of the energy type is obtained (Cveticanin, 2008)

$$\frac{\dot{x}^2}{2} + \frac{\omega_0^2}{\alpha+1}\,|x|^{\alpha+1} = \frac{\omega_0^2}{\alpha+1}\,|x_0|^{\alpha+1}. \tag{259}$$

Being both addends on the left nonnegative, the associated phase paths represent a generalized Lamé superellipse in the x — $\dot{x}$ phase plane. There is a single equilibrium point $x = \dot{x} = 0$ such that is a centre, therefore the solutions of (259) are periodic in time (see for example Cveticanin, 2009_2; Mickens, 2010; Cveticanin and Pogány, 2012).

For the new variable $|x| = |x_0|\,|s|^{1/(\alpha+1)}$ the period of vibration is

$$T_{ex} = \frac{4\,|x_0|^{(1-\alpha)/2}}{\omega_0\sqrt{2(\alpha+1)}}\int_0^1 (1-|s|)^{-1/2}s^{-\alpha/(\alpha+1)}ds. \tag{260}$$

Introducing the Euler beta function $B(m,n)$ (see Rosenberg, 1963)

$$B(m,n) = \int_0^1 (1-|s|)^{n-1}s^{m-1}ds, \tag{261}$$

the relation (260) can be rewritten as follows

$$T_{ex} = \frac{4\,|x_0|^{(1-\alpha)/2}}{\omega_0\sqrt{2(\alpha+1)}}B(\frac{1}{\alpha+1}, \frac{1}{2}). \tag{262}$$

Due to

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}, \tag{263}$$

the exact period is

$$T_{ex} = \frac{4\,|x_0|^{(1-\alpha)/2}}{\omega_0\sqrt{2(\alpha+1)}}\frac{\Gamma(\frac{1}{\alpha+1})\Gamma(\frac{1}{2})}{\Gamma(\frac{3+\alpha}{2(\alpha+1)})}, \tag{264}$$

where Γ is the Euler gamma function (Abramowitz and Stegun, 1979). Using $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ the period expression is finally

$$T_{ex} = \frac{1}{c_1 \left|x_0\right|^{(\alpha-1)/2}} \left(\frac{2\sqrt{2}\pi}{\sqrt{(\alpha+1)}} \right) \left(\frac{\Gamma(\frac{1}{\alpha+1})}{\sqrt{\pi}\Gamma(\frac{3+\alpha}{2(\alpha+1)})} \right). \tag{265}$$

Using the property of the conservative oscillator (257), for the condition $\dot{x} = 0$, according to the relation (259), the extremal amplitudes during a period of vibration are obtained as $x_{\max} = x_0$ and $\mathrm{x}_{\min} = -x_0$.

We suggest to form the approximate solution of (257) with the exact amplitude $x_{\max}$ and period of vibration T_{ex}, which is previously determined with (265). The approximate solution with the exact amplitude and frequency describes the motion which is equal for the first, second and also n-th period of vibration. Using the exact period and amplitude of vibration, the approximate solution would satisfy the following relation

$$x(0) = x(T_{ex}) = ... = x(nT_{ex}) = x_0. \tag{266}$$

It yields, that the approximate solution satisfies not only the initial conditions (258) but also the requirements of the exact amplitude and frequency and the relation (266).

Based on the previous investigation it can be concluded that the approximate solution with the exact amplitude and period of vibration has the following advantages:

- the initial values need not to be small
- there is no error accumulation due to period difference ($\Delta T = 0$) and the solution is valid in the whole time range from zero to infinity.

Remark: We must underline that in the suggested procedure for obtaining of the approximate solution there are no series expansions, averaging or simplifications of the differential equation.

The number of periodic functions which satisfy the aforementioned requirements is extremely large. We suggest an additional criterion for the choice of the approximate solution which would be based on the first time derivative of the approximate function. Namely, it is required that the maximal value of the first time derivative of the approximate solution $\dot{x}_{\max}$ be equal or close to the exact velocity of vibration. Due to the Eq. (259), it is evident that the maximal velocity of vibration exists for $x_V = 0$ and has the value

$$v_{\max} = \sqrt{\frac{2\omega_0^2}{\alpha+1} \left|x_0\right|^{\alpha+1}}. \tag{267}$$

Using (267), the suggested criterion is mathematically expressed as

$$\dot{x}_{\max}(x_V = 0) \simeq v_{\max}(x_V = 0). \tag{268}$$

Such a criterion will be useful for the choice of the most appropriate approximate solution of (257).

7.3 Types of generating solutions

In this section three periodical functions which satisfy the initial conditions (258) and the requirement of the maximal amplitude x_0, i.e., the Eq. (266), will be assumed as the approximate generating solutions of (257): the cosines Ateb function (Rosenberg, 1963), the cosines trigonometric function and the cn Jacobi elliptic function (Byrd and Friedman, 1954). It will be investigated whether these functions satisfy the requirements (266) and (268) which are the criteria for the approximate solution of (257).

7.3.1 Exact analytical solution

Let us rewrite the first integral (259) as follows

$$\dot{x} = \pm K\sqrt{1-\Big(\frac{|x|}{x_0}\Big)^{\alpha+1}}, \qquad K = \frac{\sqrt{2}\,|\omega_0| x_0^{(\alpha+1)/2}}{\sqrt{\alpha+1}}\,, \tag{269}$$

i.e.

$$\frac{\dot{x}}{\sqrt{1-\Big(\frac{|x|}{x_0}\Big)^{\alpha+1}}} = \pm K. \tag{270}$$

Now, we are ready to solve (257) i.e. (270) analytically. Let us choose the positive right–hand–side expression in (270). It will be

$$L = \int \frac{\mathrm{d}x}{\sqrt{1-\Big(\frac{|x|}{x_0}\Big)^{\alpha+1}}} = Kt + C\,, \tag{271}$$

where C denotes the integration constant. Expanding the integrand in L into a binomial series and then integrating termwise, we conclude

$$L = \int \sum_{n=0}^\infty (-1)^n \binom{-\frac12}{n} \Big(\frac{|x|}{x_0}\Big)^{(\alpha+1)n} \mathrm{d}x = x_0 \sum_{n=0}^\infty \binom{-\frac12}{n} \frac{1}{(\alpha+1)n+1} \Big(\frac{|x|}{x_0}\Big)^{(\alpha+1)n+1}.$$

Employing the Pochhammer symbol $(a)_n = a(a+1)\cdots(a+n-1), n \in \mathbb{N}$ *mutatis mutandis*

$$\binom{-\frac12}{n} = (-1)^n \frac{\big(\frac12\big)_n}{n!}$$

and

$$\frac{1}{(\alpha+1)n+1} = \frac{\frac{1}{\alpha+1}}{n+\frac{1}{\alpha+1}} = \frac{\Big(\frac{1}{\alpha+1}\Big)_n}{\Big(\frac{\alpha+2}{\alpha+1}\Big)_n},$$

hence

$$L = |x| \sum_{n=0}^\infty \frac{\big(\frac12\big)_n \Big(\frac{1}{\alpha+1}\Big)_n}{\Big(\frac{\alpha+2}{\alpha+1}\Big)_n\, n!} \Big(\frac{|x|}{x_0}\Big)^{(\alpha+1)n} = |x|\, {}_2F_1\Big[\begin{matrix} \frac12, \frac{1}{\alpha+1} \\ \frac{\alpha+2}{\alpha+1} \end{matrix} \Big| \Big(\frac{|x|}{x_0}\Big)^{\alpha+1}\Big]. \tag{272}$$

Now, by the well–known formula (see http, 2012_1 and 2012_2) such that connects the Gaussian hypergeometric ${}_2F_1$ and the incomplete Beta function B_s:

$${}_2F_1\Big[\begin{matrix} a,\ 1-b \\ a+1 \end{matrix} \Big| s\Big] = \frac{a}{s^a}\, \mathrm{B}_s(a,b), \qquad |s|<1,$$

letting

$$a = \frac{1}{\alpha+1}, \qquad b = \frac12\,,$$

we get

$$\mathrm{B}_{\big(\frac{|x|}{x_0}\big)^{\alpha+1}}\Big(\frac{1}{\alpha+1}, \frac12\Big) = \sqrt{2(\alpha+1)}\,|\omega_0|\, x_0^{(\alpha-1)/2}\, t + C\,. \tag{273}$$

Finally, the initial condition $x(0) = x_0$ clearly gives

$$\frac{(\alpha+1)|x|}{x_0}\,{}_2F_1\Big[\begin{array}{c}\frac{1}{\alpha+1},\frac{1}{2}\\ 1+\frac{1}{\alpha+1}\end{array}\Big|\Big(\frac{|x|}{x_0}\Big)^{\alpha+1}\Big] = \mathrm{B}_{\left(\frac{|x|}{x_0}\right)^{\alpha+1}}\Big(\frac{1}{\alpha+1},\frac{1}{2}\Big)$$
$$= \mathrm{B}\Big(\frac{1}{\alpha+1},\frac{1}{2}\Big) + \sqrt{2(\alpha+1)}\,|\omega_0|\,x_0^{(\alpha-1)/2}\,t\,. \tag{274}$$

This relation will be our main tool in determining the explicit solution of Cauchy problem (257) with (258).

In his classical paper Rosenberg, 1963, introduced the so–called periodic Ateb–functions concerning the problem of inversion of the half of the incomplete Beta–function

$$s \mapsto \frac{1}{2}\,\mathrm{B}_s(a,b) = \frac{1}{2}\int_0^{0\le s\le 1} s^{a-1}(1-s)^{b-1}\mathrm{d}s.$$

Obviously, we are interested in the case $a = \frac{1}{\alpha+1}, b = \frac{1}{2}$ [1]. The Ateb (kind of inverse Beta) functions in the focus of our interest are solutions of the system of ordinar differential equations

$$\begin{aligned}\dot v - u^\alpha &= 0\\ \dot u + \frac{2}{\alpha+1}\,v &= 0\,,\end{aligned} \tag{275}$$

(consult e.g. Senik, 1969). We write

$$v(s) = \mathrm{sa}(1,\alpha,s), \quad u(s) = \mathrm{ca}(\alpha,1,s)\,. \tag{276}$$

It can be easily verified that the inverse of $\frac{1}{2}\,\mathrm{B}_s(\frac{1}{2},\frac{1}{\alpha+1})$ and $v(s)$ coincide on $[-\frac{1}{2}\Pi_\alpha, \frac{1}{2}\Pi_\alpha]$, where

$$\Pi_\alpha := \mathrm{B}\left(\frac{1}{\alpha+1},\frac{1}{2}\right)\,. \tag{277}$$

Having in mind the following set of properties:

$$\mathrm{sa}(1,\alpha,z) = \begin{cases}-\mathrm{sa}(1,\alpha,-s)\\ \mp\mathrm{ca}\left(\alpha,1,\frac{1}{2}\Pi_\alpha\pm s\right)\\ \pm\mathrm{sa}(1,\alpha,\Pi_\alpha\pm s)\\ \mp\mathrm{sa}(1,\alpha,2\Pi_\alpha\mp s)\end{cases}, \tag{278}$$

we see that $\mathrm{sa}(\alpha,1,s)$ is an odd function of $s\in\mathbb{R}$; it is the so–called $2\Pi_\alpha$ – periodic sine Ateb, i.e. sa – function. Also there holds

$$\mathrm{sa}^2(1,\alpha,s) + \mathrm{ca}^{\alpha+1}(\alpha,1,s) = 1\,, \tag{279}$$

and cosine Ateb [2], that is $\mathrm{ca}(1,\alpha,s)$ function, is even and $2\Pi_\alpha$ – periodic having properties:

$$\mathrm{ca}(\alpha,1,s) = \begin{cases}\mathrm{ca}(\alpha,1,-s)\\ \mathrm{sa}\left(1,\alpha,\frac{1}{2}\Pi_\alpha\pm s\right)\\ -\mathrm{ca}(\alpha,1,\Pi_\alpha\pm s)\\ \mathrm{ca}(\alpha,1,2\Pi_\alpha\pm s)\end{cases}. \tag{280}$$

[1] Senik, 1969, constructed inverses for the incomplete Beta–function $\mathrm{B}_s(a,b)$ for

$$a = \frac{1}{n+1}, \qquad b = \frac{1}{m+1}, \qquad n,m = \frac{2\mu+1}{2\nu+1}, \quad \mu,\nu\in\mathbb{N}_0\,.$$

[2] For $\alpha = 1$ sa, ca one reduce to the sine and cosine functions. That means $\mathrm{sa}(1,1,s) = \sin s$, $\mathrm{ca}(1,1,s) = \cos s$.

By these two sets of relations we see that functions sa, ca are defined on the whole range of $\mathbb{R}$.

Now, inverting the half of the incomplete Beta–function in (274):

$$\frac{1}{2}\,\mathrm{B}_{\left(\frac{|x|}{x_0}\right)^{\alpha+1}}\left(\frac{1}{\alpha+1},\frac{1}{2}\right) = \frac{\Pi_\alpha}{2} + \frac{\sqrt{\alpha+1}\,|\omega_0|}{\sqrt{2}}\, x_0^{(\alpha-1)/2}\, t\,,$$

we clearly deduce

$$x(t) = x_0 \cdot \mathrm{sa}\left(1, \alpha, \frac{\Pi_\alpha}{2} + \frac{\sqrt{\alpha+1}\,|\omega_0|}{\sqrt{2}}\, x_0^{(\alpha-1)/2}\, t\right)\,. \tag{281}$$

Having in mind the quarter–period expansion formula (280) we arrive at

$$x(t) = x_0 \cdot \mathrm{ca}\left(\alpha, 1, \frac{\sqrt{\alpha+1}\,|\omega_0|}{\sqrt{2}}\, x_0^{(\alpha-1)/2}\, t\right)\,, \qquad t \in \mathbb{R}\,. \tag{282}$$

By $\mathrm{ca}(\alpha, 1, 0) = 1$, we see that the initial condition $x(0) = x_0$ is satisfied as well.

The obtained result (282) is the exact analytical solution of the Eq. (257) for the initial conditions (258) and is widely discussed in the paper Pogány and Cveticanin, 2012.

Recalling that (see Senik, 1969)

$$\frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{ca}(\alpha, 1, s) = -\frac{2}{\alpha+1}\,\mathrm{sa}(1, \alpha, s), \qquad \frac{\mathrm{d}}{\mathrm{d}s}\,\mathrm{sa}(1, \alpha, s) = \mathrm{ca}^\alpha(\alpha, 1, s)\,,$$

the first time derivative of the solution (282) is

$$\dot{x} = -\frac{\sqrt{2\omega_0^2}}{\sqrt{\alpha+1}}\, x_0^{(\alpha+1)/2}\, \mathrm{sa}(1, \alpha, \frac{|\omega_0|\sqrt{\alpha+1}}{\sqrt{2}}\, x_0^{(\alpha-1)/2}\, t)\,, \tag{283}$$

with the maximal value of the first derivative

$$\dot{x}_{\max} = \sqrt{\frac{2\omega_0^2}{\alpha+1}\,|x_0|^{\alpha+1}}. \tag{284}$$

The function (282) and its time derivative (283) with its maximal value (284) satisfy the requirements (258), (266) and (268) for the maximal velocity (267).

Let us consider some special cases:

1) For the linear case, when $\alpha = 1$, Eq. 257 reduces the autonomous equation

$$\ddot{x} + \omega_0^2 x = 0\,.$$

By virtue of the initial condition $x(0) = x_0$ formula (274) reduces to

$$|x|\, {}_2F_1\left[\begin{matrix} \frac{1}{2}, \frac{1}{2} \\ \frac{3}{2} \end{matrix} \,\Big|\, \left(\frac{x}{x_0}\right)^2\right] = x_0 \arcsin\left(\frac{x}{x_0}\right) = x_0\left(\frac{\pi}{2} + |\omega_0| t\right)\,,$$

therefore

$$x(t) = \pm x_0 \sin\left(\frac{\pi}{2} + |\omega_0| t\right) = x_0\, \cos\left(|\omega_0| t\right). \tag{285}$$

The first time derivative of the approximate solution (285) is

$$\dot{x} = -x_0\omega_0 \sin\left(|\omega_0| t\right),$$

with maximal amplitude

$$\dot{x}_{\max} = x_0\omega_0. \tag{286}$$

Comparing (286) with (267) for $\alpha = 1$, the relation (268) is satisfied.

