CANDIDATE DISSERTATION FOR THE DEGREE PH.D

SPLINE APPROXIMATION FOR THE SOLUTION OF TWO-DIMENSIONAL ELASTIC-PLASTIC PROBLEM

written by
ALI SHAUKET MOHAMED

supervised by
Associate Prof. RUSZLÁN FARZAN PH.D

Computer Center of
Loránd Eötvös University
Budapest

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TO MY FAMILY

MAGYAR
TUDOMÁNYOS AKADÉMIA
KÖNYVTÁRA

CONTENTS

Introduction

The mathemetical theory of elasticity (which has practicaly important applications in architecture, engineering, physics and all other useful arts in which the material of construction is solid) owes its development to demand for more realistic methods of determing the safty factors of structures or machine parts. The rapid and intensive development of computer engineering has aroused the considerable interest of researchers for the development of effective numerical methods for the solution of elasticity problems. Together with the methods in continuum mechaincs and engineering calculations.

Oneof the important and complicated cases in the elasticity problems are those in which a body has been strained by a load or subjected to very great pressure, and is set free, the set gradually diminishes. The body never returns to its primitive condition, and the ultimate deformation is the "permanent set", the part of strain that gradually disappears is called elastic after-strain. In other elastic problems some plasticity of the material appears as soon as the limit of linear elasticy is exceeded. This leads to the elastic-plastic problems which are still under considerable investigation.

The particular problem investigated in this dissertation is an elastic cylinder of rotational symmetry. The cylinder is subjected to a torque applied at both ends. The torque of sufficient magnitude is to cause portions of the material of the cylinder to yield. The material is assummed to be isotropic and yields according to the condition of von Mises. This condition Means that below the yield point the behavior is perfectly elastic and after yield the material exhibits perfect plasticity.

The yield condition requires that the maximum shearing stress has the constant value equal to the yield stress in pure shear.

Because of the axial symmetry of the cylinder the formulation of the problem can be reduced to a two-dimensional domain.

In obtaining the elastic and plastic equations it is assumed [1] that the only two non-zero components of the stress tensor are $\tau_{r\vartheta}$ and $\tau_{\vartheta z}$ and there are given in terms of stress function u by

$$\tau_{r\vartheta} = - \frac{1}{r^2} \frac{\partial u}{\partial z}, \tau_{\vartheta z} = + \frac{1}{r^2} \frac{\partial u}{\partial r}.$$

The elastic problem is expressed by the linear second-order elliptic equation with variable coefficients [1], [38], [39]. The boundary conditions are mixed: On two parts of the boundary (the central and lateral lines) the values of the function have been given (first-kind boundary conditions). On other parts of the boundary (the right and left sides) the derivative of the function have been given (second-kind boundary conditions). On one part of the boundary (central line) the coefficients of the elastic equation tends to infinity (for more detail see ch.I).

The plastic problem is expressed by the equalization of the absolut value of the plastic stress function to a constant depends on the material. This problem has the form of Cauchy problem for non-linear first-order hyperbolic equation (see ch.II).

The main problem is to obtain the free boundary separating the elastic and plastic domains. On this free boundary the solution of the elastic-plastic problem and its gradient are continuous.

We now give a short description of the contents of the dissertation.

In chapter I, we have constructed a special spline function of fourth degree for the approximation of the elastic problem in two different domains. This spline, which satisfies the equation of the problem in the every subdomain with the main boundary conditions, has only four coefficients. We have proved that such spline function exist and it is continuous over the whole given domains and not only at the grid-points of the meshs (see theorems 1.1.1 and 1.3.1). Furthermore we show that this spline gives better approximate solution to the problem than the piecewise linear spline. We have proved the convergence of the spline for the one and two-dimensional elastic problems and for the last case in two different domains corresponding to the constant and variable diameters. The only property used for the convergence that our spline belong to the W_2^1-Sobolev space. The coefficients of the spline are obtained by Ritz method with the using of Lagrange multiplyers.

In chapter II we have constructed by finite difference method two schemes-explicit and implicit-for the solution of non-linear first-order hyperbolic equation with given boundary conditions which expresses the plastic problem in a cylinder of variable diameter. The accuracies, advantages and the disadvantages of these two schemes are discussed.

Following the classical iteration methods, in chapter III, we have proposed a new iteration processes for the solution of the one-dimensional elastic-plastic problem. As a result of these iteration methods we have found out the position of the free boundary separating the elastic and plastic domains. The quadratic convergence of these iterations has been proved.

In chapter IV we have investigated the solution of the two-dimensional free boundary problem in rectangular domain with given boundary conditions as a special case of the elastic-plastic problem formulated above. The solution of the boundary value problem of the second-order elliptic diffrential equation which expresses the elastic problem has been obtained by the analytic rèpresention in the form of infinity series. Then we constructed iteration process to obtain the free boundary of the elastic-plastic problem. The convergence of the iteration is shown.

In chapter V we considered the solution of the elastic-plastic problem in the cylinder of variable diameter with the using of more general boundary conditions. Iteration process is proposed to obtain the unknown free boundary. The plastic problem need be solved only one time during all the iteration process. The elastic problem must be solved in every step of the iteration process in the corrigated domain with the new approximate boundary, using the value of the plastic function on this boundary line. The elastic and plastic problems can be solved by the methods introduced in the first two chapters. We proposed a method for obtaining the first approximation of the free boundary too. The next approximations of the free boundary can be found from the comparsion between the gradients of solutions of the elastic and plastic problems. Iteration processes based on qualititive and quantitive strategies have been constructed. The convergence of these iterations is discussed.

Most of the dissertation contents were presented in form of the papers ([38], [39], [40], [41], [42], [43]), and lectures before the following conferences and seminars:

(i) Approximate Solution for an Elastic Problem in a cylinder with Circular Section by Spline Method. Seminar held at Computer Center of ELTE, 1987.

(ii) Some Results About Solving Elastic-Plastic Problem of Free Boundary (in Russain), with R. Farzan. "Taung über Probleme und Methoden der Mathematischen Physik" Conference held in DDR, 1988.

(iii) Spline Functions Solution of Elastic Problem. "Fourth Conference of Program Designers" held in Budapest, 1988.

(iv) Spline Method for one Elastic Problem in the Body with Cylinderical Symmetry. Seminar organized by Veszprém University in cooperation with "Society of Educational Science" Veszprém county organization, "Bolyai János Matematical Society" Veszprém county branch, and "Neumann János Society of Computer Science" Veszprém county organization. Held at Veszprém University, 1989.