2) Let us determine the exact analytical solution for the differential equation of Duffing type with the pure cubic nonlinearity ($\alpha = 3$)

$$\ddot{x} + \omega_0^2 x^3 = 0.$$

Let us recall (see http, $2012_3, 2012_4$) that

$$ {}_2F_1\left[\begin{matrix}\frac14, \frac12\\ \frac54\end{matrix}\,\Big|\, s\right] = \frac{1}{\sqrt[4]{s}}\, F(\arcsin\sqrt[4]{s}\,|-1) = \frac{1}{\sqrt[4]{s}}\,\mathrm{sn}^{-1}(s|-1), \qquad |s| < 1\,, \tag{287}$$

where

$$F(s|k^2) = \int_0^{s=\mathrm{am}(s|m)} \frac{\mathrm{d}s}{\sqrt{1-k^2\sin^2 s}}, \qquad \mathrm{sn}(s|k^2) = \sin\mathrm{am}(s|k^2)\,,$$

and $F, \mathrm{am}, \mathrm{sn}$ denote the incomplete elliptic integral of the first kind, the Jacobi amplitude, the Jacobi elliptic sn function, respectively, k^2 is the modulus of the elliptic function (Byrd, 1954) and sn^{-1} stands for the inverse Jacobi elliptic sn function. Thus, for $\alpha = 3$ we have

$$|x|\, {}_2F_1\left[\begin{matrix}\frac14, \frac12\\ \frac54\end{matrix}\,\Big|\left(\frac{x}{x_0}\right)^4\right] = x_0\cdot\mathrm{sn}^{-1}\Big(\frac{|x|}{x_0}\Big| -1\Big) = C + \frac{|\omega_0|x_0^2}{\sqrt2}\,t. \tag{288}$$

Since the initial condition $x(0) = x_0$, we conclude

$$C = x_0\cdot\mathrm{sn}^{-1}(1|-1) = A\cdot K(-1)\,,$$

where $K(k^2) = F\big(\frac{\pi}{2}\big|k^2\big)$ denotes the celebrated complete elliptic integral of the first kind. The Jacobi elliptic $\mathrm{sn}(s|k^2)$ has period $\omega = 4K(k^2)$, for now

$$4K(-1) = \mathrm{B}\Big(\frac14, \frac12\Big) = \frac{\Gamma^2\big(\frac14\big)}{\sqrt{2\pi}} \approx 5.244116\,,$$

is the period of the function $\mathrm{sn}(s|-1)$. Employing the quarter–period transformation formula for the Jacobi amplitude

$$\mathrm{am}(K(k^2) - s|k^2) = \frac{\pi}{2} - \mathrm{am}\Big(\sqrt{1-k^2}s\Big|\frac{k^2}{k^2-1}\Big), \qquad k^2 \le 1$$

for $k^2 = -1$, one deduces by (288) that

$$\begin{aligned} x(t) &= x_0\cdot\mathrm{sn}\left(K(-1) + \frac{|\omega_0|x_0}{\sqrt2}\,t\Big| -1\right) = x_0\cdot\sin\mathrm{am}\left(K(-1) + \frac{|\omega_0|x_0}{\sqrt2}\,t\Big| -1\right) \\ &= x_0\cdot\sin\left\{\frac{\pi}{2} - \mathrm{am}\left(K(-1) + \frac{|\omega_0|x_0}{\sqrt2}\,t\Big| -1\right)\right\} = x_0\cdot\cos\mathrm{am}\left(x_0|\omega_0|\,t\Big|\frac12\right)\,. \end{aligned}$$

Thus

$$x(t) = x_0\cdot\mathrm{cn}\left(x_0|\omega_0|\,t\Big|\frac12\right)\,. \tag{289}$$

Here $\mathrm{cn}(s|k^2) = \cos\mathrm{am}(s|k^2)$ denotes the Jacobi elliptic cn function[3]. Being $\mathrm{cn}(0|\frac12) = 1$, formula (289) has the interpolatory property $x(0) = x_0$. It is worth to say that Lyapunov in his classical paper in 1893, introduced the Jacobi elliptic functions (cn

[3] The elliptic functions sn, cn were defined by Jacobi under the names sin am, cos am; the notation sn, cn was introduced by Gudermann and Glashier (see Du Val, 1973).

and sn) which are the special case of Ateb cosinus and Ateb sinus functions for $\alpha = 3$. The same functions are used for solving the third order nonlinear differential equation of Duffing type by Yuste and Bejarano, 1990, but also Chen and Cheung, 1996.

The first time derivative of (289) is

$$\dot{x} = -x_0^2|\omega_0| \cdot \mathrm{sn}\left(x_0|\omega_0|\, t\Big|\frac{1}{2}\right) \mathrm{dn}\left(x_0|\omega_0|\, t\Big|\frac{1}{2}\right),$$

with maximal amplitude (see Byrd and Friedman, 1954)

$$x_{\max} = \frac{\sqrt{2}}{2} x_0^2 |\omega_0|. \tag{290}$$

Comparing (290) with the maximal velocity (267) for $\alpha = 3$, it is obvious that the relation (268) is satisfied.

In spite of the fact that (282) is the exact solution of the generating differential equation (257), due to its complexity, the Ateb function is not convenient for practical application.

7.3.2 Generating solution in the form of the trigonometric function

Let us assume the approximate solution in the form of a trigonometric function

$$x_a = x_0 \cos(\Omega t), \tag{291}$$

where for the exact period of vibration (264) the frequency of vibration is

$$\Omega = \frac{2\pi}{T_{ex}} = \Omega_\alpha x_0^{(\alpha-1)/2} \sqrt{\omega_0^2}, \tag{292}$$

with

$$\Omega_\alpha = \sqrt{\frac{\alpha+1}{2}} \frac{\sqrt{\pi}\Gamma(\frac{3+\alpha}{2(\alpha+1)})}{\Gamma(\frac{1}{\alpha+1})}. \tag{293}$$

Substituting the assumed generating solution (291) and its first and second time derivatives into (257), it follows

$$\frac{d^2}{dt^2}(A\cos\psi) + \omega_0^2 (A\cos\psi)^{\frac{\alpha-1}{2}} \approx 0. \tag{294}$$

For the first time derivative of (291)

$$\dot{x}_a = -x_0 \Omega \sin(\Omega t), \tag{295}$$

the maximal value is

$$\dot{x}_{a\max} = \Omega_\alpha x_0^{(\alpha+1)/2} \sqrt{\omega_0^2} = \sqrt{\frac{2\omega_0^2}{\alpha+1} |x_0|^{\alpha+1}} \left[\frac{\sqrt{\pi}}{2}\right] \frac{(\alpha+1)\Gamma(\frac{3+\alpha}{2(\alpha+1)})}{\Gamma(\frac{1}{\alpha+1})}. \tag{296}$$

It is evident that the obtained maximal velocity (296) differs from the exact maximal velocity (267). Comparing these two values it is evident that the difference depends on the order of nonlinearity and is zero only for the linear case, i.e., $\alpha = 1$. The trigonometric function is the most widely used one for describing the oscillatory motion.

7.3.3 Generating solution in the form of the Jacobi elliptic function

Let us assume another approximate solution of the same problem in the form of the Jacobi elliptic function (Byrd, 1954)

$$x_p = x_0 \mathrm{cn}(\Omega_1 t, k^2). \tag{297}$$

where Ω_1 is the frequency and k^2 is the modulus of the Jacobi elliptic function. The suggested approximate solution satisfies the given initial conditions and the requirement of the exact amplitude.

As it is known (see Gradstein and Rjizhik, 1971) the relation between the frequency and modulus of the Jacobi elliptic function, and the vibration period and frequency of vibration, is as follows

$$\Omega_1 = \frac{4K(k^2)}{T} = \frac{4K(k^2)}{2\pi}\Omega, \tag{298}$$

where $K(k^2)$ is the complete elliptic integral of the first kind (Byrd, 1954). Substituting the exact vibration frequency (292) into the relation (298), we obtain the frequency of the Jacobi elliptic function

$$\Omega_1 = \Omega_1^* x_0^{(\alpha-1)/2}\sqrt{\omega_0^2}. \tag{299}$$

where

$$\Omega_1^* = \frac{2K(k^2)}{\pi}\Omega_\alpha. \tag{300}$$

It is evident that the frequency of the Jacobi elliptic function depends on the modulus k^2 which has to be determined.

The modulus k^2 will be calculated from the condition of the maximal vibration velocity (267) for $x_V = 0$. Namely, using the condition $x_V = 0$ and the relation (297), it is obtained that the maximal vibration velocity is for

$$t_V = \frac{(2n-1)K(k^2)}{\Omega_1}, \tag{301}$$

where $n = 1, 2, 3, ...$The first time derivative of the approximate solution (297) is

$$\dot{x}_p = -x_0\Omega_1 \mathrm{sn}(\Omega_1 t, k^2)\mathrm{dn}(\Omega_1 t, k^2). \tag{302}$$

Substituting (301) into (302) it is

$$\dot{x}(t_V) = -x_0\Omega_1 \mathrm{sn}((2n-1)K(k^2), k^2)\mathrm{dn}((2n-1)K(k^2), k^2), \tag{303}$$

where

$$\mathrm{sn}((2n-1)K(k^2), k^2) = \pm 1, \qquad \mathrm{dn}((2n-1)K(k^2), k^2) = k' = \sqrt{1-k^2}, \tag{304}$$

i.e.,

$$\mathrm{sn}((2n-1)K(k^2), k^2)\mathrm{dn}((2n-1)K(k^2), k^2) = \pm k', \tag{305}$$

and k' is the complementary modulus of the Jacobi elliptic function. Due to (303), (305) and (299) with (293), the maximal value of the first time derivative is

$$\dot{x}_{p\,\max} \equiv \dot{x}(t_V) = x_0\Omega_1 k' = \frac{2K(k^2)k'}{\pi}\Omega_\alpha x_0^{(\alpha+1)/2}\sqrt{\omega_0^2}. \tag{306}$$

Equating the relation (306) with the maximal velocity of vibration (267) and using the criterion (189), it follows

$$k'K(k^2) = \frac{\sqrt{\pi}}{(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})}{\Gamma(\frac{3+\alpha}{2(\alpha+1)})}. \tag{307}$$

The solution of the Eq. (307) gives the value of the modulus of the Jacobi elliptic function k^2. For that certain value of k^2, the frequency of the Jacobi elliptic function (299) is calculated. Substituting the corresponding value of k^2 and (299) into (297) and (302), the approximate solution of (257) and its first time derivative are obtained.

Analyzing the relation (307) it is obvious that the modulus of the Jacobi elliptic function depends only on the order of the nonlinearity and is $k^2(\alpha)$. Solving the nonlinear algebraic equation (307) for various values of α, the modulus k^2 is calculated and given in Table 5. In the Table 5, also the constant of the frequency of vibration Ω_α (293) and the constant of the frequency of the Jacobi elliptic function Ω_1^* (300) for certain value of α are given. Knowing the order of nonlinearity α and taking the correspondent modulus of the Jacobi elliptic function k^2 and the constants Ω_α and Ω_1^*, the frequency of vibration (292) and the frequency of the Jacobi elliptic function (299) are calculated.

Table 5. The constant Ω_α of the frequency of vibration, the constant of the frequency of the Jacobi elliptic function Ω_1^* and the modulus of the Jacobi elliptic function k^2 as the function of the order of nonlinearity α.

α	Ω_α	k^2	Ω_1^*
0.0	1.110720	-1.807855	0.843971
0.1	1.098258	-1.381162	0.873824
0.2	1.086125	-1.062291	0.898978
0.3	1.074316	-0.816762	0.920225
0.4	1.062822	-0.622992	0.938194
0.5	1.051636	-0.466883	0.953392
0.6	1.040748	-0.338896	0.966232
0.7	1.030150	-0.232378	0.977054
0.8	1.019832	-0.142567	0.986138
0.9	1.009785	-0.065976	0.993721
1.0	1.000000	0.000000	1.000000
1.1	0.990467	0.057340	1.005143
1.2	0.981179	0.107572	1.009292
1.3	0.972126	0.151894	1.012572
1.4	0.963300	0.191252	1.015083
1.5	0.954694	0.226408	1.016925
1.6	0.946298	0.257978	1.018169
1.7	0.938106	0.286466	1.018886
1.8	0.930110	0.312288	1.019137
1.9	0.922304	0.335789	1.018974
2.0	0.914681	0.357261	1.018444
3.0	0.847213	0.500000	1.000000
4.0	0.792402	0.575240	0.970417
5.0	0.746834	0.621298	0.938189
6.0	0.708220	0.652256	0.906434
7.0	0.674977	0.674439	0.876303
8.0	0.645978	0.691091	0.848161

α	Ω_α	k^2	Ω_1^*
9.0	0.620397	0.704038	0.822048
10	0.597617	0.714386	0.797865
20	0.455377	0.760706	0.630870
30	0.382158	0.775993	0.536665
40	0.335713	0.783594	0.474776
50	0.302909	0.788140	0.430235
60	0.278155	0.791164	0.396230
70	0.258622	0.793320	0.369179
80	0.242698	0.794935	0.346998
90	0.229395	0.796191	0.328384
100	0.218063	0.797194	0.312474

Besides, the in-built subroutine `JacobiCN[`s,k^2`]` in *Mathematica* enables the exact curve plotting of the Jacobi elliptic functions (see Du Val, 1973).

Considering the relation (307) the following is concluded:

1) For the linear oscillator, when $\alpha = 1$, the right hand side of the equation (307) is $\pi/2$ and according to (307) the corresponding value of the modulus of the Jacobi elliptic function is $k^2 = 0$. Then, the cn Jacobi elliptic function transforms into the cosines trigonometric function.

2) For $0 < \alpha < 1$ the modulus of the Jacobi elliptic function is according to (307) negative. For $k^2 < 0$, the cn Jacobi elliptic function transforms into cd $\equiv$cn/dn Jacobi elliptic function (see Gradstein and Rjizhik, 1971)

$$\mathrm{cn}(\Omega_1 t, k^2) = \mathrm{cd}(\Omega_1 t\sqrt{1+|k^2|}, \frac{|k^2|}{1+|k^2|}). \quad (308)$$

3) If α tends to zero the right hand side of the equation tends to 2, and $k^2 = -1.807854731$. Using the transformation (308) the corresponding modulus of the cd Jacobi elliptic function tends to $k^2 = 0.643\,86$.

4) Introducing the new parameter $s = 1/(\alpha+1)$ which tends to zero when α tends to infinity, the right hand side of the relation (307) is

$$\lim_{\alpha\to\infty} \frac{\sqrt{\pi}}{(\alpha+1)} \frac{\Gamma(\frac{1}{\alpha+1})}{\Gamma(\frac{3+\alpha}{2(\alpha+1)})} = \lim_{s\to 0} \sqrt{\pi} \frac{s\Gamma(s)}{\Gamma(s+\frac{1}{2})} = \lim_{s\to 0} \sqrt{\pi} \frac{\Gamma(s+1)}{\Gamma(s+\frac{1}{2})} = \sqrt{\pi}\frac{\Gamma(1)}{\Gamma(\frac{1}{2})} = 1. \quad (309)$$

Then, the value of the modulus of the Jacobi elliptic function tends to a constant value: $k^2 = 0.806192642$.

5) For $\alpha = 3$ the modulus of the Jacobi elliptic function is $k^2 = 1/2$ as it is already shown in (289).

Analyzing the data in the Table 1 it is evident:

1) For the linear case, when $\alpha = 1$, the both constants: Ω_α and Ω_1^* are equal to 1.

2) The value of Ω_α decreases from 1 by increasing of the order of nonlinearity α, and increases with decreasing of the order of nonlinearity α.

Remark 6 *Comparing the approximate solutions (297) and (291) with the exact solution (282) and also the maximal values of the first derivatives (306) and (296) with the exact vibration velocity (267), it can be concluded that the relation (297) is more appropriate than (291) to be considered as the approximate solution of (257).*

7.4 Approximate trial solutions

Using the procedure published in Cveticanin, 2012, the approximate trial solutions of (253) are introduced.

7.4.1 Solution with Ateb function

Having in mind the solution (282) and its first time derivative (283), the trial solution of the differential equation (254) and its first time derivative are assumed in the same form, but with time variable parameters, i.e.,

$$x = A(t) \cdot \mathrm{ca}\,(\alpha, 1, \psi(t))\,, \tag{310}$$

and

$$\dot{x} = -\frac{\sqrt{2}\,\omega(\tau)}{\sqrt{\alpha+1}}\,[A(t)]^{(\alpha+1)/2}\,\mathrm{sa}(1,\alpha,\psi(t))\,, \tag{311}$$

with

$$\dot{\psi}(t) := \omega(\tau)\frac{\sqrt{\alpha+1}}{\sqrt{2}}\,[A(t)]^{(\alpha-1)/2} + \dot{\theta}(t)\,,$$

and $A \equiv A(t), \psi \equiv \psi(t), \theta \equiv \theta(t)$ and $\omega \equiv \omega(\tau)$. Comparing the first time derivative of (310)

$$\dot{x} = -\frac{\sqrt{2}\,\omega(\tau)}{\sqrt{\alpha+1}}\,[A(t)]^{(\alpha+1)/2}\,\mathrm{sa}(1,\alpha,\psi(t)) + \dot{A}(t)\cdot\mathrm{ca}\,(\alpha,1,\psi(t)) - \frac{2A(t)\dot{\theta}(t)}{\alpha+1}\,\mathrm{sa}(1,\alpha,\psi(t))$$

with (311), it is concluded that they are equal if

$$\dot{A}\mathrm{ca}(\alpha,1,\psi) - \frac{2A\dot{\theta}}{\alpha+1}\,\mathrm{sa}(1,\alpha,\psi) = 0\,. \tag{312}$$

Substituting the second derivative of $\ddot{x}$ and the function x 310 into ODE 254, we get

$$\ddot{x} + \omega^2 x|x|^{\alpha-1} = -|\omega|\frac{\sqrt{\alpha+1}}{\sqrt{2}}\,A^{(\alpha-1)/2}\,\mathrm{sa}(1,\alpha,\psi)\cdot\dot{A} - \frac{|\omega|\sqrt{2}}{\sqrt{\alpha+1}}\,A^{(\alpha+1)/2}\,\mathrm{ca}^{\alpha}(\alpha,1,\psi)\cdot\dot{\theta}$$
$$= \varepsilon f(\tau, A\mathrm{ca}(\alpha,1,\psi), -\frac{\sqrt{2}\,\omega}{\sqrt{\alpha+1}}\,A^{(\alpha+1)/2}\,\mathrm{sa}(1,\alpha,\psi)) =: \varepsilon f\,,$$

such that becomes a first order differential equation

$$\dot{A}\mathrm{sa}(1,\alpha,\psi) + \frac{2\,A\dot{\theta}}{\alpha+1}\,\mathrm{ca}^{\alpha}(\alpha,1,\psi) = -\frac{\sqrt{2}}{\sqrt{\alpha+1}}\,\frac{\varepsilon f A^{(1-\alpha)/2}}{\omega}\,, \tag{313}$$

Coupled with (312) this system of ODEs in $\dot{A}, \dot{\theta}$ results in

$$\dot{A} = -\frac{\sqrt{2}\,A^{(1-\alpha)/2}}{\omega\sqrt{\alpha+1}}\,\varepsilon f\mathrm{sa}(1,\alpha,\psi), \tag{314}$$

$$A\dot{\theta} = -\frac{\sqrt{\alpha+1}\,A^{(1-\alpha)/2}}{\omega\sqrt{2}}\,\varepsilon f\mathrm{ca}(\alpha,1,\psi)\,. \tag{315}$$

The obtained first order differential equations (314) and (315) represent the rewritten version of the second order differential equation (254) in the new variables A and θ. To solve the differential equations (314) and (315) is not an easy task. Bearing on mind that any T–periodic ($f(t+T) = f(t)$) integrable function f possesses the property

$$\int_{\gamma}^{T+\gamma} f'(s) f^{\beta}(s)\mathrm{d}s = 0, \qquad \beta > 0,\, \gamma \in \mathbb{R},$$

by letting $f(s) = \mathrm{ca}(\alpha, 1, s)$, $T = 2\Pi_\alpha$, the averaging procedure immediately gives

$$\dot{A} = -\frac{A^{(1-\alpha)/2}}{\omega\Pi_\alpha\sqrt{2(\alpha+1)}} \int_0^{2\Pi_\alpha} \varepsilon f \mathrm{sa}(1,\alpha,\psi)\mathrm{d}\psi,$$

and

$$A\dot{\theta} = -\frac{\sqrt{\alpha+1}\,A^{(1-\alpha)/2}}{2\Pi_\alpha\omega\sqrt{2}} \int_0^{2\Pi_\alpha} \varepsilon f \mathrm{ca}(\alpha,1,\psi)\mathrm{d}\psi\,.$$

To find the value of the inner–most integral we apply a result by Drogomirecka, 1997, reads as follows

$$\int_0^{2\Pi_\alpha} \mathrm{sa}^p(n,m,\psi)\,\mathrm{ca}^q(m,n,\psi)\,\mathrm{d}\psi = \frac{1}{2}\big(1+(-1)^p+(-1)^q+(-1)^{p+q}\big)\,\mathrm{B}\left(\frac{p+1}{n+1},\frac{q+1}{m+1}\right), \tag{316}$$

for all $p, q \in \{\frac{r}{l} : r \in \mathbb{Z},\, l = 2k-1, k \in \mathbb{N}\}$, i.e. the powers are rational, having integer numerator and odd positive denominator.

Let us consider the vibrations of the oscillator with slow mass variation and the reactive force

$$\varepsilon f = -\frac{\varepsilon}{m(\tau)}\frac{dm(\tau)}{d\tau}\dot{x}. \tag{317}$$

The differential equation of motion (254) transforms into

$$\ddot{x} + \omega^2(\tau)x\,|x|^{\alpha-1} = -\frac{\varepsilon}{m(\tau)}\frac{dm(\tau)}{d\tau}\dot{x}, \tag{318}$$

with (256).

According to (314) and (315), the Eq. (318) is rewritten in the form

$$\dot{A} = -\frac{2A}{\alpha+1}\frac{\varepsilon}{m(\tau)}\frac{dm(\tau)}{d\tau}\,\mathrm{sa}^2(1,\alpha,\psi), \tag{319}$$

$$A\dot{\theta} = -\frac{\varepsilon A}{m(\tau)}\frac{dm(\tau)}{d\tau}\,\mathrm{sa}(1,\alpha,\psi)\mathrm{ca}(\alpha,1,\psi)\,. \tag{320}$$

Using the averaging procedure (316) the averaged differential equations are

$$\begin{aligned}\dot{A} &= -\frac{A}{\alpha+1}\frac{\varepsilon}{m(\tau)}\frac{1}{\Pi_\alpha}\frac{dm(\tau)}{d\tau}\int_0^{2\Pi_\alpha}\mathrm{sa}^2(1,\alpha,\psi)\,\mathrm{d}\psi\\ &= -\frac{2A}{\alpha+1}\frac{\varepsilon}{m(\tau)}\frac{1}{\Pi_\alpha}\frac{dm(\tau)}{d\tau})\,\mathrm{B}\left(\frac{3}{2},\frac{1}{\alpha+1}\right),\end{aligned} \tag{321}$$

$$A\dot{\theta} = -\frac{\varepsilon A}{m(\tau)}\frac{dm(\tau)}{d\tau}\frac{1}{2\Pi_\alpha}\int_0^{2\Pi_\alpha}\mathrm{sa}(1,\alpha,s)\,\mathrm{ca}(\alpha,1,\psi)\,\mathrm{d}\psi = 0. \tag{322}$$

Integrating the relation (322), it is $\theta = \theta(0)$ =const. The uncoupled differential equation (321) is rewritten as

$$\frac{dA}{A} = -P\frac{dm}{m}, \tag{323}$$

where the constant P with (277) for $\Pi_\alpha := \mathrm{B}\left(\frac{1}{\alpha+1},\frac{1}{2}\right)$ is

$$P = \frac{1}{\alpha+1}\frac{\mathrm{B}\left(\frac{1}{\alpha+1},\frac{3}{2}\right)}{\mathrm{B}\left(\frac{1}{\alpha+1},\frac{1}{2}\right)} = \frac{1}{3+\alpha}. \tag{324}$$

For the initial condition $x(0) = x_0$ and initial mass $m(0) = m_0$, the solution of the differential equation (323) is

$$A = x_0 \left(\frac{m_0}{m}\right)^{\frac{1}{3+\alpha}}. \tag{325}$$

The amplitude of vibration increases if the mass decreases, and it decreases if the mass increases. The amplitude variation is proportional to $(m_0/m)^{1/(3+\alpha)}$ and depends on mass variation, but also on the order of nonlinearity. For the same mass variation the amplitude variation is for the linear oscillator $x_0(m_0/m)^{1/4}$ (this result is previously obtained by Bessonov, 1967) and for the high order of nonlinearity ($\alpha \to \infty$) the amplitude tends to be constant i.e., $A \approx x_0$.

Introducing (256) and (325) into (311), the variation of the maximal vibration velocity is obtained

$$\dot{x}_{\max A} = \frac{\sqrt{2}\,\omega_0}{\sqrt{\alpha+1}} x_0^{(\alpha+1)/2} \left(\frac{m_0}{m}\right)^{\frac{\alpha+2}{\alpha+3}}. \tag{326}$$

Comparing (326) with (267), it is

$$\dot{x}_{\max A} = v_{\max} \left(\frac{m_0}{m}\right)^{\frac{\alpha+2}{\alpha+3}}. \tag{327}$$

The maximal velocity of vibration decreases in time if the mass increases, and increases for the case of the , mass decrease. The velocity of variation depends on the order of nonlinearity. For the same mass variation the velocity variation is for the linear oscillator proportional to $(m_0/m)^{3/4}$ and for extremely high order of nonlinearity ($\alpha \to \infty$) it is $(m_0/m)^1$.

The most often used nonlinearity is the pure cubic one. For the mass variable oscillator of Duffing type using the results (325) and (326), the amplitude and maximal velocity of vibration are expressed as

$$A = x_0 \left(\frac{m_0}{m}\right)^{\frac{1}{6}}, \qquad \dot{x}_{\max A} = v_{\max} \left(\frac{m_0}{m}\right)^{\frac{5}{6}}. \tag{328}$$

7.4.2 Solution with trigonometric function

Let us assume the trial solution of the Eq. (254) in the form of the generating solution (291) and its first time derivative (295) but with time variable amplitude, frequency and phase angle, i.e.,

$$x = A(t)\cos\psi(t), \tag{329}$$
$$\dot{x} = -A(t)\Omega(\tau, A(t))\sin\psi(t), \tag{330}$$

where

$$\dot{\psi} = \dot{\theta} + \Omega(\tau, A(t)), \tag{331}$$

and $A(t)$, $\theta(t)$ and $\psi(t)$ are unknown functions. According to (292) the frequency function is

$$\Omega(\tau, A(t)) = \Omega_\alpha \omega(\tau)(A(t))^{\frac{\alpha-1}{2}}, \tag{332}$$

where Ω_α =const. given with Eq. (293). Calculating the first time derivative of (329) and equating it with expression (330), it follows

$$\dot{A}(t)\cos\psi(t) - A(t)\dot{\theta}(t)\sin\psi(t) = 0. \tag{333}$$

Substituting (329), (330) and the time derivative of (330) into (254) and using the relation (294), we obtain

$$-\dot{A}\Omega\sin\psi - A\dot{A}\frac{\partial\Omega}{\partial A}\sin\psi - \varepsilon A\frac{\partial\Omega}{\partial\omega}\frac{\partial\omega}{\partial\tau}\sin\psi - A\Omega\dot{\theta}\cos\psi = \varepsilon f(\tau, A\cos\psi, -A\Omega\sin\psi), \tag{334}$$

where $\omega \equiv \omega(\tau), A \equiv A(t), \theta \equiv \theta(t), \psi \equiv \psi(t)$. Hence, two first order differential equations (333) and (334) replace the second order differential equation (254). Solving (333) and (334) with respect to $\dot{A}$ and $\dot{\theta}$, we have

$$\dot{A}(\Omega - A\frac{\partial\Omega}{\partial A}\sin^2\psi) = -\varepsilon A\frac{\partial\Omega}{\partial\omega}\frac{\partial\omega}{\partial\tau}\sin^2\psi - \varepsilon f(\tau, A\cos\psi, -A\Omega\sin\psi)\sin\psi, \tag{335}$$

$$A\dot{\theta}(\Omega - A\frac{\partial\Omega}{\partial A}\sin^2\psi) = -\varepsilon A\frac{\partial\Omega}{\partial\omega}\frac{\partial\omega}{\partial\tau}\sin\psi\cos\psi - \varepsilon f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi. \tag{336}$$

Averaging the differential equations in the period 2π we obtain the following equations

$$\dot{A} = -\frac{2\varepsilon A}{(5-\alpha)\omega}\frac{d\omega}{d\tau} - \frac{4\varepsilon A^{\frac{1-\alpha}{2}}}{(5-\alpha)\omega\Omega_\alpha}\frac{1}{2\pi}\int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\sin\psi d\psi, \tag{337}$$

$$A\dot{\theta} = -\frac{2\varepsilon A^{\frac{1-\alpha}{2}}}{(5-\alpha)\omega\Omega_\alpha}\frac{1}{2\pi}\int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi d\psi, \tag{338}$$

and from (331) and (332)

$$\dot{\psi} = \Omega_\alpha\omega A^{\frac{\alpha-1}{2}} - \frac{2\varepsilon A^{\frac{1-\alpha}{2}}}{(5-\alpha)\omega\Omega_\alpha}\frac{1}{2\pi}\int_0^{2\pi} f(\tau, A\cos\psi, -A\Omega\sin\psi)\cos\psi d\psi. \tag{339}$$

Solving the averaged differential equation (337) and substituting the obtained solution for A into (339) the approximate function ψ is obtained which gives the solution (329).

Small linear damping force acts For the special case when beside the reactive force also the linear damping force acts

$$F_x = -\varepsilon b\dot{x}, \tag{340}$$

where εb is the small damping coefficient, the function f is

$$f = -\left(\frac{b}{m} + \frac{1}{m}\frac{dm}{d\tau}\right)\dot{x}. \tag{341}$$

As the mass variation is slow and the damping coefficient is small, the reactive and damping force are also small in comparison to the elastic force. Substituting (256) and (341) into (337) and (339) the differential equation (253) transforms into a system of two averaged first order differential equations

$$\frac{\dot{A}}{A} = -\frac{\varepsilon}{(5-\alpha)m}\left(\frac{dm}{d\tau}\right) - \frac{2\varepsilon b}{(5-\alpha)\,m}, \tag{342}$$

$$\dot{\psi}(t) = \Omega_\alpha A^{\frac{\alpha-1}{2}} \sqrt{\frac{k_\alpha}{m}}, \tag{343}$$

In general, the averaged amplitude variation is the solution of (342)

$$A = A_0 \left(\frac{m_0}{m}\right)^{\frac{1}{5-\alpha}} \exp\left(-\frac{2\varepsilon b}{5-\alpha}\int \frac{dt}{m}\right), \tag{344}$$

which gives the phase angle function

$$\psi = \psi_0 + \Omega_\alpha \sqrt{k_\alpha}\left(A_0 m_0^{\frac{1}{5-\alpha}}\right)^{\frac{\alpha-1}{2}} \int m^{-\frac{\alpha-1}{2(5-\alpha)}-\frac{1}{2}} \left(\exp\left(-\frac{2\varepsilon b}{5-\alpha}\int \frac{dt}{m}\right)\right)^{\frac{\alpha-1}{2}} dt. \tag{345}$$

The amplitude and the phase of vibration vary in time due to damping, but also due to mass variation. The order of nonlinearity has a significant influence on the velocity of amplitude and phase increase or decrease.

Linear mass variation Let us consider the case when the mass variation is linear, as it is suggested by Yuste (1991)

$$m = m_0 + m_1\tau = m_0 + \varepsilon m_1 t, \tag{346}$$

where m_1 is a constant and ε is a small parameter. According to (342), we obtain the differential equation for the amplitude variation

$$\frac{\dot{A}}{A} = -\frac{\varepsilon(2b + m_1)}{(5-\alpha)m} \tag{347}$$

a) For the special parameter values, when $m_1/b = -2$ the amplitude of vibration is constant i.e.,

$$A = A_0 = const. \tag{348}$$

and the relation (345) transforms into

$$\psi = \psi_0 + \frac{2}{\varepsilon m_1}\Omega_\alpha A_0^{\frac{\alpha-1}{2}} \sqrt{k_\alpha}\left(m^{1/2} - m_0^{1/2}\right) \tag{349}$$

For this special case in spite of the action of the linear damping the amplitude of vibration is constant due to the fact that the linear mass separation makes the compensation to the effect of damping. Using the series expansion of the function ψ we have

$$\psi = \psi_0 + 2\Omega_\alpha A_0^{\frac{\alpha-1}{2}} \sqrt{\frac{k_\alpha}{m_0}} t. \tag{350}$$

The approximate value of the period of vibration is independent on the mass variation and damping coefficient, and is given as follows

$$T = \frac{2\pi}{2\Omega_\alpha A_0^{\frac{\alpha-1}{2}} \sqrt{\frac{k_\alpha}{m_0}}}. \tag{351}$$

The approximate period value depends only on the order of nonlinearity.

b) For $m_1/b \neq -2$ the amplitude-time and phase-time functions are

$$A = A_0 \left(\frac{m}{m_0}\right)^{-\frac{1}{5-\alpha}\left(1+\frac{2b}{m_1}\right)}, \tag{352}$$

and

$$\psi = \frac{\sqrt{m_0 k_\alpha}}{\varepsilon m_1} \frac{A_0^{\frac{\alpha-1}{2}} \Omega_\alpha}{\frac{1}{2} - \frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)} \left(\left(\frac{m}{m_0}\right)^{\frac{1}{2} - \frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)} - 1 \right) + \psi_0, \qquad (353)$$

which give the approximate solution (329)

$$\begin{aligned} x &= A_0 \left(\frac{m}{m_0}\right)^{-\frac{1}{5-\alpha}\left(1+\frac{2b}{m_1}\right)} \qquad (354) \\ &\cos\left(\frac{\sqrt{m_0 k_\alpha}}{\varepsilon m_1} \frac{A_0^{\frac{\alpha-1}{2}} \Omega_\alpha}{\frac{1}{2} - \frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)} \left(\left(\frac{m}{m_0}\right)^{\frac{1}{2} - \frac{\alpha-1}{2(5-\alpha)}\left(1+\frac{2b}{m_1}\right)} - 1 \right) + \psi_0 \right). \end{aligned}$$

The amplitude and phase variation depend on the relation m_1/b, parameter m_1 and order of nonlinearity α.