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CHAPTER I

Aproximate Solution For An Elastic Problem By Spline Functions

1.0. Comments and literatures. In developing the polinomial splines one may consider the approximating functions

$$S(x) = c_1\phi_1(x) + c_2\phi_2(x) +...+c_n\phi_n(x),$$

that satisfy a set of mixed (i. e. both interpolitory and smoothness) constraints. Let the functions $\phi_i(i = \overline{1,N})$ are linearly independent on the interval $[a,b]$ so they span a N-dimensional subspace X_N of $C[a,b]$. One can suppress the smoothness constraints by building the desired smoothness into the basis ϕ_i, leaving a set of pure interpolatory constraints to fulfill. At other times one can start with the space X_N and ask what kind of interpolation problems can be uniquely solved by members of this linear space. Following these ways one can construct different types of polynomial splines.

A spline of order 2n is simple when there is at most a jump discontinuity in the (2n-1)th derivative at a mesh point. When jumps in derivative of order greater than 2n-k-1 are permitted at an interior mesh point x_i, the spline is said to be of deficiency k at x_i. If the spline is of deficiency k at all interior mesh points, it is said to be of deficiency k where $0 \le k \le n$.

The effectiveness of the spline in approximation can be explained to a considerable extend by its striking convergence properties. If $f^{(q)}(x)$ (q th derivative of the function f) is continuous on $[a,b]$ (q = 0,1,2, or4), it is found that $S_\Delta(f;x)$ converges to $f(x)$ on a sequence of mesh at least as rapidly as the approach to zero of qth power of the mesh norm $||\Delta|| = \max_j h_j$. Similarly, $S_\Delta^{(p)}(f;x)$ converges to $f^{(p)}(x)$ $(0 \le p \le q)$ at least as rapidly as the (q-p)th power of the mesh norm. The construction and convergence of splines in two-dimensional domains are similar to that of one-dimensional splines. For the spline theory and its applications one may refer to the work of [9], [10], [11], [12], [13].

The elastic problem under consideration is represented by the following minimization problem: The unknown u satisfies

$$u \in H_A \text{ and } I(u) = \min_{\forall v \in H_A} I(v),$$

where the set H_A of admissible functions is closed convex subset of Hilbert space W, and the functional I of the system takes the form

$$I(v) = a[v,v] - 2f(v),$$

where a [,] is a symmetric bilinear form (a(v,w) = a(w,v)) and f is a linear form, both defined and continuous over the Hilbert space W. As regards existence and uniqueness properties of the solution of this problem, it is assumed in addition that

(i) the space W is complete.

(ii) the bilinear form a[,] is symmetric and v elliptic, in the sense that

$$\exists\, \alpha > 0,\ \forall v \in W,\ \alpha||v||^2 \le a[v,v].$$

Ritz method is used to obtain the coefficients of the spline with the use of Lagrange multiplyers, that is we minimize the functional

$$I(v) = a[v,v] - 2f(v) + \sum \lambda_i g_i$$

with respect to the coefficients of the spline, where λ_i are the Lagrange multiplyers, g_i are functions which expresses the conditions of the continuity of the splines (see 1.2.39).
For Ritz method and the minimization problems we refer to the books [2], [5], [6], [14].

1.1. Introduction. In this chapter we shall use spline functions method to obtain an approximate solution for an elastic axisymmetric cylinder problem. The cylinder subjected to a torque T applied at both ends (FIG.1.1).

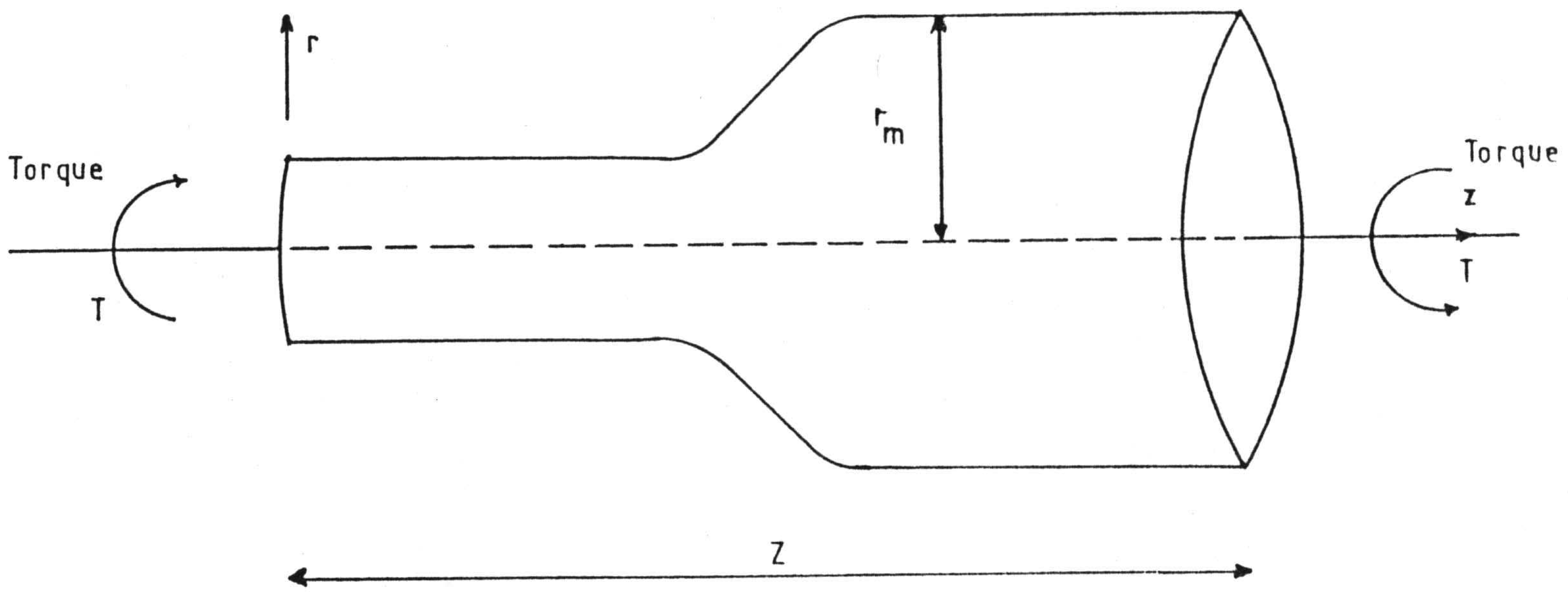

FIGURE 1.1. An axisymmetric cylinder

The formulation of the problem [1] can be done in a two-dimensional domain in the rz-plane because of supposed axial symmetry. The problem reduced to find function u which must satisfy the elliptic differential equation:

$$\text{(1.1.1)} \qquad Au := -\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial u}{\partial z}\right) = 0, \quad r,z \in \Omega,$$

together with the boundary conditions:

$$\text{(1.1.2)} \qquad u\big|_{\partial\Omega} = F(z), \quad \text{on the surfaces of domain,}$$

$$\text{(1.1.3)} \qquad -\frac{\partial u}{\partial z} = \phi_1(r), \quad \text{on the left side of the domain,}$$

$$\text{(1.1.4)} \qquad \frac{\partial u}{\partial z} = \phi_2(r), \quad \text{on the right side of the domain.}$$

Notice the above boundary conditions are more general than those given in [1] where $F(z)$ = constant, $\phi_1(r) = \phi_2(r) = 0$.

1.2.1. It is desirable to consider the elastic axisymmetric problem in the case when the diameter of the domain is constant (FIG.2.1).

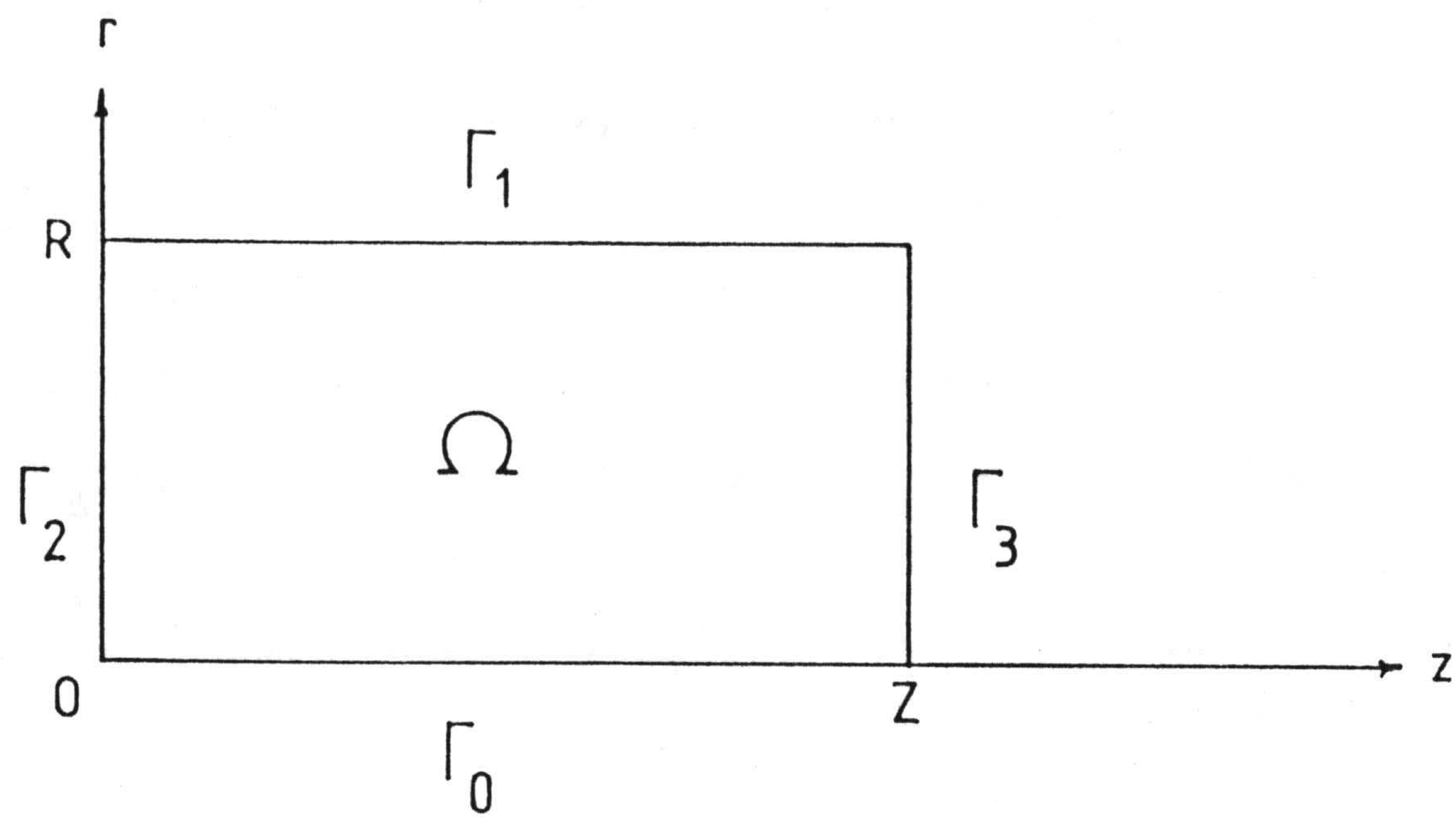

FIGURE 2.1. The domain Ω

The problem can be summarized as follows: The function u must satisfy equation(1.1.1) on Ω with the boundary conditions

(1.2.1) $\quad u = 0, \text{ on } \Gamma_0,$

(1.2.2) $\quad u = F(z), \text{ on } \Gamma_1,$

(1.2.3) $$\frac{\partial u}{\partial n} = - \frac{\partial u}{\partial z} = \phi_1(r) \text{ , on } \Gamma_2,$$

(1.2.4) $$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = \phi_2(r) \text{ , on } \Gamma_3,$$

where Γ_i $(i = \overline{0,3})$ are as shown in FIG.2.1.

1.2.2. The construction of spline functions. Let the domain Ω (FIG.2.1) is divided into (n x m) rectangular subdomains by lines parallel to the rectangular coordinates. Let

$$(1.2.5) \qquad \Omega \cup \left(\bigcup_{i=0}^{3} \Gamma_i\right) = \bar{\Omega} = \bigcup_{i,j} \bar{G}_{i,j}, \; i = \overline{0,n-1}, \; j = \overline{0,m-1},$$

$n \geq 1$, $m \geq 2$, where $\bar{G}_{i,j} = \left\{z,r \,|\, z_i \leq z \leq z_{i+1}, \; r_j \leq r \leq r_{j+1}, \right.$

$\left. z_i = i.h_z, \; r_j = j.h_r \right\}$; $z_o = r_o = 0$, $z_n = Z$, $r_m = R$.

Let the solution of (1.1.1) be approximated by the spline function:

$$(1.2.6) \qquad u(z,r) \approx S_\Delta(z,r) := S_{ij}(z,r), \text{ in } \bar{G}_{i,j}.$$

In constructing $S_\Delta(z,r)$ the following steps have been considered

(i) Suppose $S_\Delta(z,r)$ in every subdomain G_{ij} is a polynomial of two variables.

(ii) Let this polynomial satisfy (1.1.1) in the interior of the subdomains:

$$(1.2.7) \qquad A\, S_{i,j}(z,r) = 0, \text{ in } G_{i,j}.$$

(iii) The minimal degree polynomial of two variables obtained that satisfy (1.2.7) nontrivially is

$$(1.2.8) \qquad S_{i,j}(z,r) = A_{i,j}\, r^4 + B_{i,j}\, r^4 z + C_{i,j}\, z + D_{i,j}$$

where $A_{i,j}$, $B_{i,j}$, $C_{i,j}$ and $D_{i,j}$ are constants.

(iv) The spline functions (1.2.8) are supposed to be continuous in Ω.