Let us consider a numerical example were the order of nonlinearity is $\alpha = 4/3$, the rigidity $k_{4/3} = 1$ and the mass decrease is $m = 1 - 0.01t$, where $m_0 = 1$, $m_1 = 1$ and $\varepsilon = 0.01$. The differential equation of motion is

$$\ddot{x} + \frac{x\,|x|^{1/3}}{1 - 0.01t} = 0.01\,(1-b)\,\dot{x}, \qquad (355)$$

where b is the damping coefficient. For the initial conditions $x(0) = A_0 = 0.1$ and $\dot{x}(0) = 0$ the analytical solution (354) has the form

$$\begin{aligned} x &= \frac{0.1}{(1-0.01t)^{0.27273(1-2b)}} \qquad (356) \\ &\cos\left(\frac{66.028}{0.5\,(1 - 0.0909(1-2b))} \left(1 - (1-0.01t)^{0.5(1-0.0909(1-2b))}\right) \right). \end{aligned}$$

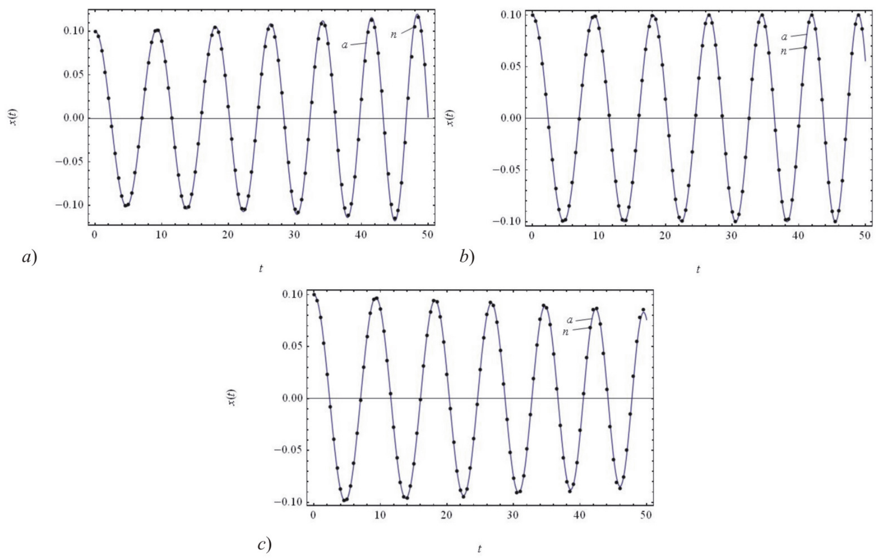


Fig.21. The $x - t$ diagrams obtained analytically (a – full line) and numerically (n – dot line) for: a) $b = 0$, b) $b = 1/2$ and c) $b = 1$.

In Fig.21 the approximate solution (356) and the numerical solution of (355), obtained by using of the Runge-Kutta procedure, are plotted. The $x-t$ diagrams for various values of the damping parameter b are shown.

It can be concluded that for $b = 1/2$ the amplitude of vibration is constant as it is previously stated (see Eq. (345)). For the case when the damping is neglected ($b = 0$), due to mass decrease and existence of the reactive force, the amplitude of vibration increases. For certain damping ($b = 1$) which is higher than the limit value ($b = 1/2$) the amplitude of vibration decreases. The analytical solution is in a very good relation to the numeric one in spite of the long time interval of consideration.

Linear oscillator For the linear oscillator when $\alpha = 1$

$$A = A_0 \left(\frac{m_0}{m}\right)^{\frac{1}{4}(1+\frac{2b}{m_1})} \tag{357}$$

and

$$\psi = \frac{2\sqrt{k_1}}{\varepsilon m_1} \left(\sqrt{m} - \sqrt{m_0}\right) + \psi_0 \tag{358}$$

If the damping parameter is zero the amplitude variation is $A = A_0 \left(\frac{m_0}{m}\right)^{1/4}$. Using the series expansion of the functions in (358) the approximate frequency of vibration is $\sqrt{k_1/m_0}$ which corresponds to the systems with constant mass and without damping.

Influence of the reactive force Let us analyze the influence of the reactive force described with the function

$$f = -\frac{1}{m}\frac{dm}{d\tau}\dot{x}. \tag{359}$$

Substituting (359) into (344) and (345) we obtain the variation of the amplitude

$$A = x_0 \left(\frac{m_0}{m}\right)^{\frac{1}{5-\alpha}}, \tag{360}$$

and of the phase angle function

$$\psi = \psi_0 + \Omega_\alpha \sqrt{k_\alpha} \left(x_0 m_0^{\frac{1}{5-\alpha}}\right)^{\frac{\alpha-1}{2}} \int m^{-\frac{\alpha-1}{2(5-\alpha)}-\frac{1}{2}} dt. \tag{361}$$

For the certain order of nonlinearity α the amplitude of vibration increases with decreasing of the mass in time. If the mass increases, the amplitude of vibration decreases for the oscillator of the certain degree of nonlinearity. For the linear oscillator, when $\alpha = 1$, the amplitude variation is

$$A = x_0 \left(\frac{m_0}{m}\right)^{1/4}, \tag{362}$$

and for the pure cubic oscillator with cubic nonlinearity (see Cveticanin, 1992)

$$A = x_0 \left(\frac{m_0}{m}\right)^{1/2}. \tag{363}$$

If the mass increases, the amplitude decreases faster for higher order of nonlinearity. If the mass decreases the amplitude increases faster for smaller order of nonlinearity.

Using (330), (332), (293) and also (360), the variation of the vibration velocity is

$$\dot{x}_{\max A} = \dot{x}_{a\max} \left(\frac{m_0}{m}\right)^{\frac{3}{5-\alpha}}. \tag{364}$$

The velocity correction function due to mass variation is $(m_0/m)^{3/(5-\alpha)}$.

For the case when the relative velocity of the adding or separated mass is zero and the reactive force is zero, or for the case when the reactive force is sufficiently small and can be omitted, the amplitude and phase angle functions are according to (337), (339) and (256)

$$\dot{A} = \frac{A\dot{m}}{(5-\alpha)m}, \tag{365}$$

and

$$\dot{\psi} = \Omega_\alpha \sqrt{\frac{k_\alpha}{m}} A^{\frac{\alpha-1}{2}}. \tag{366}$$

Integrating the relations (365) and (366) for the initial amplitude A_0, phase angle ψ_0 and mass m_0, it follows

$$A = A_0 \left(\frac{m}{m_0}\right)^{\frac{1}{5-\alpha}}, \tag{367}$$

$$\psi = \psi_0 + \Omega_\alpha \sqrt{k_\alpha} \left(A_0 m_0^{-\frac{1}{5-\alpha}}\right)^{\frac{\alpha-1}{2}} \int (m)^{\frac{\alpha-1}{2(5-\alpha)}-\frac{1}{2}} dt. \tag{368}$$

Due to (329) and the relations (367) and (368), the approximate solution is

$$x = A_0 \left(\frac{m}{m_0}\right)^{\frac{1}{5-\alpha}} \cos\left(\psi_0 + \Omega_\alpha \sqrt{k_\alpha} \left(A_0 m_0^{-\frac{1}{5-\alpha}}\right)^{\frac{\alpha-1}{2}} \int (m)^{\frac{\alpha-1}{2(5-\alpha)}-\frac{1}{2}} dt\right). \tag{369}$$

Analyzing the relation (367) it is obvious that for the same order of nonlinearity α, the amplitude of vibration increases by increasing of the mass. Besides, for the same mass variation, the amplitude increases faster for higher orders of nonlinearity.

7.4.3 Solution in the form of the Jacobi elliptic function

Let us assume the solution of (254) in the form of the Jacobi elliptic function

$$x = Acn(\psi, k^2), \tag{370}$$

with the first time derivative

$$\dot{x} = -A\Omega_1 sndn, \tag{371}$$

where $A = A(t)$ is the unknown time variable amplitude,

$$\psi = \psi(t) = \int_0^t \Omega_1(t)dt + \theta(t), \tag{372}$$

$\theta = \theta(t)$ is the unknown time variable phase and

$$\Omega_1 = \frac{2K(k^2)}{\pi} \Omega_\alpha A^{(\alpha-1)/2} \sqrt{\frac{k_\alpha}{m(\tau)}}. \tag{373}$$

The first time derivative of (370) is

$$\dot{x} = -A\Omega_1 sndn + \dot{A}cn - A\dot{\theta} sndn. \tag{374}$$

Comparing the time derivatives (371) and (374) it is evident that they are equal for

$$\dot{A}cn - A\dot{\theta} sndn = 0. \tag{375}$$

Introducing (370), (371) and the time derivative of (371) into (254), and using the relation

$$-A\Omega_1^2 cn(1-2k^2+2k^2cn^2)+\omega^2 Acn \approx 0, \tag{376}$$

we obtain a first order differential equation

$$-\dot{A}\Omega_1 sndn - A\dot{\Omega}_1 sndn - A\Omega_1\dot{\theta}cn(1-2k^2+2k^2cn^2) = \varepsilon\frac{f}{m}+\frac{\dot{m}}{m}A\Omega_1 sndn. \tag{377}$$

The relations (375) and (377) represent the re-written version of the second order nonlinear differential equation () into a system of two coupled first order nonlinear differential equations. After some modification, we have

$$\dot{A}\Omega_1[sn^2dn^2+cn^2(1-2k^2+2k^2cn^2)] = -\varepsilon\frac{f}{m}sndn-\frac{\dot{m}}{m}A\omega sn^2dn^2-A\dot{\Omega}_1 sn^2dn^2, \tag{378}$$

$$A\dot{\theta}\Omega_1[sn^2dn^2+cn^2(1-2k^2+2k^2cn^2)] = -\varepsilon\frac{f}{m}cn-\frac{\dot{m}}{m}A\Omega_1 cnsndn-A\dot{\Omega}_1 cnsndn. \tag{379}$$

Solving the Eqs. (378) and (379), the two unknown functions A and θ are calculated. For simplification the averaging the differential equations is done. The differential equations are averaged over the period of vibration and they are

$$\dot{A}\left\langle sn^2dn^2+cn^2(1-2k^2+2k^2cn^2)\right\rangle = -\frac{\varepsilon}{m\Omega_1}\left\langle fsndn\right\rangle - A(\frac{\dot{m}}{m}+\frac{\dot{\Omega}_1}{\Omega_1})\left\langle sn^2dn^2\right\rangle, \tag{380}$$

$$A\dot{\theta}\left\langle sn^2dn^2+cn^2(1-2k^2+2k^2cn^2)\right\rangle = -\frac{\varepsilon}{m\Omega_1}\left\langle fcn\right\rangle, \tag{381}$$

where $\langle ...\rangle = \frac{1}{4K(k^2)}\int\limits_0^{4K(k^2)}(...)\,d\psi$. After integration, the Eqs. (380) and (381) transform into

$$\begin{aligned}\dot{A}\left[\left(1-k^2\right)+k^2C_4\right] &= -\frac{\varepsilon}{m\Omega_1}\left\langle fsndn\right\rangle - \\ &\quad A(\frac{\dot{m}}{m}+\frac{\dot{\Omega}_1}{\Omega_1})\left[\left(1-k^2\right)-\left(1-2k^2\right)C_2-k^2C_4\right], \\ A\dot{\theta}\left[\left(1-k^2\right)+k^2C_4\right] &= -\frac{\varepsilon}{m\Omega_1}\left\langle fcn\right\rangle,\end{aligned} \tag{382}$$

where (see [?])

$$C_2 = \frac{1}{k^2}\left(\frac{E}{K}-1+k^2\right)\approx\frac{1}{2}+\frac{k^2}{16}+... \tag{383}$$

$$C_4 = \frac{1}{3k^2}\left[2(2k^2-1)C_2+(1-k^2)\right]\approx\frac{3}{8}+\frac{k^2}{32}+... \tag{384}$$

Oscillator without reactive force For the case when the absolute velocity of mass adding or separation is zero, the differential equation of motion is

$$m(\tau)\ddot{x}+k_\alpha x\left|x\right|^{\alpha-1}=0. \tag{385}$$

According to (**??**) and (382), the amplitude and phase variation are

$$\theta = const., \qquad A = A_0(\frac{\Omega_{10}}{\Omega_1})^{\frac{\left[\left(1-k^2\right)-\left(1-2k^2\right)C_2-k^2C_4\right]}{\left[\left(1-k^2\right)+k^2C_4\right]}} = A_0(q)^{-s}. \tag{386}$$

For the linear oscillator the amplitude-time function is

$$A = x_0(\frac{m}{m_0})^{1/4}, \tag{387}$$

and for the cubic oscillator it is

$$A = A_0(\frac{m}{m_0})^{0.285\,86}. \tag{388}$$

For the case when mass increases, the higher the added quantity, the higher the amplitude.

Oscillator with the reactive force If the terms εf are omitted and zero, the solutions of the differential equations (**??**) and (382) are

$$\theta = \text{const.}, \qquad A = x_0(\frac{m_0}{m})^{s/(2+(\alpha-1)s)}, \tag{389}$$

where

$$s = \frac{\left[\left(1-k^2\right)-\left(1-2k^2\right)C_2-k^2C_4\right]}{\left[\left(1-k^2\right)+k^2C_4\right]}. \tag{390}$$

Using (389) and the frequency of the Jacobi elliptic function (373), the approximate frequency of vibration of the oscillator is

$$\Omega(\tau) = q\Omega, \tag{391}$$

where q is the correction function

$$q = (\frac{m_0}{m})^{\frac{s(\alpha-1)}{2(2+(\alpha-1)s)}+\frac{1}{2}}. \tag{392}$$

For the linear oscillator the amplitude variation is

$$A = x_0(\frac{m_0}{m})^{1/4}, \tag{393}$$

as is already given (see Eq. (362)). The frequency of vibration is

$$\Omega = \sqrt{\frac{k_1}{m}}, \tag{394}$$

as it is previously published in Bessonov, 1967.

The maximal velocity variation function is according to (371), (373), (306) and (389)

$$\dot{x}_{\max} = \dot{x}_{p\max}(\frac{m_0}{m})^P, \tag{395}$$

where

$$P = \frac{s\alpha+1}{2+(\alpha-1)s}.$$

and $\dot{x}_{p\max} = \frac{2K(k^2)k'}{\pi}\Omega_\alpha x_0^{(\alpha+1)/2}\sqrt{\omega_0^2}$.

Using the previous results, the oscillator with cubic nonlinearity (398) , when $\alpha = 3$ and $k^2 = 1/2$ has the following amplitude and velocity variation

$$A = x_0(\frac{m_0}{m})^{5/32}, \tag{396}$$

and

$$\dot{x}_{\max} = \dot{x}_{p\max}(\frac{m_0}{m})^{13/16} \tag{397}$$

where $\dot{x}_{p\max} = \frac{\sqrt{2}K(1/2)}{\pi}\Omega_\alpha$.

Comparing (393) and (387), it can be concluded that for the same mass variation the amplitude variation depends on the existence of the reactive force. If the mass increases and there is not a reactive force, the amplitude of vibration increases, but the existence of the reactive force causes the amplitude decrease.

Example 1. Let us consider the strong nonlinear cubic mass variable oscillator with reactive force. The mathematical model is

$$\frac{d}{dt}(m\dot{x}) + x^3 = 0, \tag{398}$$

with the initial conditions $x(0) = x_0 = 1$ and $\dot{x}(0) = 0$, and mass variation $m = (1+0.1t)$. Applying the Runge-Kutta procedure, the numerical $x-t$ solution and also the $\dot{x}-t$ relation is obtained. Using the suggested solving procedures the approximate solutions of (398) are calculated. For the case when the approximate solution is in the form of the Ateb function, we express the amplitude and maximal velocity of vibration variation functions (328) as

$$A = (1+0.1t)^{-0.16667}, \qquad \dot{x}_{\max A} = \frac{\sqrt{2}}{2}(1+0.1t)^{-0.83333}. \tag{399}$$

In Fig.22a the numerical $x-t$ and approximate $A-t$, and in Fig.22b the numerical $\dot{x}-t$ and the approximate $\dot{x}_{\max A}-t$ curves are plotted. It can be seen that the curves (399) are on the top of the numeric solution. The difference is negligible not only for the amplitudes of vibration, but also for the first time derivatives.

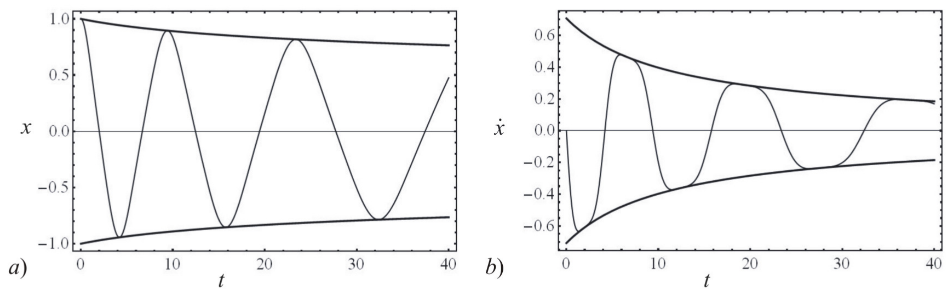


Fig.22. Comparison of the approximate Ateb solution with the numeric one: a) the $x-t$ (thin line) and $A-t$ (thick line) curves, and b) $\dot{x}-t$ (thin line) and $\dot{x}_{\max A}-t$ (thick line) for the cubic oscillator with linear mass variation.

Using the approximate solution in the form of the trigonometric function, due to (363) and (364) the amplitude and maximal velocity functions are given as

$$A = (1+0.1t)^{-0.5}, \qquad \dot{x}_{\max A} = 0.84721(1+0.1t)^{-3/2}. \tag{400}$$

In Fig.23a the numerical $x - t$ and approximate $A - t$, and in Fig.23b the numerical $\dot{x} - t$ and the approximate $\dot{x}_{\max A} - t$ curves are plotted. It can be seen that the curves (400) significantly differ from the numeric solutions. The difference is not only for the amplitudes of vibration, but also for the first time derivatives. The analytical solutions can be applied only for the qualitative analysis of the problem

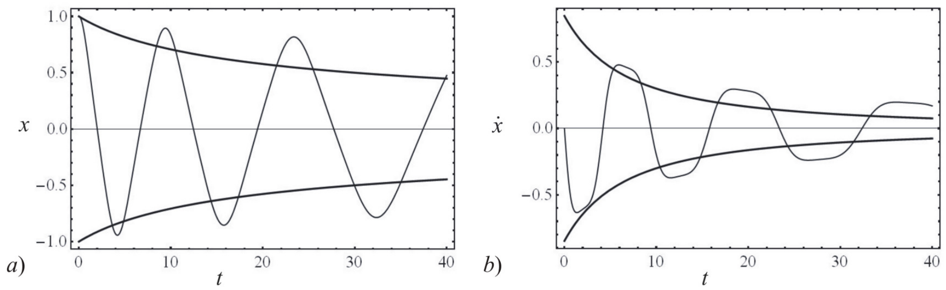


Fig.23. Comparison of the approximate trigonometric solution with the numeric one: a) the $x - t$ (thin line) and $A - t$ (thick line) curves, and b) $\dot{x} - t$ (thin line) and $\dot{x}_{\max A} - t$ (thick line) for the cubic oscillator with linear mass variation.

Using the approximate solution in the form of the Jacobi elliptic function, according to (396) and (397) the amplitude and maximal velocity functions are given as

$$A = (1 + 0.1t)^{-0.15625}, \qquad \dot{x}_{\max A} = 0.70711(1 + 0.1t)^{-0.8125}. \tag{401}$$

In Fig.24a the numerical $x - t$ and approximate $A - t$, and in Fig.24b the numerical $\dot{x} - t$ and the approximate $\dot{x}_{\max A} - t$ curves are plotted. It can be seen that the curves (401) are close to the numeric solutions, not only for the amplitudes of vibration, but also for the first time derivatives.

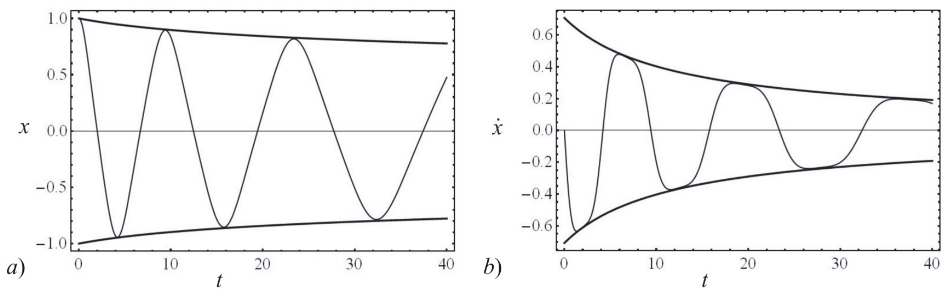


Fig.24. Comparison of the approximate solution in the form of the Jacobi elliptic function with the numeric one: a) the $x - t$ (thin line) and $A - t$ (thick line) curves, and b) $\dot{x} - t$ (thin line) and $\dot{x}_{\max A} - t$ (thick line) for the cubic oscillator with linear mass variation.

Example 2. Let us consider the strong nonlinear cubic mass variable oscillator with reactive force. The mathematical model is

$$\frac{d}{dt}(m\dot{x}) + x\,|x|^{1/2} = 0, \tag{402}$$

with the initial conditions $x(0) = x_0 = 1$ and $\dot{x}(0) = 0$, and mass variation $m = (1+0.1t)^2$. Applying the Runge-Kutta procedure, the numerical $x-t$ solution and also the $\dot{x}-t$ rela-tion is obtained. Using the suggested solving procedures the approximate solutions of (402) are calculated. For the case when the approximate solution is in the form of the Ateb function, we express the amplitude (325) and maximal velocity (326) of vibration variation functions as

$$A = (1 + 0.1t)^{-2/9}, \qquad \dot{x}_{\max A} = \frac{2}{\sqrt{5}}(1 + 0.1t)^{-7/9}. \tag{403}$$

In Fig.25a the numerical $x - t$ and approximate $A - t$, and in Fig.25b the numerical $\dot{x} - t$ and the approximate $\dot{x}_{\max A} - t$ curves are plotted. It can be seen that the curves (399) are on the top of the numeric solution. The difference is negligible not only for the amplitudes of vibration, but also for the first time derivatives.

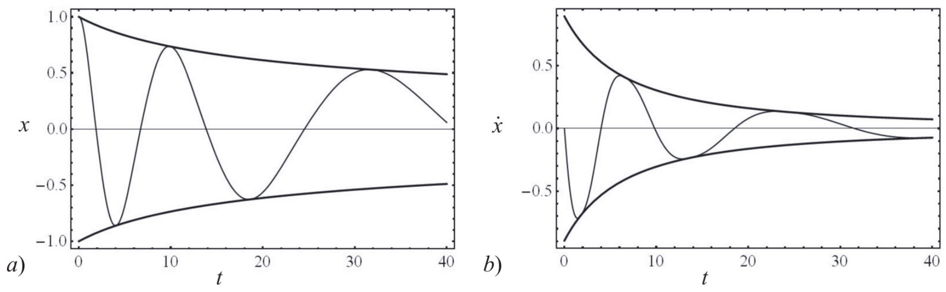


Fig.25. Comparison of the approximate Ateb solution with the numeric one: a) the $x-t$ (thin line) and $A-t$ (thick line) curves, and b) $\dot{x}-t$ (thin line) and $\dot{x}_{\max A}-t$ (thick line) for the cubic oscillator with linear mass variation.

Finally, it can be concluded that the approximate solution based on the Ateb function and on the Jacobi elliptic function are suitable for application for solving of the problem of vibrations of the mass variable oscillators. Namely, the obtained solutions satisfy not only the initial conditions and the amplitude variation properties of the oscillator, but also agrees with the function which describes the velocity variation. Comparing the relations (396) and (397) with (386) and (395) it is concluded that the former are more appropriate for practical use due to their simplicity.

7.5 Conclusion

Due to previous consideration it can be concluded:

1. The vibration of the oscillator with monotone time variable parameter has time variable amplitude and phase. The free vibrations for all of the oscillators with a strong nonlinearity of any order and with the certain monotone slow time variable parameters are qualitatively the same independently on the order of the nonlinearity. The order of nonlinearity quantitatively changes the amplitude and the phase of vibrations but has no influence on the character of vibrations. Namely, for certain parameter variation the higher the order of nonlinearity, the faster or slower is the amplitude and phase increase or decrease. The tendency of increase or decrease of amplitude and phase i.e., frequency of vibration variation is not directed by the order of nonlinearity but with the type of time parameter variation.

2. It is evident that in the oscillator with variable mass for the special relation between the coefficient of damping and parameter of mass variation (which affects the reactive force) the amplitude of vibration is constant, but the phase angle varies independently on the order of nonlinearity.

3. The approximate solution of the nonlinear differential equation with strong nonlinearity of any order (integer or non-integer) and time variable parameter can be obtained analytically.

4. The approximate analytic method for solving the differential equation based on the exact solution of the corresponding differential equation with constant parameters and strong nonlinearity of any order (integer or non-integer) gives very accurate results in comparison to the numerical one.

5. The solving method based on the approximate solution with exact period of vibration of the corresponding oscillator with constant parameter gives very convenient results for the oscillator with time variable parameters. For technical purpose the solution is accurate enough and appropriate for practical use. This solution has the form of trigonometric function and satisfies the requirements for simplicity and usefulness for application in techniques.

The vibrations of the mass variable systems are widely investigated by Leach, 1983; Abdalla, 1986_1; Abdalla, 1986_2; Crespo *et al.*, 1990, Xie *et al.*, 1995; Sanchez-Otiz and Salas Brits, 1995; Flores *et al.*, 2003., too. The results published in this Chapter and in the mentioned papers are applied in dynamics of mechanisms and rotors with time variable mass (see Cveticanin, 1998_2).

7.6 Van der Pol oscillator with time variable mass

One of the most often analyzed oscillator is of the Van der Pol type. This mechanical oscillator has the corresponding electric analogy where the parameters of the electrical circuit are time variable. It requires the analyses of the Van der Pol oscillators as the mass variable system. Mathematical model of the Van der Pol oscillator with time variable mas is

$$m\ddot{x} + k_\alpha x\,|x|^{\alpha-1} = \varepsilon(b - cx^2)\dot{x} - \dot{m}\dot{x}, \tag{404}$$

with

$$x(0) = x_0, \qquad \dot{x}(0) = 0, \tag{405}$$

where εb and εc are constants. If the mass variation is the function of the slow time $\tau = \varepsilon t$, where $\varepsilon << 1$ is the small parameter, the right hand side terms of the Eq. (404) are small. Using the previously mentioned procedure, the differential equation is rewritten into two first order averaged differential equations

$$\begin{aligned}\dot{A}\left[(1-k^2)+k^2C_4\right] &= -\frac{\varepsilon c A^3\Omega_1 sn^2cn^2dn^2}{m\Omega_1}\left[(1-k^2)\,C_2-(1-2k^2)\,C_4-k^2C_6\right]\\ &\quad -A(\frac{\dot{m}}{m}+\frac{\dot{\Omega}_1}{\Omega_1}-\frac{\varepsilon b}{m})\left[(1-k^2)-(1-2k^2)\,C_2-k^2C_4\right] \quad (406)\\ \dot{\theta} &= 0, \quad (407)\end{aligned}$$

where

$$C_6 = \frac{4(2k^2-1)C_4+3(1-k^2)C_2}{5k^2} \approx \frac{5}{16}+\frac{5}{256}k^2+... \tag{408}$$

Integrating the Eq. (407) it is obvious that the phase angle is constant, i.e. θ =const. Using the relation (373) and its time derivative

$$\frac{\dot{\Omega}_1}{\Omega_1} = -\frac{1}{2}\frac{\dot{m}}{m}+\frac{\alpha-1}{2}\frac{\dot{A}}{A}, \tag{409}$$

the differential equation (406) transforms into

$$\dot{A}a_1 + A(\frac{1}{2}\dot{m}-\varepsilon b)\frac{a_2}{m}+\varepsilon c A^3\frac{a_3}{m} = 0, \tag{410}$$

where

$$\begin{aligned} a_1 &= \left(1-k^2\right)+k^2C_4+\frac{\alpha-1}{2}a_2, &(411)\\ a_2 &= \left(1-k^2\right)-\left(1-2k^2\right)C_2-k^2C_4, &(412)\\ a_3 &= \left(1-k^2\right)C_2-\left(1-2k^2\right)C_4-k^2C_6. &(413)\end{aligned}$$

The differential equation (410) is of the Bernoulli type (see Kamke, 1971). Introducing the new variable

$$p = A^{-2}, \tag{414}$$

the nonlinear differential equation (410) is transformed into the linear one

$$p' - \frac{\dot{m}-2\varepsilon b}{m}\frac{a_2}{a_1}p = \frac{2a_3\varepsilon c}{a_1 m}. \tag{415}$$

In general, the solution of (415) with (414) is

$$\frac{1}{A^2} \equiv p = \exp(-F)\left(C+\frac{2a_3\varepsilon c}{a_1}\int_0^t \exp(F)\frac{dt}{m}\right), \tag{416}$$

where

$$F = \frac{2\varepsilon b a_2}{a_1}\int_0^t \frac{dt}{m} - \ln(\frac{m}{m_0})^{\frac{a_2}{a_1}}, \tag{417}$$

and C is the arbitrary constant dependent on the initial conditions.

For the linear mass variation

$$m(\tau) = m_0(1+\tau), \tag{418}$$

with $\dot{m} = \varepsilon m_0 = const.$, and

$$F = \ln(\frac{m}{m_0})^{\frac{(2b-m_0)a_2}{m_0 a_1}}, \tag{419}$$

the solution of Eq. (416) is in general (see Polyanin and Zaitsev, 2003)

$$\frac{1}{A^2} = (\frac{m_0}{m})^q\left(C+\frac{2a_3 c}{a_1 q}\left((\frac{m}{m_0})^q-1\right)\right), \tag{420}$$

where

$$q = \frac{(2b-m_0)a_2}{m_0 a_1}, \tag{421}$$

and m_0 is the initial mass. Finally, using the initial conditions (405), we have

$$\frac{m^q}{A^2} - \frac{m_0^q}{x_0^2} = \frac{2ca_3}{m_0 a_1 q}(m^q - m_0^q). \tag{422}$$

7.6.1 Discussion of the result and a numerical example

1. Substituting the mass variation (418), the Eq. (422) is rewritten in the form

$$A^2 = \frac{(1+\varepsilon t)^q x_0^2 m_0 a_1 q}{m_0 a_1 q + 2ca_3 x_0^2((1+\varepsilon t)^q - 1)}. \tag{423}$$

The amplitude of vibration depends on q (421), which gives the relation between the negative damping coefficient b and the initial mass m_0. Namely, this relation takes into the consideration the effect of reactive force on the vibrations of the Van der Pol oscillator. The following three cases are evident:

a) For $2b = m_0$, when $q = 0$, the amplitude of vibration tends to the initial amplitude x_0 independently on the order of nonlinearity of the oscillator.

b) According to (421) and (423), the amplitude-time curve is for $2b > m_0$ given as

$$A = \sqrt{\frac{(1+\varepsilon t)^{\frac{(2b-m_0)a_2}{m_0 a_1}} x_0^2 (2b-m_0)a_2}{(2b-m_0)a_2 + 2ca_3 x_0^2 ((1+\varepsilon t)^{\frac{(2b-m_0)a_2}{m_0 a_1}} - 1)}}. \tag{424}$$

The amplitude of vibration is time variable and tends to a steady-state value

$$A_S = \sqrt{\frac{m_0 a_1 q}{2ca_3}} = \sqrt{\frac{(2b-m_0)a_2}{2ca_3}}, \tag{425}$$

which does not depend on the initial displacement x_0.

c) For the case when $2b < m_0$, i.e., $q < 0$, the amplitude-time relation is according to (421) and (423)

$$A = \sqrt{\frac{(1+\varepsilon t)^{-\frac{(m_0-2b)a_2}{m_0 a_1}} x_0^2 (m_0-2b)a_2}{(m_0-2b)a_2 + 2ca_3 x_0^2 (1 - (1+\varepsilon t)^{\frac{(m_0-2b)a_2}{m_0 a_1}})}}.$$

The amplitude of vibration is time variable and tends to zero

$$A_S = 0. \tag{426}$$

Namely, independently on the initial conditions, properties of the Van der Pol oscillator and mass variation, after some time the vibration disappear.

Remarks

1. For the case of the linear mass variation the reactive force acts as the positive damping force.

2. For the case when the reactive force is zero, the Eq. (422) is

$$\frac{m^{q_1}}{A^2} - \frac{m_0^{q_1}}{A_0^2} = \frac{2ca_3}{m_0 a_1 q_1}(m^{q_1} - m_0^{q_1}), \tag{427}$$

where

$$q_1 = \frac{2ba_2}{m_0 a_1}. \tag{428}$$

The amplitude varies in time according to the relation

$$A = \sqrt{\frac{2b(1+\varepsilon t)^{\frac{2ba_2}{m_0 a_1}} x_0^2 a_2}{2ba_2 + 2ca_3 x_0^2 ((1+\varepsilon t)^{\frac{2ba_2}{m_0 a_1}} - 1)}}, \tag{429}$$

and tends to the steady state one

$$A_S = \sqrt{\frac{ba_2}{ca_3}}, \tag{430}$$

which is independent on the value of the initial mass and mass variation. For all values of the initial displacements and mass variation, only one steady state motion exists. The steady state amplitude depends on the properties of the oscillator and is independent on mass variation properties.