The main reason of choosing the fourth-degree polynominal for constructing the spline is deduced from the following: By formula (1.2.8) the exact solution $u = Fr^4/R^4$ is obtained, for the one-dimensional elastic problem $\left(Au = -\partial/\partial r(1/r^3 \partial u/\partial r) = 0,\right.$ with the boundary condition $u(0) = 0$, $\left. u(R) = F\right)$, and for the problem $\big((1.1.1),(1.2.1)-(1.2.4)\big)$ in the case that $F(z) = \text{const.}$, $\phi_1(r) = \phi_2(r) = 0$. Therefore, we can suppose that (1.2.8) gives a better approximate solution to (1.1.1) with the boundary conditions (1.2.1)-(1.2.4), than a piecewise bilinear approximation (standard two-dimensional bilinear function with four coefficients).

Let us rewrite formula (1.2.8) in the analogous form

(1.2.9) $S_{i,j}(z,r) = a_{i,j}(r^4 - r_j^4) + b_{i,j}(r^4 - r_j^4)(z - z_i) + c_{i,j}(z - z_i) + d_{i,j}$, $i = \overline{0,n-1}$, $j = \overline{0,m-1}$,

where $a_{i,j}$, $b_{i,j}$, $c_{i,j}$ and $d_{i,j}$ are constants.

Suppose that (1.2.9) satisfies the main boundary conditions (1.2.1),(1.2.2), but not by all means the natural boundary conditions (1.2.3),(1.2.4) [2]. Therefore, the formula (1.2.9) have the following forms in the first and the last rows in $\bar{\Omega}$

(1.2.10) $S_{i,0}(z,r) = a_{i,0}r^4 + b_{i,0}r^4(z-z_i)$, $i = \overline{0,n-1}$,

(1.2.11) $S_{i,m-1}(z,r) = a_{i,m-1}(r^4-R^4) + b_{i,m-1}(r^4-R^4)(z-z_i) + \tilde{F}(z)$, $i = \overline{0,n-1}$,

where $\tilde{F}(z)$ is the piecewise linear approximation of the function $F(z)$.

For the other rows in $\bar{\Omega}$ (i.e. $j = \overline{1,m-2}$) the spline functions have the form (1.2.9).

THEOREM 1.2.1. The continuous spline functions (1.2.9) exist in $\bar{\Omega}$ and satisfy the main boundary conditions (1.2.1)-(1.2.2).

PROOF. The conditions that the spline functions (1.2.9) be continuous in $\bar{\Omega}$ and take on the main boundary function values are expressed by the following system of equations

(1.2.12) $a_{i,j} + b_{i,j}(z_{i+1}-z_i) - a_{i+1,j} = 0$, $i = \overline{0,n-2}$, $j = \overline{0,m-1}$,

(1.2.13) $c_{i,j}(z_{i+1}-z_i) - d_{i,j} - d_{i+1,j} = 0$, $i = \overline{0,n-2}$, $j = \overline{1,m-2}$,

(1.2.14) $a_{i,0}\, r_1^4 - d_{i,1} = 0$, $i = \overline{0,n-1}$, (1.2.15) $b_{i,0} r_1^4 - c_{i,1} = 0, \quad i = \overline{0,n-1},$

(1.2.16) $a_{i,j} (r_{j+1}^4 - r_j^4) + d_{i,j} - d_{i,j+1} = 0, i = \overline{0,n-1}, j = \overline{1,m-3},$

(1.2.17) $b_{i,j} (r_{j+1}^4 - r_j^4) + c_{i,j} - c_{i,j+1} = 0, i = \overline{0,n-1}, j = \overline{0,1,m-3},$

(1.2.18) $a_{i,m-2}(r_{m-1}^4 - r_{m-2}^4) + d_{i,m-2} - a_{i,m-1}(r_{m-1}^4 - r_m^4) =$

$= F(z_i), \ i = \overline{0,n-1},$

(1.2.19) $b_{i,m-2}(r_{m-1}^4 - r_{m-2}^4) + c_{i,m-2} - b_{i,m-1}(r_{m-1}^4 - r_m^4) =$

$$= \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}, \ i = \overline{0,n-1}.$$

The system (1.2.12)-(1.2.19) has 2(2mn-m-2n+1) equations with 4(mn-n) coefficients. The analysis shows that there exist (mn-n-m+1) dependent equations in the system. By deleting the dependent equations from the system (using the equations (1.2.14),(1.2.16),(1.2.18) for i=0 only) we get the following $\overline{M}$ system (of $\overline{N}' = (3mn-3n-m+1)$ independent equations with 4(mn-n) unknown coefficients)

$a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0, \quad i = \overline{0,n-2}, \quad j = \overline{0,m-1},$

$c_{i,j}(z_{i+1} - z_i) - d_{i,j} - d_{i+1,j} = 0, \quad i = \overline{0,n-2}, \quad j = \overline{1,m-2},$

$a_{0,0} r_1^4 - d_{0,1} = 0,$

$b_{i,0} r_1^4 - c_{i,1} = 0, \quad i = \overline{0,n-1},$

$a_{0,j} (r_{j+1}^4 - r_j^4) + d_{0,j} - d_{0,j+1} = 0, \quad j = \overline{1,m-3},$

$b_{i,j} (r_{j+1}^4 - r_j^4) + c_{i,j} - c_{i,j+1} = 0, \quad i = \overline{0,n-1}, \quad j = \overline{1,m-3},$

$a_{0,m-2}(r_{m-1}^4 - r_{m-2}^4) + d_{0,m-2} - a_{0,m-1}(r_{m-1}^4 - r_m^4) = F(0),$

$$b_{i,m-2}(r_{m-1}^4 - r_{m-2}^4) + c_{i,m-2} - b_{i,m-1}(r_{m-1}^4 - r_m^4) = \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i},$$

$i = \overline{0,n-1}.$

By induction we get that the rank of matrix P (P is the matrix of coefficients for the system $\bar{M}$) is equal to the rank of matrix $\bar{P}$ ($\bar{P}$ is obtained by adjoining to matrix P the column made up of the right hand side terms of the system $\bar{M}$). Therefore the system $\bar{M}$ is consistent [3] and its solution exists but not unique. Hence we have now a family of continuous splines in $\bar{\Omega}$.

1.2.3. Let us give some useful results from the theory of the bilinear spline functions by which we can construct spline functions solve the one and two-dimensional elastic problems mentioned before. These splines are equivalent to the form (1.2.9) but they have another forms.

(i) The one variable continuous piecewise linear spline functions on the interval [O,R] can be defined as follows [4]

$$(1.2.20) \qquad S_\Delta(r) = a_o r + b_o + \sum_{i=1}^{n-1} a_j(r - r_j)_+ ,$$

where

$$(r - r_j)_+ := \begin{cases} r - r_j, & r \geq r_j \\ 0, & r < r_j \end{cases} .$$

The spline functions constructed above, for one-dimensional elastic problem, can be written as follows

$$(1.2.21) \qquad S_\Delta(r) = a_o(r^4 - r_j^4) + b_o + \sum_{j=1}^{m-1} a_j(r^4 - r_j^4)_+ .$$

(ii) The two variables continuous piecewise bilinear spline functions on the domain [O,Z] x [O,R] can be defined as follows [4]

$$(1.2.22) \qquad S_\Delta(z,r) = P_{o,o}(z,r) + \sum_{j=1}^{m-1} Q_i(r)(z - z_i)_+ +$$

$$+ \sum_{j=1}^{m-1} P_j(z)(r - r_j)_+ + \sum_{i=1}^{n-1}\sum_{j=1}^{m-1} e_{i,j}(z - z_i)_+(r - r_i)_+ ,$$

where $(z - z_i)_+ := \begin{cases} z - z_i, & z \geq z_i \\ 0, & z < z_i \end{cases}$, and $(r - r_j)_+$ as defined above,

and

$$P_{0,0}(z,r) := a_{0,0} r + b_{0,0} r z + c_{0,0} z + d_{0,0} ,$$

$$Q_i(r) := a_i r + c_i , \quad P_j(z) := b_j z + d_j .$$

The spline functions constructed for the two-dimensional problem (1.1.1) can be written as follows

$$(1.2.23) \qquad S_\Delta(z,r) = \bar{P}_{0,0}(z,r) + \sum_{i=1}^{n-1} (a_i r^4 + c_i)(z - z_i)_+ + \sum_{j=1}^{m-1} (b_j z + d_j)(r^4 - r_j^4)_+ + \sum_{i=1}^{n-1} \sum_{j=1}^{m-1} e_{i,j} (z - z_i)_+ (r^4 - r_j^4)_+ ,$$

where $\quad \bar{P}_{0,0}(z,r) := a_{0,0} r^4 + b_{0,0} r^4 z + c_{0,0} z + d_{0,0}$.

Formula (1.2.23) satisfies the main boundary conditions (1.2.1)-(1.2.2), and so the following splines we get in the first and the last rows in $\bar{\Omega}$

$$(1.2.24) \qquad S_{i,0}(z,r) = a_{0,0} r^4 + b_{0,0} r^4 z + \sum_{i=1}^{n-1} a_i r^4 (z - z_i)_+ ,$$

$$(1.2.25) \qquad S_{i,m-1}(z,r) = (b_{m-1} z + d_{m-1}) \left(\frac{r^4}{R^4} - 1 \right)_+ r_{m-1}^4 + \sum_{i=1}^{n-1} e_{i,m-1} \left(\frac{r^4}{R^4} - 1 \right)_+ (z - z_i)_+ r_{m-1}^4 + \tilde{F}(z) \frac{r^4}{R^4} .$$

It can be shown that the spline functions (1.2.23)-(1.2.25) are equivalent to the spline functions (1.2.9)-(1.2.11) and have the same number of unknown coefficients (mn+m-n-1).

1.2.4. The existence of general solution of analytical problem. We have a space of continuous spline functions. These splines are satisfying the main boundary conditions and contain (mn-n+m-1) free coefficients. Ritz method has been chosen to obtain the approximate solution of (1.1.1) together with using the natural boundary conditions (1.2.3)-(1.2.4).

It is need to show that the generalized solution of (1.1.1) exists in the Hilbert space. For this reason we rewrite the original problem to an equivalent form as follows: Let

(1.2.26) $u = \bar{u} + \tilde{u}$, $\tilde{u} = F(z)\ r^4/R^4$,

and so

(1.2.27) $Au = A\bar{u} + A\tilde{u} = 0$.

From (1.1.1),(1.2.1)-(1.2.4),(1.2.26), and (1.2.27) the following non-homogenous problem for the function $\bar{u}$ is obtained

(1.2.28) $A\bar{u} = \bar{F} := r/R^4 .\ d^2F(z)/dz^2$,

(1.2.29) $\bar{u}(z,0) = \bar{u}(z,R) = 0$,

(1.2.30) $\frac{\partial \bar{u}}{\partial z}(0,r) = -\bar{\phi}_1(r) := -\phi_1(r) - r^4/R^4\ \frac{dF}{dz}(0)$,

(1.2.31) $\frac{\partial \bar{u}}{\partial z}(Z,r) = \bar{\phi}_2(r) := \phi_2(r) - r^4/R^4\ \frac{dF}{dz}(Z)$.

Obviously this problem is equivalent to the original problem. Therefore the domain of definition of the operator A of (1.1.1) become

(1.2.32) $D(A) = \left\{ v \mid Av \in L_2(\Omega),\ v(z,0) = v(z,R) = 0 \right\}$.

Let us define the bilinear functional as follows [2,5,6]

(1.2.33) $[u,v] = \int_\Omega \frac{1}{r^3} \left[\frac{\partial u}{\partial r} \frac{\partial v}{\partial r} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \right] dz\ dr$.

This bilinear functional has all the properties of the inner product in the Hilbert space. Therefore the energetical Hilbert space H_A can be defined as follows (H_A can be also denoted as the weighted Sobolev space [1]).

(1.2.34) $H_A = \left\{ v \mid \frac{i}{r^{3/2}} \left[\left(\frac{\partial v}{\partial r}\right)^2 + \left(\frac{\partial v}{\partial z}\right)^2 \right]^{1/2} \in L_2(\Omega),\ v(z,0) = \right.$

$\left. = v(z,R) = 0 \right\}$.

If $u,v, \in H_A$ then the product (1.2.33) makes sense [2,5,6].

Let the norm of v in H_A be as follows

(1.2.35) $[v] = [v,v]^{1/2}$.

Therefore if, for every $v \in H_A$, $\bar{u}$ satisfies the equality

(1.2.36) $[\bar{u},v] = (\bar{F},v):= -\int\limits_{\Omega} \frac{r}{R^4}\frac{dF}{dz}\frac{\partial v}{\partial z}\,dz\,dr +$

$$+\int_0^R \frac{1}{r^3}\phi_1(r)\,v(0,r)dr + \int_0^R \frac{1}{r^3}\phi_2(r)\,v(Z,r)\,dr,$$

then $\bar{u}$ is the generalized solution of (1.2.28)-(1.2.31) [2,5]. It can be seen than the operator $(\bar{F},v)$ is bounded in H_A

(1.2.37) $|(F,v)| \le c[v],$

if the following are satisfied

(i) $F(z) \in W_2^1[0,Z]$,

(ii) $\frac{1}{r^{3/2}}\phi_K(r) \in L_2[0,R]$, $k = 1,2.$

Now we can say that the generalized solution $\bar{u}$ of (1.2.28) exists in the Hilbert space H_A[2]. According to that the generalized solution $u = \bar{u} + \tilde{u}$ of (1.1.1) exists too in the energetical space.