3. For the oscillator with reactive force and with linear negative damping, i.e., Eq. (404) for $c = 0$,

$$m\ddot{x} + k_\alpha x \left|x\right|^{\alpha-1} = \varepsilon b\dot{x} - \dot{m}\dot{x}, \tag{431}$$

the amplitude-time relation is according to (423)

$$A = x_0 (\frac{m}{m_0})^{\frac{(2b-m_0)a_2}{2m_0 a_1}}.$$

For the linear mass variation (418), when $(m/m_0) \geqslant 1$, the amplitude of vibration increases if $(2b - m_0) > 0$ and decreases for $(2b - m_0) < 0$. For $(2b - m_0) = 0$, in spite of mass variation, the amplitude has the constant value equal to the initial amplitude.

If in the oscillator with time variable mass the reactive force is zero, due to the fact that the relative velocity of the mass which is added of separated is zero, but the negative linear damping acts, the amplitude of vibration increases for all values of the initial mass and mass variation parameters and yields,

$$A = x_0 (\frac{m}{m_0})^{\frac{2ba_2}{m_0 a_1}}. \tag{432}$$

4. If in the mass variable oscillator the reactive force and only the nonlinear (positive) damping acts ($b = 0$ and $c \neq 0$), we have the mathematical model

$$m\ddot{x} + k_\alpha x \left|x\right|^{\alpha-1} = -(\varepsilon c x^2 - \dot{m})\dot{x}. \tag{433}$$

According to (423), the amplitude-time relation is

$$A = \sqrt{\frac{x_0^2 m_0 a_2}{(m_0 a_2 + 2ca_3 x_0^2)(1+\varepsilon t)^{\frac{a_2}{a_1}} - 2ca_3 x_0^2}}. \tag{434}$$

Analyzing the relation (434), it is evident that the amplitude of vibration decreases in time and tends to zero.

For the case of mass variable oscillator without reactive force, but with negative damping, the amplitude-time relation is according to (416)

$$A = \sqrt{\frac{x_0^2 a_1 m_0}{a_1 m_0 + 2ca_3 x_0^2 \ln\left|1+\varepsilon t\right|}}, \tag{435}$$

and the vibration decreases.

5. The obtained values can be compared with those obtained for the Van der Pol oscillator with constant mass

$$m\ddot{x} + x\left|x\right|^{2/3} = \varepsilon(b - cx^2)\dot{x}. \tag{436}$$

Thus, the relation (415) gives the amplitude-time variation

$$A^2 = \frac{ba_2 x_0^2}{ca_3 x_0^2 + (ba_2 - ca_3 x_0^2)\exp(-\frac{2\varepsilon b}{m}\frac{a_2}{a_1}t)}, \tag{437}$$

and the steady-state amplitude

$$A_S = \sqrt{\frac{ba_2}{ca_3}}, \tag{438}$$

which is independent on the initial displacement. For parameter values $b = 1$ and $c = 1$, the steady state amplitude (438) for the linear oscillator ($\alpha = 1$) is $A_S = 2\sqrt{3}$, as it is previously published by Nayfeh and Mook, 1979, while for the pure cubic nonlinear one ($\alpha = 3$) is $A_S = 1.7076$. The value, given by Mickens, 2010, is $A_S = 2$. Comparing the solution of (436) obtained numerically, applying the Runge Kutta procedure, with the approximate amplitude (438) it can be concluded that the suggested solution in the form of the Jacobi elliptic function gives the more accurate result.

Example. To illustrate the obtained results, let us consider an example where the mass variation is linear

$$m = m_0(1 + 0.01t), \tag{439}$$

and the coefficients of the Van der Pol oscillator are $\varepsilon b = 0.01$ and $\varepsilon c = 0.01$. Substituting the suggested values into (404) it follows

$$m_0(1 + 0.01t)\ddot{x} + x\,|x|^{2/3} = -0.01m_0\dot{x} + 0.01(1 - x^2)\dot{x}. \tag{440}$$

The Eq. (440) is solved numerically by applying of the Runge-Kutta solving procedure and the solutions are compared with the approximate analytical results for the amplitude-time relation (422). In Figs. 26 - 29 the numerically calculated $x - t$ and analytically obtained $A - t$ curves are plotted for various values of the initial displacement x_0 and initial mass m_0.

In Fig.26 the amplitude-time $A - t$ curves obtained by solving of the analytical relation (423) for the values of the initial mass $m_0 = 0.5$, 1 and 1.5, respectively, and various values of the initial amplitude x_0 are plotted. As $m_0 < 2$ the steady-state amplitude of vibration satisfies the relation (425) and have the values $A_S = 1.691$, $A_S = 1.381$ and $A_S = 0.976$, respectively. Besides, for $x_0 < A_S$, the amplitude of vibration increases to the limit value A_S, and for $x_0 > A_S$ it decreases to the limit value. For $x_0 = A_S$ the motion is with constant amplitude.

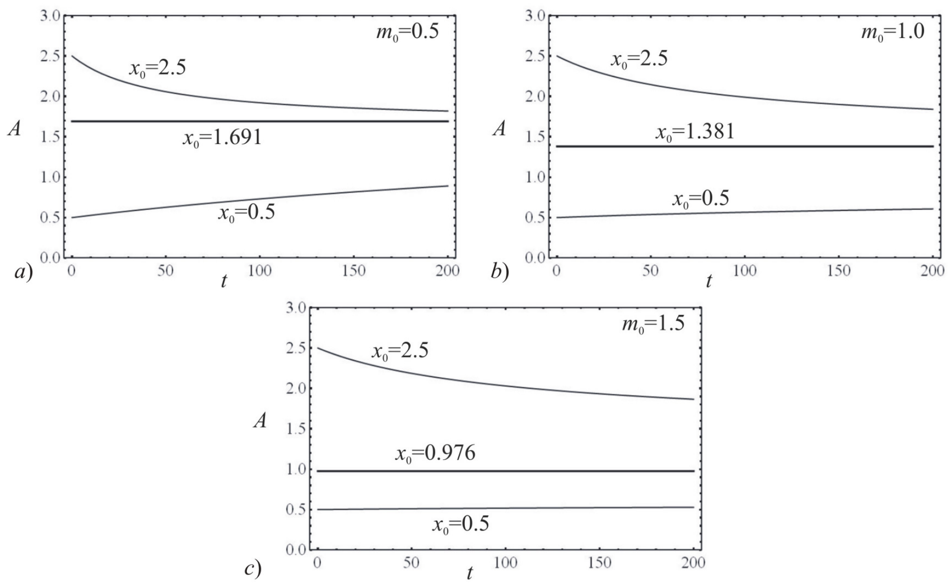


Fig.26. The amplitude-time curves for various initial displacement and various initial masses: a) $m_0 = 0.5$; b) $m_0 = 1$ and c) $m_0 = 1.5$.

In Fig.27. the $x-t$ curve obtained numerically by solving (440) and amplitude-time $A-t$ curves (425) for $m_0 = 1$ and the initial displacements: a) $x_0 = 2.5$; b) $x_0 = 1.381$; c) $x_0 = 0.5$ are plotted. It is evident that the analytically obtained results are on the top of the numerical ones. The difference is negligible.

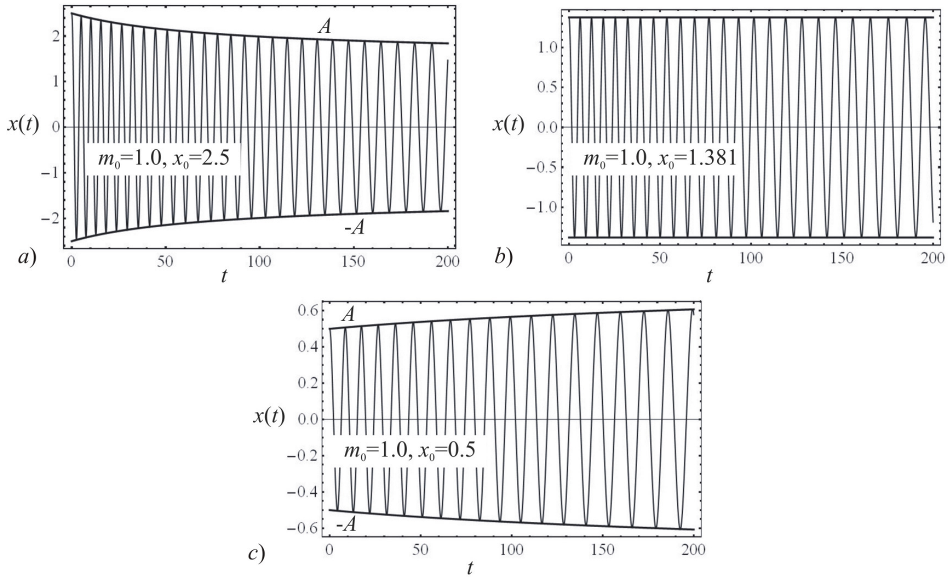


Fig.27. The $x - t$ and $A - t$ curves for $m_0 = 1$ and initial displacements: a) $x_0 = 2.5$; b) $x_0 = 1.381$; c) $x_0 = 0.5$.

In Fig.28, the $A-t$ curves for $m_0 = 3$ and initial displacements $x_0 = 0.5$, 1.5 and 2.5 are plotted. For this case, where $m_0 > 2$, the amplitude of vibration decreases to zero, independently on the initial amplitude.

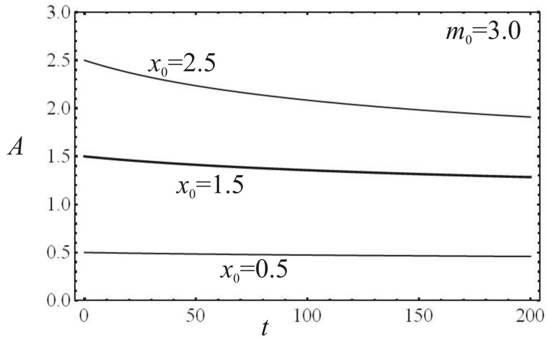


Fig.28. The $A - t$ curves for $m_0 = 3$ and initial displacements $x_0 = 0.5$, 1.5 and 2.5.

In Fig.29, beside the $A - t$ curves also the $x - t$ curves for $m_0 = 3$ and initial displacements: a) $x_0 = 0.5$ and b)$x_0 = 2.5$ are plotted.The $A - t$ curves are the envelopes of the $x-t$ curves. The difference between the numeric and analytic results is negligible.

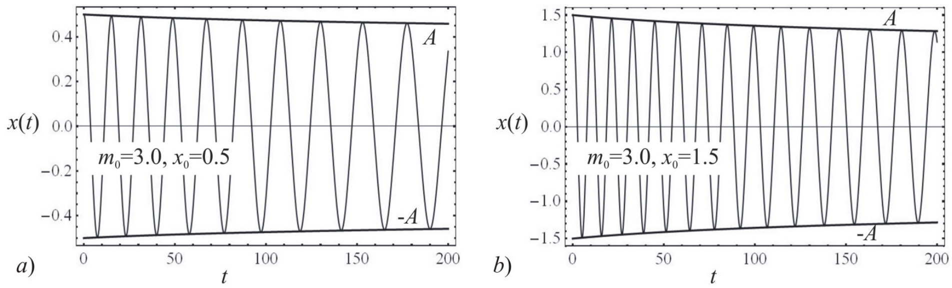


Fig.29. The $x - t$ and $A - t$ curves for $m_0 = 3$ and initial displacements: a) $x_0 = 2.5$ and b) $x_0 = 0.5$.

7.7 Vibration of the Laval rotor: an one-mass system with two-degrees-of-freedom

The most often considered rotor is of Laval type. It represents a symmetrically supported shaft-disc system (see Fig.16), where the shaft is elastic and with mass which can be neglected in comparison to the mass of the disc m. The position of the disc center S is given with the polar coordinates ρ and φ, which describe the position of the rotor center S to the fixed point O and the angle position of OS to the fixed axis x. The vibration of the disc occurs due to the elastic force in the shaft

$$\mathbf{F}_e = \mathbf{F}_e(\rho) = F_e(\rho)\mathbf{r}_0, \tag{441}$$

where $\mathbf{r}_0$ is the unit vector in the radial direction. If the mass of the disc is slowly varying in time, $m = m(\tau)$, where the slow time $\tau = \varepsilon t$, the mathematical model of vibration of the rotor is according to (149)

$$m\frac{d\mathbf{v}_S}{dt} + \mathbf{F}_e(\rho) = \mathbf{F}_r + \varepsilon\frac{dm}{d\tau}(\mathbf{u} - \mathbf{v}_S). \tag{442}$$

It is assumed that the motion of the rotor center S is in the Oxy plane and the corresponding differential equations of motion are due to (442)

$$m\frac{d\dot{x}}{dt} + F_{ex} = F_{rx} + \varepsilon\frac{dm}{d\tau}(u_x - \dot{x}), \tag{443}$$

$$m\frac{d\dot{y}}{dt} + F_{ey} = F_{ry} + \varepsilon\frac{dm}{d\tau}(u_y - \dot{y}). \tag{444}$$

Substituting the projections of the elastic force (441) on the x and y axis into (443) and (444), we have

$$m\frac{d\dot{x}}{dt} + F_e(\rho)\cos\varphi = F_{rx} + \varepsilon\frac{dm}{d\tau}(u_x - \dot{x}), \tag{445}$$

$$m\frac{d\dot{y}}{dt} + F_e(\rho)\sin\varphi = F_{ry} + \varepsilon\frac{dm}{d\tau}(u_y - \dot{y}). \tag{446}$$

The connection between the Cartesian coordinates x, y and the polar coordinates is (see Fig.16)

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi, \tag{447}$$

i.e.,

$$\rho = \sqrt{x^2 + y^2}, \qquad \varphi = \tan^{-1}\left(\frac{y}{x}\right). \tag{448}$$

Substituting (448) into (445), (446) we have

$$m\frac{d\dot{x}}{dt} + xF_e'(\sqrt{x^2+y^2}) = F_{rx} + \varepsilon\frac{dm}{d\tau}(u_x - \dot{x}), \tag{449}$$

$$m\frac{d\dot{y}}{dt} + yF_e'(\sqrt{x^2+y^2}) = F_{ry} + \varepsilon\frac{dm}{d\tau}(u_y - \dot{y}), \tag{450}$$

where $F'(\rho) = F(\rho)/\rho$. Introducing the complex deflection function $z = x + iy$ and the complex conjugate function $\bar{z} = x - iy$ the differential equation of vibration of the rotor is

$$m\ddot{z} + zF_e'(\sqrt{z\bar{z}}) = F_z + \varepsilon\frac{dm}{d\tau}(u_z - \dot{z}), \tag{451}$$

where $F_z = F_{rx} + iF_{ry}$, $u_z = u_x + iu_y$ and $i = \sqrt{-1}$ is the imaginary unit.

7.7.1 Rotor with small nonlinearities

For the case when the elastic force is linear and the additional forces in the system are small in comparison to the linear ones, the differential equation (451) transforms into

$$m\ddot{z} + k_1 z = \varepsilon F_z(z, \dot{z}, cc, \tau) + \varepsilon\frac{dm}{d\tau}(u_z - \dot{z}), \tag{452}$$

i.e.

$$\ddot{z} + \omega^2(\tau)z = \varepsilon f(z, \dot{z}, cc, \tau) + \frac{\varepsilon}{m}\frac{dm}{d\tau}(u_z - \dot{z}), \tag{453}$$

where k_1 is a positive coefficient, cc are the complex conjugate functions and

$$\omega(\tau) = \sqrt{\frac{k_1}{m(\tau})}, \qquad f = \frac{F}{m(\tau)}. \tag{454}$$

For the case when $\varepsilon = 0$, Eq. (453) transforms into a linear one with the constant parameter $\omega_0 = \sqrt{\frac{k_1}{m_0}}$,i.e.,

$$\ddot{z} + \omega_0^2 z = 0. \tag{455}$$

Comparing the Eq. (453) with the Eq. (455) it can be concluded that the Eq. (453) is the perturbed version of the Eq. (455). It gives us the possibility to assume the approximate solution of the Eq. (453) to be based on the solution of the Eq. (455), usually called the 'generating equation'.

The solution of the generating equation (455) and its first time derivative are

$$z = A_1 e^{i(\omega_0 t + \theta_1)} + A_2 e^{-i(\omega_0 t + \theta_2)}, \tag{456}$$

$$\dot{z} = iA_1\omega_0 e^{i(\omega_0 t + \theta_1)} - iA_2\omega_0 e^{-i(\omega_0 t + \theta_2)}, \tag{457}$$

where A, A_2, θ_1 and θ_2 are unknown constants which have to satisfy the initial conditions of motion. For the initial distance $\rho(0)$ and initial angle $\varphi(0)$, and also initial radial and circular velocities, $\dot{\rho}(0)$ and $\rho(0)\dot{\varphi}(0)$ the initial conditions for the vibration of the rotor are

$$z(0) = z_0 \equiv x(0) + iy(0) = \rho(0)e^{i\varphi(0)}, \tag{458}$$

$$\dot{z}(0) = \dot{z}_0 \equiv \dot{x}(0) + i\dot{y}(0) = e^{i\varphi(0)}(\dot{\rho}(0) + i\rho(0)\dot{\varphi}(0)). \tag{459}$$

The four unknown constants A_1, A_2, θ_1 and θ_2 have to satisfy the four algebraic equations (458) and (459).