1.2.5. The properties of spline functions. The approximation of the generalized solution is sought in a set of splines by the Ritz method. From the variational method theory [2,6], $\bar{u}$ is minimizing the functional $I(v)$

(1.2.38) $I(v) = [v,v]-2(\bar{F},v).$

To use the conditions of continuity of the set of the splines S_Δ, we rewrite the form of the functional (1.2.38) as follows

(1.2.39) $I(S_\Delta) = [S_\Delta,S_\Delta] - 2(\bar{F},S_\Delta) + \sum_{k=1}^{\bar{N}} \lambda_k g_k,$

where λ_k $\left(k = \overline{1,\bar{N}}\right)$ are the Lagrange multiplyers, g_k are left hand side functions of the independent system of equations $\bar{M}$. So the approximate solution of the original problem is the spline functions which minimizes the functional $I(S_\Delta)$. Notice that in the system of equations, which will appear as a consequence of minimizing (1.2.39), it is needed to use the equations $g_k = 0$ too according to the theory of Lagrange multiplyers.

Remark. We don't need the Lagrange multiplyers in case of using the spline in form of (1.2.23). But we know as mentioned befor that the spline in forms of (1.2.9)-(1.2.11) is more useful for our purposes.

Let us consider the spline functions density for the one and the two-dimensional problems. First we are going to introduce the one dimensional problem briefly. The problem reduced to find the function u which must satisfy the differential equation [1]

$$Au = -\frac{d}{dr}\left(\frac{1}{r^3}\frac{du}{dr}\right) = 0, \tag{1.2.40}$$

with the boundary conditions $u(0) = 0$, $u(R) = F$.

The generalized solution of (1.2.40) is approximated by the following spline functions

$$S_\Delta(r) = S_j(r) = a_j(r^4 - r_j^4) + d_j, \quad r_j<r<r_{j+1}, \quad d_0 = 0. \tag{1.2.41}$$

S_Δ is supposed to be continuous on (o,R). From the connections between the polynomials (1.2.41) the following system of equations is obtained

$$a_j(r_{j+1}^4 - r_j^4) + d_j = d_{j+1}. \tag{1.2.42}$$

The properties and definitions introduced to the spline functions in the two-dimensional problem can be considered in the one-dimensional problem too. Hence, the approximate solution of (1.2.42) can be obtained by using the similar functional to that of (1.2.39). The functions g_K appeared in (1.2.39) now have the following form

$$g_K = a_K\,(r_{K+1}^4 - r_K^4) + d_K - d_{K+1} \tag{1.2.43}$$

Note. It is easy to see that $S_\Delta(r) \in W_2^1 \cap C$ and $S_\Delta(r) \in H_A \cap C$.

THEOREM 1.2.2. The set of the splines $S_\Delta(r)$ (1.2.41) in a limit sense are dense in H_A. That is for every $f(r) \in H_A$ there exists $\bar{S}_\Delta(r)$ such that

$$[f(r)-\bar{S}_\Delta(r)] \to 0 \text{ if } h_r \to 0 \ (m \to \infty),$$

in the mean time for $f(r)$ and $\bar{S}_\Delta(r)$ the following are true

$$||f(r) - \bar{S}_\Delta(r)||_{L_2} \to 0 \quad \text{if} \quad h_r \to 0,$$

$$||f(r) - \bar{S}_\Delta(r)||_{W_2^1} \to 0 \quad \text{if} \quad h_r \to 0,$$

where $h_r = \max |r_{j+1} - r_j|$.

PROOF. Let the form of the spline functions on (r_j, r_{j+1}) as follow

$$(1.2.44) \quad S_\Delta(r) = \bar{S}_j(r) := f_j + \frac{f_{j+1} - f_j}{r^4_{j+1} - r^4_j} (r^4 - r^4_j),$$

where $f_j = f(r_j)$. It is obvious that (1.2.44) is continuous on all the interval (0,R).

We want to remark that from lemma 1.2.1.(the prove is in page **31**) it follows that $f(r) \in W_2^1$. Therefore there exists at least one point r_c on the interval (r_j, r_{j+1}) such that $\frac{df}{dr}(r_c)$ exists and is bounded. The Taylor's expansion with the Peano's form of the remainder at the neighborhood of r_c on (r_j, r_{j+1}) [7]

$$(1.2.45) \qquad f(r) = f(r_c) + (r - r_c) \frac{df}{dr}(r_c) + o(|r - r_c|).$$

Formula (1.2.45) is used to get the expansions of f_j, f_{j+1} in (1.2.44). Subtracting (1.2.44) from (1.2.45) we get

$$(1.2.46) \qquad |f(r) - \bar{S}_\Delta(r)| = \frac{df}{dr}(r_c)\, O(h_r^2) + o(h_r).$$

For the L_2-norm on all the interval (0,R), we get

(1.2.47) $$||f(r) - \bar{S}_\Delta(r)||_{L_2} = o(h_r).$$

The same results can be obtained for the C-norm too.

Let us differentiate (1.2.44) and (1.2.45)

(1.2.48) $$\frac{d\,\bar{S}_j(r)}{dr} = \frac{f_{j+1} - f_j}{r^4_{j+1} - r^4_j}\,4r^3.$$

(1.2.49) $$\frac{d\,f(r)}{dr} = \frac{df}{dr}(r_c) + \frac{1}{|r - r_c|}\,o\,(|r - r_c|).$$

Subtracting (1.2.48) from (1.2.49) we obtain

(1.2.50) $$\left|\frac{d\,f(r)}{dr} - \frac{d\,S_j(r)}{dr}\right| = \frac{df}{dr}(r_c)\,O(h_r) + h_r^{-1}\,o(h_r).$$

Since $\frac{d\,\bar{S}_j(r)}{dr}$ is bounded on $(0,R)$ and $f(r) \in W_2^1$ we have

(1.2.51) $$\left\|\frac{d\,f(r)}{dr} - \frac{d\,\bar{S}_j(r)}{dr}\right\|_{L_2(0,R)} = h_r^{-1}\,o(h_r).$$

Clearly from (1.2.47) and (1.2.51) we get

(1.2.52) $$||f(r) - \bar{S}_\Delta(r)||_{W_2^1} = h_r^{-1}\,o(h_r).$$

For the norm in the H_A space we have

(1.2.53) $$[f(r) - \bar{S}_\Delta(r)]^2 := \int_0^R \frac{1}{r^3}\left[\frac{d}{dr}\,(f(r) - \bar{S}_\Delta(r))\right]^2 dr.$$

The norm (1.2.53) can be written as a sum of integrals over the subdomains (r_j, r_{j+1}). Because $f(r) \in H_A$ the Taylor's expansion of f in the neighborhood of $r = 0$ become:

(1.2.54) $$f(r) = b_3 r^3 + b_4 r^4 + \ldots$$

From (1.2.48)-(1.2.49) and (1.2.53)-(1.2.54) we obtained

(1.2.55) $$\int_0^{h_1} \frac{1}{r^3}\left[\frac{d}{dr}\,(f(r) - \bar{S}_0(r))\right]^2 dr = O(h_r^2).$$

For j>0 the following integral exists and is bounded

$$(1.2.56)\qquad \int_{r_j}^{r_{j+1}} \frac{1}{r^3}\left[\frac{d}{dr}\left(f(r) - \bar{S}_j(r)\right)\right]^2 dr \le \frac{1}{r_j^3}\int_{r_j}^{r_{j+1}} h_r^{-2}\, o(h_j^2)dr =$$

$= h_r^{-1}\, o(h_r^2)$.