The solution of (453) and its time derivative are assumed in the form of (456) and (457), but with time variable amplitude and phase

$$\begin{aligned} z &= A_1(t)e^{i\psi_1(t)} + A_2(t)e^{-i\psi_2(t)}, && (460) \\ \dot{z} &= iA_1(t)\omega e^{i\psi_1(t)} - iA_2(t)\omega e^{-i\psi_2(t)}, && (461) \end{aligned}$$

where

$$\psi_1(t) = \int_t \omega(\tau)dt + \theta_1(t), \qquad \psi_2(t) = \int_t \omega(\tau)dt + \theta_2(t). \tag{462}$$

Let us obtain the time derivative of (460)

$$\dot{z} = e^{i\psi_1}(\dot{A}_1 + iA_1(\omega + \dot{\theta}_1)) + e^{-i\psi_2}(\dot{A}_2 - iA_2(\omega + \dot{\theta}_2)). \tag{463}$$

The equality of (461) and (463) requires the following relation

$$e^{i\psi_1}(\dot{A}_1 + iA_1\dot{\theta}_1) + e^{-i\psi_2}(\dot{A}_2 - iA_2\dot{\theta}_2) = 0. \tag{464}$$

Substituting (460) and the time derivative of (461)

$$\begin{aligned} \ddot{z} &= i\dot{A}_1\omega e^{i\psi_1} - A_1(\omega + \dot{\theta}_1)\omega e^{i\psi_1} + \varepsilon i A_1 \frac{d\omega}{d\tau} e^{i\psi_1} \\ &\quad - i\dot{A}_2\omega e^{-i\psi_2} - A_2\omega(\omega + \dot{\theta}_2)e^{-i\psi_2} - i\varepsilon A_2 \frac{d\omega}{d\tau} e^{-i\psi_2}, && (465) \end{aligned}$$

into (453), it follows

$$\begin{aligned} & (i\dot{A}_1 - A_1\dot{\theta}_1)e^{i\psi_1} - (i\dot{A}_2 + A_2\dot{\theta}_2)e^{-i\psi_2} \\ = & \frac{\varepsilon f}{\omega} + \frac{\varepsilon}{m\omega}\frac{dm}{d\tau}(u_z - iA_1\omega e^{i\psi_1} + iA_2\omega e^{-i\psi_2}) - \frac{\varepsilon i}{\omega}\frac{d\omega}{d\tau}(A_1 e^{i\psi_1} - A_2 e^{-i\psi_2}), && (466) \end{aligned}$$

where

$$\begin{aligned} \varepsilon f &= \varepsilon f(A_1 e^{i\psi_1} + A_2 e^{-i\psi_2}, iA_1\omega e^{i\psi_1} - iA_2\omega e^{-i\psi_2}, A_1 e^{-i\psi_1} && (467) \\ &\quad + A_2 e^{i\psi_2}, -iA_1\omega e^{-i\psi_1} + iA_2\omega e^{i\psi_2}, \tau). \end{aligned}$$

The differential equations (464) and (466) represent the rewritten version of the differential equation of motion (453) into the new variables A_1, A_2, θ_1 and θ_2. After some transformation and assuming that the absolute velocity of the adding or separating is zero, we have

$$i\dot{A}_1 - A_1\dot{\theta}_1 = \frac{\varepsilon f}{2\omega}e^{-i\psi_1} - \frac{\varepsilon i}{4m}\frac{dm}{d\tau}(A_1 e^{i\psi_1} - A_2 e^{-i\psi_2})e^{-i\psi_1}, \tag{468}$$

$$i\dot{A}_2 + A_2\dot{\theta}_2 = -\frac{\varepsilon f}{2\omega}e^{i\psi_2} + \frac{\varepsilon i}{4m}\frac{dm}{d\tau}(A_1 e^{i\psi_1} - A_2 e^{-i\psi_2})e^{i\psi_2}. \tag{469}$$

Splitting (468) and (469) into real and imaginary parts the following four equations are obtained

$$\begin{aligned} \dot{A}_1 &= \frac{\varepsilon}{2\omega}\operatorname{Im}(fe^{-i\psi_1}) - \frac{\varepsilon}{4m}\frac{dm}{d\tau}(A_1 - A_2\cos(\psi_1 + \psi_2)), && (470) \\ A_1\dot{\theta}_1 &= -\frac{\varepsilon}{2\omega}\operatorname{Re}(fe^{-i\psi_1}) - \frac{\varepsilon A_2}{4m}\frac{dm}{d\tau}\sin(\psi_1 + \psi_2), && (471) \\ \dot{A}_2 &= -\frac{\varepsilon}{2\omega}\operatorname{Im}(fe^{i\psi_2}) + \frac{\varepsilon}{4m}\frac{dm}{d\tau}(A_1\cos(\psi_1 + \psi_2) - A_2), && (472) \\ A_2\dot{\theta}_2 &= -\frac{\varepsilon}{2\omega}\operatorname{Re}(fe^{i\psi_2}) - \frac{\varepsilon A_1}{4m}\frac{dm}{d\tau}\sin(\psi_1 + \psi_2). && (473) \end{aligned}$$

where Re and Im are the real and imaginary parts of the functions $fe^{\pm i\psi_1}$ and $fe^{\pm i\psi_2}$.To solve the system of differential equations (470)-(473) is not an easy task. It is at this point the averaging procedure over the period of vibration, $\langle\cdots\rangle = (1/4\pi^2)\int_0^{2\pi}\int_0^{2\pi}\cdots d\psi_1 d\psi_2$, is introduced. The averaged differential equations are

$$\begin{aligned}
\dot{A}_1 &= \frac{\varepsilon}{2\omega}\left\langle \mathrm{Im}(fe^{-i\psi_1})\right\rangle - \frac{\varepsilon}{4m}\frac{dm}{d\tau}A_1, && (474)\\
A_1\dot{\theta}_1 &= -\frac{\varepsilon}{2\omega}\left\langle \mathrm{Re}(fe^{-i\psi_1})\right\rangle, && (475)\\
\dot{A}_2 &= -\frac{\varepsilon}{2\omega}\left\langle \mathrm{Im}(fe^{i\psi_2})\right\rangle - \frac{\varepsilon}{4m}\frac{dm}{d\tau}A_2, && (476)\\
A_2\dot{\theta}_2 &= -\frac{\varepsilon}{2\omega}\left\langle \mathrm{Re}(fe^{i\psi_2})\right\rangle. && (477)
\end{aligned}$$

From Eqs. (474)-(477) the approximate amplitude-time and phase-time functions are obtained.

Analyzing the relations (475) and (477) it is evident that due to averaging the explicit influence of the reactive force on the phase angle variation is eliminated. However, the reactive force causes the amplitude variation, and the amplitude variation gives the phase variation, implicit. If the vibration of the rotor is only due to mass variation and the elastic force in the shaft, the Eqs. (474)-(477) simplify into

$$\begin{aligned}
\dot{A}_1 &= -\frac{\varepsilon}{4m}\frac{dm}{d\tau}A_1, \qquad A_1\dot{\theta}_1 = 0, && (478)\\
\dot{A}_2 &= -\frac{\varepsilon}{4m}\frac{dm}{d\tau}A_2, \qquad A_2\dot{\theta}_2 = 0. && (479)
\end{aligned}$$

The phase angles are constant and equal to the initial phase angles, i.e., $\theta_1 = \theta_1(0) = const$, and $\theta_2 = \theta_2(0) = const$. The amplitude-time variations are separated for the forward and backward motion

$$A_1 = A_1(0)\sqrt[4]{\frac{m(0)}{m(\tau)}}, \qquad A_2 = A_2(0)\sqrt[4]{\frac{m(0)}{m(\tau)}}, \tag{480}$$

where $A_1(0)$ and $A_2(0)$ are initial amplitudes and $m(0)$ is the initial mass. If the mass increases the amplitudes decrease: the faster the mass increase, the slower the amplitude decrease.

For $\theta_1(0) = \theta_2(0)$ and $A_1(0) = A_2(0)$ the approximate solution (460) is

$$z = 2A_1(0)\sqrt[4]{\frac{m(0)}{m(\tau)}}\cos((\theta_1(0) + \int\omega(\tau)dt). \tag{481}$$

The oscillation is along the x axis. If the mass variation if quadratic, $m(\tau) = (1+\tau)^2$ the oscillatory motion is according to (481)

$$x = 2A_1(0)\sqrt[4]{\frac{m(0)}{m(\tau)}}\cos((\theta_1(0) + \frac{\sqrt{k_1}}{\varepsilon}\ln|1+\tau|). \tag{482}$$

The period of vibration depends on mass variation: the period decreases with increasing of the mass.

7.7.2 Single frequency solution

Let us assume the initial position of the rotor center S in the polar coordinates as

$$\rho(0) = A_1, \qquad \varphi(0) = \theta_1. \tag{483}$$

The initial radial velocity $v_r(0)$ and circular velocity $v_c(0)$ are

$$v_r(0) = \dot{\rho}(0) = 0, \qquad v_c(0) = \rho(0)\dot{\varphi}(0), \tag{484}$$

where the angular velocity $\dot{\varphi}(0)$ is equal to the eigenfrequency of the oscillator (455), i.e., $\dot{\varphi}(0) = \omega_0 \equiv \sqrt{k_1/m_0}$. For these initial conditions

$$z(0) = \rho(0)e^{i\varphi(0)} \equiv A_1 e^{i\theta_1}, \qquad \dot{z}(0) = i\rho(0)\dot{\varphi}(0)e^{i\varphi(0)} = i\omega_0 z(0), \tag{485}$$

the solution (460) of the Eq (453) simplifies into a single frequency one, i.e.,

$$z = A_1 e^{i(\omega t + \theta_1)}. \tag{486}$$

Using (486) the Eq.(453) transforms into the system of two first order differential equations

$$\dot{A}_1 = \frac{\varepsilon}{2\omega}\,\mathrm{Im}(fe^{-i\psi_1}) - \frac{\varepsilon}{4m}\frac{dm}{d\tau}A_1, \tag{487}$$

$$A_1\dot{\theta}_1 = -\frac{\varepsilon}{2\omega}\,\mathrm{Re}(fe^{-i\psi_1}), \tag{488}$$

where $\varepsilon f = \varepsilon f(A_1 e^{i\psi_1}, iA_1\omega e^{i\psi_1}, A_1 e^{-i\psi_1}, -iA_1\omega e^{-i\psi_1}, \tau)$. Averaging the differential equations (487) and (488) over the period of vibration, we have

$$\dot{A}_1 = -\frac{\varepsilon}{4m}\frac{dm}{d\tau}A_1 + \frac{\varepsilon}{2\omega}\int_0^{2\pi}\mathrm{Im}(fe^{-i\psi_1})d\psi_1, \tag{489}$$

$$A_1\dot{\theta}_1 = -\frac{\varepsilon}{2\omega}\int_0^{2\pi}\mathrm{Re}(fe^{-i\psi_1})d\psi_1. \tag{490}$$

From the Eqs. (489) and (490) the amplitude-time and phase-time functions are obtained.

7.7.3 Rotor with strong nonlinear elastic force

Let us consider the rotor with strong nonlinear elastic force and small additional forces εF_z. The mathematical model (451) transforms into

$$m(\tau)\ddot{z} + zF_e'(\sqrt{z\bar{z}}) = \varepsilon F_z + \varepsilon\frac{dm}{d\tau}(u_z - \dot{z}), \tag{491}$$

i.e.

$$\ddot{z} + z\omega^2(\tau, \sqrt{z\bar{z}}) = \varepsilon f + \frac{\varepsilon}{m}\frac{dm}{d\tau}(u_z - \dot{z}) \tag{492}$$

where

$$\omega(\tau, \sqrt{z\bar{z}}) \equiv \omega = \sqrt{\frac{F_e'(\sqrt{z\bar{z}})}{m(\tau)}}, \tag{493}$$

and

$$f = \frac{F_z}{m(\tau)}. \tag{494}$$

For the case when $\varepsilon = 0$, the generating differential equation is obtained

$$m_0 \ddot{z} + z F_e'(\sqrt{z\bar{z}}) = 0, \tag{495}$$

where $m(0) = m_0 = const$. For the initial conditions (485) we assume the generating solution (486) with one frequency

$$\omega_0 = \omega(0) = \sqrt{\frac{F_e'(A_1)}{m_0}} \equiv \sqrt{\frac{F_e'(\rho(0))}{m_0}}. \tag{496}$$

The frequency ω_0 depends on the amplitude of vibration. Based on (486), the solution of (492) and its first time derivative are assumed as

$$z = A_1 e^{i\psi_1}, \tag{497}$$

and

$$\dot{z} = A_1 i\omega e^{i\psi_1}, \tag{498}$$

where

$$\psi_1 = \int \omega(\tau, A_1) dt + \theta_1(t), \tag{499}$$

and $A_1 \equiv A_1(t), \psi_1 \equiv \psi_1(t)$ and $\theta_1 \equiv \theta_1(t)$. The first time derivative of (497)

$$\dot{z} = (A_1 i\omega + \dot{A}_1 + iA_1\dot{\theta}_1) e^{i\psi_1},$$

is equal to (498) for

$$e^{i\psi_1}(\dot{A}_1 + iA_1\dot{\theta}_1) = 0. \tag{500}$$

Substituting (497) and (498), and also the time derivative of (498), the following first order differential equations is obtained

$$\dot{A}_1 i\omega e^{i\psi_1} + A_1 i\dot{\omega} e^{i\psi_1} - A_1\omega\dot{\theta}_1 e^{i\psi_1} = \varepsilon f + \frac{\varepsilon}{m}\frac{dm}{d\tau}(u_z - \dot{z}). \tag{501}$$

After some modification of (500) and (501) we yield the rewritten version of the differential equation (492) into two first order differential equations

$$2\dot{A}_1\omega = \mathrm{Im}(\varepsilon f e^{-i\psi_1}) - \frac{\varepsilon}{m}\frac{dm}{d\tau}(u_z \sin\psi_1 + A_1\omega) - A_1\dot{\omega}, \tag{502}$$

$$-2\omega A_1\dot{\theta}_1 = \mathrm{Re}(\varepsilon f e^{-i\psi_1}) + \frac{\varepsilon}{m}\frac{dm}{d\tau} u_z \cos\psi_1. \tag{503}$$

If the absolute velocity of adding or separating of the mass is zero, we obtain

$$\dot{A}_1 = \frac{1}{2\omega}\mathrm{Im}(\varepsilon f e^{-i\psi_1}) - \frac{\dot{m}}{2m}A_1 - \frac{\dot{\omega}}{2\omega}A_1, \tag{504}$$

$$\dot{\theta}_1 = -\frac{1}{2\omega A_1}\mathrm{Re}(\varepsilon f e^{-i\psi_1}). \tag{505}$$

where $f \equiv f(\tau, A_1 e^{i\psi_1}, A_1 e^{-i\psi_1}, A_1 i\omega e^{i\psi_1}, -A_1 i\omega e^{-i\psi_1})$. After averaging of (504) and (505) over the period of vibration of the rotor 2π, the averaged differential equations of vibration are

$$\dot{A}_1 = \frac{1}{2\omega}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Im}(\varepsilon f e^{-i\psi_1})d\psi_1 - \frac{\dot{m}}{2m}A_1 - \frac{\dot{\omega}}{2\omega}A_1, \quad (506)$$

$$\dot{\theta}_1 = -\frac{1}{2\omega A_1}\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}(\varepsilon f e^{-i\psi_1})d\psi_1. \quad (507)$$

If the additional small forces are zero, and for the initial amplitude $A_1(0)$ and initial phase angle $\theta_1(0)$, the amplitude-time and phase-time relations are

$$\theta_1 = \theta_1(0) = const. \quad (508)$$

$$A_1^4(F_e'(A_1)) = \frac{1}{m(\tau)}A_1^4(0)m(0)F_e'(A_1(0)), \quad (509)$$

Example. Let us consider the oscillations of the nonlinear rotor with linear mass variation $m = 1 + 0.01t$.The model of the oscillation is

$$m(\tau)\ddot{z} + z(z\bar{z}) = -\dot{m}\dot{z} \quad (510)$$

where according to the initial polar coordinates of the rotor center S: $\rho(0) = 1$ and $\theta(0) = \pi/4$, and velocities $\dot{\rho}(0) = 0$ and $\dot{\varphi}(0) = \omega(0) = (m_0)^{-1/2}$, the initial conditions (485) in complex form are

$$z(0) = 1e^{\frac{i\pi}{4}} = \frac{\sqrt{2}}{2}(1+i), \quad (511)$$

$$\dot{z}(0) = i\omega(0)z(0) = \frac{\sqrt{2}}{2}(-1+i). \quad (512)$$

Substituting the aforementioned values, the amplitude-time function is due to (509)

$$A_1 = \sqrt[5]{\frac{1}{1+\tau}} = \sqrt[5]{\frac{1}{1+0.01t}}. \quad (513)$$

The projections of the differential equation (510) on the axis x and y, the two coupled differential equations follow as

$$(1+0.01t)\ddot{x} + x(x^2+y^2) = -0.01\dot{x}, \quad (514)$$

$$(1+0.01t)\ddot{y} + y(x^2+y^2) = -0.01\dot{y}, \quad (515)$$

with initial conditions

$$x(0) = y(0) = 0.70711,$$

$$\dot{x}(0) = -0.70711, \qquad \dot{y}(0) = 0.70711. \quad (516)$$

The differential equations (514) and (515) are solved numerically by Runge Kutta procedure for the initial conditions (516). It is evident that the approximate analytical relation for amplitude variation $A_1 = \sqrt[5]{1/(1+0.01t)}$ is the envelope of the $x-t$ and $y-t$ diagrams (Fig.30).