Finally the norm for all the intervals can be written as follows

$$(1.2.57)\qquad \left[\left(f(r) - \bar{S}_j(r)\right)\right]^2 = \int_0^{h_1} \frac{1}{r^3}\left[\frac{d}{dr}\left(f(r) - \bar{S}_0(r)\right)\right]^2 dr +$$

$$+ \sum_{j=1}^{m-1} \int_{r_j}^{r_{j+1}} \frac{1}{r^3}\left[\frac{d}{dr}\left(f(r) - \bar{S}_j(r)\right)\right]^2 dr = O(h_r^2) + \sum_{j=1}^{m-1} h_r^{-1}\, o(h_r^2) =$$

$= h_r^{-2}\, o(h_r^2)$.

Obviously from (1.2.35) and (1.2.57) we have

$$(1.2.58)\qquad [f(r) - \bar{S}_0(r)] = h_r^{-1}\, o(h_r).$$

and $h_r^{-1}\, o(h_r) \to 0$ if $h_r \to 0$.

If $f(r) \in W_2^2 \cap H_A$, we can get a better estimations than that obtained in theorem 1.2.2. To show this we introduce the following theorem.

THEOREM 1.2.3. If $f(r) \in W_2^2 \cap H_A$ there exist $\bar{S}_\Delta(r)$ such that the following estimations are true

$$[f(r) - \bar{S}_\Delta(r)] = O(h_r),$$

$$\|f(r) - \bar{S}_\Delta(r)\|_{L_2} = O(h_r^2),$$

$$\|f(r) - \bar{S}_\Delta(r)\|_{W_2^1} = O(h_r).$$

PROOF. Because $f(r) \in W_2^2$ there exists at least one point r_c on the interval (r_j, r_{j+1}) such that $\frac{d^2f}{dr^2}(r_c)$ exists and is bounded.

The Taylor's expansion of $f(r)$ at the neighborhood of r_c on (r_j, r_{j+1})

$$(1.2.59) \qquad f(r) = f(r_c) + \lambda_r h_r \frac{df}{dr}(r_c) + Q_1(r),$$

where $\lambda_r h_r = r - r_c$, $|\lambda_r| \le 1$.

The Peano's form of the remainder in (1.2.59) [7,8] is

$$(1.2.60) \qquad Q_1(r) = \lambda_r^2 h_r^2 \frac{d^2f}{dr^2}(r_c) + o(h_r^2).$$

Hence we have that $|Q_1(r)| = O(h_r^2)$.

Subtract the spline functions (1.2.44) from (1.2.57) we have

$$(1.2.61) \qquad |f(r) - \bar{S}_\Delta(r)| = O(h_r^2).$$

For the L_2-norm on all the interval $(0,R)$ we get

$$(1.2.62) \qquad ||f(r) - \bar{S}_\Delta(r)||_{L_2} = O(h_r^2).$$

Notice that because $f(r) \in W_2^2$ it follows that $f(r) \in C[o,R]$, and so it is true that $||f(r) - \bar{S}_\Delta(r)||_{C[0,R]} = O(h_r^2)$.

Subtracting $\frac{d\bar{S}_\Delta(r)}{dr}$ from $\frac{df(r)}{dr}$, where the Peano's form for the remainder in the expansion of $\frac{df}{dr}$ is $|Q_2(r)| = O(h_r)$, we obtain

$$(1.2.63) \qquad \left|\frac{df(r)}{dr} - \frac{d\bar{S}_\Delta(r)}{dr}\right| = O(h_r).$$

Since $d\bar{S}_\Delta(r)/dr$ are bounded and piece-wise continous on $(0,R)$, and $f \in W_2^2$ we have

$$(1.2.64) \qquad ||f(r) - \bar{S}_\Delta(r)||_{L_2(0,R)} = O(h_r).$$

Therefore, from (1.2.62) and (1.2.64) we get

$$(1.2.65)\qquad ||f(r) - \bar{S}_\Delta(r)||_{V_2^1} = O(h_r).$$

Following theorem 1.2.2. we get the norm in H_A space as follows

$$(1.2.66)\qquad [f(r) - \bar{S}_\Delta(r)]^2 = \int_0^{h_1} \frac{1}{r^3}\left[\frac{d}{dr}\ (f(r) - \bar{S}_0(r))\right]^2 dr +$$

$$+ \sum_{j=1}^{m-1} \int_{r_j}^{r_{j+1}} \frac{1}{r^3}\left[\frac{d}{dr}\ (f(r) - \bar{S}_j(r))\right]^2 dr = O(h_r^2) + \sum_{j=1}^{m-1} h_r\ O(h_r^2) = O(h_r^2).$$

Clearly from (1.2.35) and (1.2.66) we have

$$(1.2.67)\qquad [f(r) - \bar{S}_\Delta(r)] = O(h_r).$$

Next we are going to introduce the spline functions density for the two-dimensional problem.

THEOREM 1.2.4. The set of the spline functions $S_\Delta(z,r)$ (1.2.9) in a limit sense are dense in H_A. That is for every $f(z,r) \in H_A$ there exists $\bar{S}_\Delta$ such that

$$[f(z,r) - \bar{S}_\Delta(z,r)] \to 0 \text{ if } h_z, h_r \to 0,\ (n,m \to \infty),$$

in the mean time for $f(z,r)$ and $\bar{S}_\Delta(z,r)$ the following are true

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{L_2} \to 0 \text{ if } h_z, h_r \to 0,$$

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{V_2^1} \to 0 \text{ if } h_z, h_r \to 0,$$

where $h_z = \max(z_{i+1} - z_i)$, $h_r = \max(r_{j+1} - r_j)$.

PROOF. Let the form of the splines in $G_{i,j}$ be as follows

$$(1.2.68) \qquad \bar{S}_{\Delta}(z,r) = \bar{S}_{i,j}(z,r) := \frac{f_{i,j+1} - f_{i,j}}{r^4_{j+1} - r^4_j}(r^4 - r^4_j) +$$

$$+ \frac{f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j} + f_{i,j}}{h_z(r^4_{j+1} - r^4_j)}(r^4 - r^4_j)(z - z_i) +$$

$$+ \frac{f_{i+1,j} - f_{i,j}}{h_z}(z - z_i) + f_{i,j},$$

where $f_{i,j} = f(z_i, r_j)$. Clearly (1.2.68) is continuous in Ω.