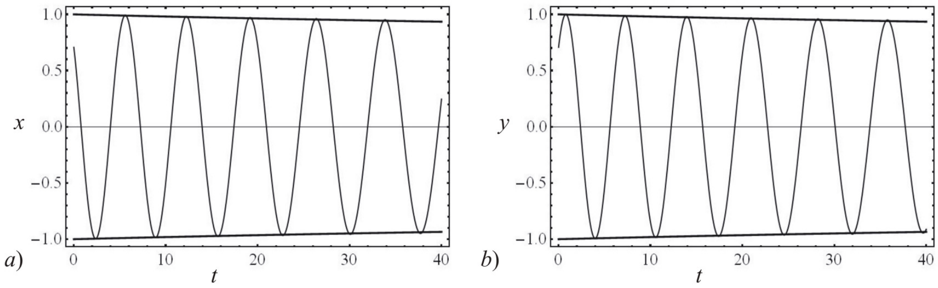


Fig.30. The $A-t$ diagram (513) as the envelope curve (thick line) of the numeric solution of (514) and (515): a) $x-t$ (thin line) and b) $y-t$ (thin line).

Remark: The procedure shown in this section can be applied to the equation

$$m(\tau)\ddot{z} + zF_e'(\sqrt{z\bar{z}}) - 2i\Omega_p g\dot{z} = \varepsilon F_z + \varepsilon\frac{dm}{d\tau}(u_z - \dot{z}),$$

where the term $(-2i\Omega_p\dot{z}g)$ defines the gyroscopic effect, Ω_p is the constant angular velocity of the rotor and g is the gyroscopic constant. The solution has the form (497) with amplitude (506) and phase (507), where

$$m\Omega^2 - 2\Omega_p g\Omega - k_1 F_e'(A_1) = 0.$$

7.8 Vibration of the two mass variable bodies system with a nonlinear connection and reactive forces

In Fig.31, a system of two mass variable bodies connected with a nonlinear connection is shown. The mass of the bodies m is continually varying in time.

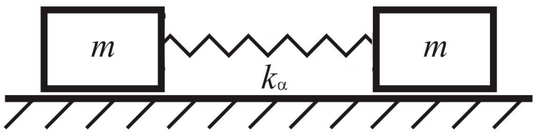


Fig.31. Model of a two-body connected system

It is assumed that The mass variation is slow and is the function of the 'slow time' $\tau = \varepsilon t$, where $\varepsilon << 1$ is a small parameter and t is time. The system has two degrees of freedom. The generalized coordinates of the system are x and y, respectively. The mathematical model of the motion is a system of two coupled second order differential equations

$$m(\tau)\ddot{x} + k_\alpha(x-y)\left|x-y\right|^{\alpha-1} = -\varepsilon\frac{dm}{d\tau}\dot{x}, \tag{517}$$

$$m(\tau)\ddot{y} + k_\alpha(y-x)\left|y-x\right|^{\alpha-1} = -\varepsilon\frac{dm}{d\tau}\dot{y}, \tag{518}$$

with initial conditions

$$x(0) = x_0, \qquad y(0) = y_0, \qquad \dot{x}(0) = 0, \qquad \dot{y}(0) = 0, \tag{519}$$

where k_α is the rigidity constant and the last term in the equations (517) and (518) are the reactive forces which act on the body. The reactive forces are caused by variation of the body mass in time.

Let us introduce the new variables

$$q = x + y, \qquad p = y - x. \tag{520}$$

The rewritten differential equations (517) and (518) are

$$\ddot{q} = -\frac{\varepsilon}{m}\frac{dm}{d\tau}\dot{q}, \tag{521}$$

$$\ddot{p} + \omega^2(\tau)p\,|p|^{\alpha-1} = -\frac{\varepsilon}{m}\frac{dm}{d\tau}\dot{p}, \tag{522}$$

and the initial conditions (519)

$$\begin{aligned} p(0) &= p_0 = x_0 - y_0, \qquad \dot{p}(0) = 0, \\ q(0) &= q_0 = x_0 + y_0, \qquad \dot{q}(0) = 0, \end{aligned} \tag{523}$$

where $\omega^2(\tau) = 2k_\alpha/m(\tau)$ and $m \equiv m(\tau)$. The two differential equations (521) and (522) are uncoupled and can be solved separately for the initial conditions (523). The differential equation (522) is already solved and discussed in the Sec.7.1.3. The suggested approximate solution for (522) can be assumed in the form of the Ateb function (310), trigonometric function (329) or Jacobi elliptic function (370). In the previous consideration it is concluded that the appropriate solutions are the first two.

Let us introduce the notation of the solution in general as

$$p = A(t)\mathrm{cg}(p(t)), \tag{524}$$

where $\mathrm{cg}(p(t))$ is a solution function whatever: Ateb, trigonometric, Jacobi elliptic, and $p(t)$ includes all of the time variable parameters, while $A(t)$ is the time variable amplitude of vibration. The Eq. (521) is rewritten as

$$\frac{d}{dt}(m\dot{q}) = 0, \tag{525}$$

and gives for the the initial conditions (523) the first time integral

$$m\dot{q} = 0. \tag{526}$$

Integrating (526) and using the initial conditions (523), we have

$$q = q_0. \tag{527}$$

Finally, using (520) and the solutions (524) and (527), the $x - t$ and $y - t$ functions are obtained

$$y = \frac{p+q}{2} = \frac{x_0 + y_0}{2} + \frac{A(t)}{2}\mathrm{cg}cn(p(t)), \tag{528}$$

$$x = \frac{q-p}{2} = \frac{x_0 + y_0}{2} - \frac{A(t)}{2}\mathrm{cg}cn(p(t)). \tag{529}$$

Example. Let us consider a numerical example where the mass variation of the bodies is $m = 1 + 0.01t$ and the connection between the bodies is with cubic nonlinearity. The differential equations of motion are

$$(1+0.01t)\ddot{x}+(x-y)^3 = -0.01\dot{x}, \quad (530)$$
$$(1+0.01t)\ddot{y}+(y-x)^3 = -0.01\dot{y}, \quad (531)$$

with initial conditions

$$x(0)=1, \qquad \dot{x}(0)=0, \qquad y(0)=3, \qquad \dot{y}(0)=0. \quad (532)$$

According to the suggested solving procedure, the solution of the auxiliary differential equation 522

$$(1+0.01t)\ddot{p}+p^3=-0.01\dot{p}, \quad (533)$$

is assumed in the form of the Ateb function and gives the amplitude vibration (328)

$$A_p = p_0\left(\frac{1}{1+0.01t}\right)^{\frac{1}{6}}, \quad (534)$$

where according to (523), $p_0=-2$. Thereby, due to (534) and the relations (528) and (529), the amplitude-time relation of the vibration is

$$A=-\left(\frac{1}{1+0.01t}\right)^{\frac{1}{6}}. \quad (535)$$

The vibrations of the masses occur around the steady state position $(x_0+y_0)/2=2$. Finally, the envelope curve for the vibrations $x-t$ and $y-t$ is

$$\bar{A}=-\left(\frac{1}{1+0.01t}\right)^{\frac{1}{6}}. \quad (536)$$

In Fig.32, the numerical solution of the system of differential equations (530) and (531) is plotted. Besides, the analytically obtained $\bar{A}-t$ curve (536) is shown. It can be seen that the difference between the numerical solution and the analytically obtained one is negligible.

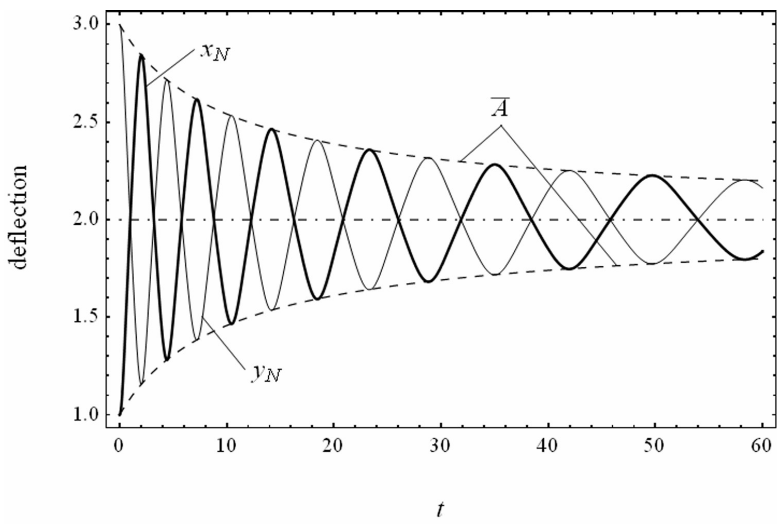


Fig.32. The x_N-t and y_N-t curves obtained numerically and the analytically obtained $\bar{A}-t$ curve for the two-mass systems with two degrees of freedom.

7.8.1 Conclusion

Based on the relations (528) and (529) and the results obtained in example, it is obvious:

1. The solution procedure developed for a single oscillator, is suitable after some exte-nsion and adoption, for solving of the motion of a system of two connected bodies with variable mass.

2. The motion of the both bodies represents the oscillations around the averaged value of the two initial displacements but in opposite directions. The oscillations of both bodies is equal but with phase distortion of 180 degrees. The properties of the oscillatory motion and the influence of the reactive force are previously discussed in Sec.7.4.

8 Conclusion, contribution in the dissertation and remarks for future investigation

In this dissertation the dynamics of the system of bodies with variable mass is considered. The dynamics of the system of particles with variable mass is generalized to the dynamics of the system of bodies with variable mass. Namely, not only the mass variation of the body but also the variation of the moment of inertia is included into investigation. It gives some additional terms in the mathematical model and available the more realistic description of the phenomena which occur during mass and moment of inertia variation.

In this dissertation various mechanisms and machines, in which the mass and the moment of inertia are varying, are shown. The practical use of the phenomena of mass and moment of inertia variation in time are applied to realize some working action and processes in industry and techniques.

In general the linear momentum and the angular momentum of the system of bodies with mass variation is expressed. The following assumption was introduced: for the mass separation, the separated body and the final (remainder) body after mass variation give an unique system, while for the case of mass addition, the added mass and the initial mass before mass variation form a close system. This assumption was the basic one for the investigations that follow. The difference between the linear momentums of the system before and after the process of mass variation are equal to the impulses caused by the external forces and torque. It is suggested to omit these impulses as they are negligible in comparison to the impulses caused by forces and torques during mass variation. It is concluded that the mass centre of the final body after body separation and also of the initial body before body addition is practically unmovable, but there is a jump like variation of the velocity of mass centre and of the angular velocity of the bodies due to body separation or addition. The dynamics of body addition or separation to the other body lasts for a very short time. It is a discontinual occurrence during which a body with the new dynamic properties is formed. Besides, during the body addition two bodies form only one. In this dissertation the dynamics of addition of bodies are treated as the plastic impact, where the impact impulses are inner values. In contrary, the dynamics of the body separation is introduced to be the inverse plastic impact.

In this dissertation due to the generalization from the system of particles with variable mass to the system of bodies with variable mass and moment of inertia, the most general motion of the bodies is the free one. So, in general the free motion of the bodies during separation or addition are treated. Namely, during body addition it is assumed that the initial body and also the added body move freely and produce the free motion of the final body. The same conclusion is evident for body separation: if the initial body has the free motion, after the separation the separated and also the final bodies have free motion. The properties of motion of the final body after separation or addition are investigated. As the special case the in-plane motion of the bodies is treated. In the dissertation for various types of motion of the body which is separating or adding the motion of the final body is analyzed. A numerous cases are shown.

The dissertation considers a very important problem of disjoining of the rotor. Namely, the initial body is a disc which moves in-plane. It is well known that due to fatigue in the material a separation of a part of the disc occurs. The separated body moves with the velocity of the initial body and the motion is in the plane of the disc. In this paper the motion of the final reminder body is determined. The obtained results give us an opportunity to prescience the motion of the final body as the function of dynamic properties of the initial disc and the separated part. The results are of interest for engineers and technicians.

In this dissertation the analytical procedure is developed for obtaining of the output parameters (velocity and angular velocity) of the final body after separation or addition. The procedure is based on the variation of the kinetic energy of the system before and after body separation or addition. The methodology is applied for the problem of separation of a pendulum rotating around a fixed axle normal to the pendulum plane.

Using the relations for variation of the linear momentum and of the angular momentum of the system of bodies before and after body separation and addition and introducing the limiting process, the mathematical model of motion of the continual mass and moment of inertia variation is obtained. The mathematical model is a system of second order differential equations with time variable parameters. It is worth to say that beside the reactive force (which was known for the particle with variable mass) also the reactive torque appears. Namely, due to mass variation in time, a reactive force occurs, and due to variation of the moment of inertia in time, a reactive torque is present. This result is also a new one in comparison to those given in the literature. In this paper the motion of the rotor on which the band is winding up is considered. Mass and moment of inertia of the disc are varying continually in time. The disc, with time variable mass and moment of inertia, as the main part of the rotor has an in-plane motion. The influence of the reactive force on the motion is discussed. The elastic properties of the shaft on the motion of the rotor with variable mass are also considered.

As a special case of motion the vibration of the body with continual mass variation is investigated. In this dissertation a new solving procedure for vibrations of the mass variable oscillators is developed. The more stronger criteria for the approximate solution to the differential equations with time variable parameters is introduced. Namely, in this paper it is required that the approximate solution satisfy not only the initial conditions but also to have the amplitude of vibration and the period which are equal or very close to the exact ones. For the first time, as it is seen in literature, the additional requirement is added: the extremal values of the first time derivative of the approximate solution have to be equal or very close to the exact velocity of vibration. The following periodic functions were applied for the asymptotic solution of vibration: the Ateb function, the trigonometric function and the Jacobi elliptic function. It is concluded that the most often used trigonometric function gives the most inaccurate result. The solution based on the trigonometric function is satisfactory for qualitative analysis of the problem, while it is not adequate for the quantitative analysis. The asymptotic solution based on the Jacobi elliptic function is much more appropriate than the trigonometric one, as the so calculated amplitude of vibration and velocity of vibration retrace the numerically obtained vibration properties of the oscillator. The main disadvantage of the solution is its complexity in calculation. The approximate solution based on the Ateb function, which is the exact solution of the corresponding differential equation with constant parameters, gives the best results. In spite of its complexity connected with some calculation difficulties, the final relations for the amplitude and frequency variation in time are quite simple and applicable in practical use. They are suggested to be used by engineers and technicians. Namely, it is concluded that the amplitude and period of vibration but also the velocity of vibration variations, according to mass change, depend only on the mass variation and the order of nonlinearity. The procedure developed in this dissertation is applied for solving the problem of vibration of the one-degree-of-freedom oscillator with time variable mass. The new results are obtained in the analysis of the influence of the reactive force on the vibration of the nonlinear oscillator with variable mass. If the mass increases the amplitude of vibration decreases. The velocity of amplitude decrease on the order of nonlinearity. If the relative velocity of mass variation is zero, the amplitude of vibration increases with mass increase. The order of nonlinearity has a significant influence on the velocity of amplitude variation.

In this dissertation the Van der Pol oscillator with time variable parameters is also investigated. In the literature the Van der Pol oscillator with constant parameters is considered. This type of oscillator has an analogy in the electrical circuits, where as it is known, the parameters (like capacity) are time variable. It suggested that the Van der Pol oscillator with time variable parameters has to be investigated, too. The new results show that the motion of the oscillator deeply depends on the properties of mass variation. A limit value for the initial mass has to be obtained applying the procedure suggested in this paper: the higher the values of the initial mass than the limit one, the amplitude of vibration of the Van der Pol oscillator with variable mass tend to zero independently on the initial displacements, while, for the case when the initial mass is smaller than the limit value, the limit cycle motion with the steady state amplitude occurs, independently on the initial displacement.

In this dissertation a contribution to the dynamics of the rotors with variable mass is given. The rotor is analyzed as an one-mass system with two-degrees-of-freedom. In general, the common amplitude and frequency of vibration is determined. The influence of the reactive force is investigated, too.

The results obtained for the mass variable one-degree-of-freedom oscillator are extended for analyzing the motion of the two-mass variable body system, with nonlinear connection and two degrees of freedom. It is proved that the motion of the bodies is the oscillatory one, around the position which depends on the initial displacements of the bodies. The oscillations are equal for the both bodies but are opposite. The analytical procedure for obtaining of the oscillatory properties of the bodies is developed. It was of special interest to determine the amplitude-time variation.

The future investigation may be directed to excited oscillatory system with time variable mass. The correlation and interaction between the parameters of the excitation and of the mass variation on the motion of the body have to be investigated.

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