From lemma 1.2.2 (the prove is in page 32) it follows that $f(z,r) \in W^1_2$. Therefore there exists at least one point (z_c, r_c) on the subdomains $G_{i,j}$ such that $\frac{\partial f}{\partial z}(z_c, r_c)$ and $\frac{\partial f}{\partial r}(z_c, r_c)$ exist and are bounded . The Taylor's expansion of $f(z,r)$ at the neighborhood of (z_c, r_c) in $G_{i,j}$ [7] is

$$(1.2.69) \qquad f(z,r) = f(z_c, r_c) + (z - z_c)\frac{\partial f}{\partial z}(z_c, r_c) +$$

$$+ (r - r_c)\frac{\partial f}{\partial r}(z_c, r_c) + o\left(\sqrt{(z - z_c)^2 + (r - r_c)^2}\right).$$

It is easy to show that $\sqrt{(z - z_c)^2 + (r - r_c)^2} \le |z - z_c| + + |r - r_c|$ and $o\left(\sqrt{(z - z_c)^2 + (r - r_c)^2}\right) = o(|z - z_c|) + o(|r - r_c|)$.

Formula (1.2.69) is used to get the expansions of $f_{i,j}$ $f_{i+1,j}$, $f_{i,j+1}$ and $f_{i+1,j+1}$ in (1.2.68). Subtracting (1.2.68) from (1.2.69) we get

$$(1.2.70) \qquad |f(z,r) - \bar{S}_{\Delta}(z,r)| = \frac{\partial f}{\partial r}(z_c, r_c)\, O(h_r^2) + o(h_z + h_r).$$

For the L_2 norm on the domain Ω we get

$$(1.2.71) \qquad ||f(z,r) - \bar{S}_{\Delta}(z,r)|| = o(h_z + h_r).$$

Let us differentaiate (1.2.68) and so we obtain

(1.2.72) $$\frac{\partial \bar{S}_{i,j}}{\partial z} = \frac{f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j} + f_{i,j}}{h_z\left(r^4_{j+1} - r^4_j\right)} \left(r^4 - r^4_j\right) +$$

$$+ \frac{f_{i+1,j} + f_{i,j}}{h_z},$$

(1.2.73) $$\frac{\partial \bar{S}_{i,j}}{\partial r} = \frac{f_{i+1,j} - f_{i,j}}{r^4_{j+1} - r^4_j} 4r^3 +$$

$$+ \frac{f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j} + f_{i,j}}{h_z\left(r^4_{j+1} - r^4_j\right)} 4r^3 \left(z - z_i\right).$$

For the derivatives of f(z,r) the Taylor's expansion become

(1.2.74) $$\frac{\partial f(z,r)}{\partial z} = \frac{\partial f}{\partial z}(z_c, r_c) + \left((z - z_c)^2 +\right.$$

$$\left. + (r - r_c)^2\right)^{-1/2} \cdot o\left[\sqrt{(z - z_c)^2 + (r - r_c)^2}\right].$$

(1.2.75) $$\frac{\partial f(z,r)}{\partial r} = \frac{\partial f}{\partial r}(z_c, r_c) + \left((z - z_c)^2 +\right.$$

$$\left. + (r - r_c)^2\right)^{-1/2} \cdot o\left[\sqrt{(z - z_c)^2 + (r - r_c)^2}\right].$$

Subtracting (1.2.72) from (1.2.74) and (1.2.73) from (1.2.75) we obtain

(1.2.76) $$\left|\frac{\partial f(z,r)}{\partial z} - \frac{\partial \bar{S}_{i,j}(z,r)}{\partial z}\right| = \left(h_z^2 + h_r^2\right)^{-1/2} \cdot o\left[\sqrt{(h_z^2 + h_r^2)}\right].$$

(1.2.77) $$\left|\frac{\partial f(z,r)}{\partial r} - \frac{\partial \bar{S}_{i,j}(z,r)}{\partial r}\right| = \left(h_z^2 + h_r^2\right)^{-1/2} \cdot o\left[\sqrt{(h_z^2 + h_r^2)}\right].$$

Since $\frac{\partial \bar{S}_\Delta(z,r)}{\partial z}$, $\frac{\partial \bar{S}_\Delta(z,r)}{\partial r}$ are bounded, piecewise continuous in Ω, and $f(z,r) \in W^1_2$ we have

$$(1.2.78) \qquad \left\|\frac{\partial f(z,r)}{\partial r} - \frac{\partial \bar{S}_\Delta(z,r)}{\partial r}\right\|_{L_2(\Omega)} = (h_z^2 + h_r^2)^{-1/2}\cdot o\left[\sqrt{(h_z^2 + h_r^2}\right],$$

$$(1.2.79) \qquad \left\|\frac{\partial f(z,r)}{\partial z} - \frac{\partial \bar{S}_\Delta(z,r)}{\partial z}\right\|_{L_2(\Omega)} = (h_z^2 + h_r^2)^{-1/2}\cdot o\left[\sqrt{(h_z^2 + h_r^2}\right].$$

Clearly from (1.2.71), (1.2.78) and (1.2.79) we get

$$(1.2.80) \qquad \|f(z,r) - \bar{S}_\Delta(z,r)\|_{V_2^1} = (h_z^2 + h_r^2)^{-1/2}\cdot o\left[\sqrt{(h_z^2 + h_r^2}\right].$$

The right side of (1.2.80) tends to zero if h_z and h_r tend to zero.

For the norm in the H_A space we consider the following

$$(1.2.81) \qquad [f(z,r) - \bar{S}_\Delta(z,r)]^2 = \iint_\Omega \frac{1}{r^3}\left[\left(\frac{\partial}{\partial z}(f - \bar{S}_\Delta)\right)^2 + \right.$$

$$\left. + \left(\frac{\partial}{\partial r}(f - \bar{S}_\Delta)\right)^2\right] dz\, dr = \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} \iint_{G_{i,j}} \frac{1}{r^3}\left[\left(\frac{\partial}{\partial z}(f - \bar{S}_{i,j})\right)^2 + \right.$$

$$\left. + \left(\frac{\partial}{\partial r}(f - \bar{S}_{i,j})\right)^2\right] dz\, dr.$$

Because $f(z,r) \in H_A$ the Taylor's expansion of $f(z,r)$ in the nieghborhood of $r = 0$:

$$(1.2.82) \qquad f(z,r) = a_3(z)\, r^3 + a_4(z)\, r^4 + \ldots..$$

Let the spline functions in the nieghborhood of $r = 0$:

$$(1.2.83) \qquad \bar{S}_{i,0}(z,r) = \left(\frac{a_3(z_i)}{r_1} + a_4(z_i)\right) r^4 +$$

$$+ \left[\frac{a_3(z_{i+1}) - a_3(z_i)}{h_z\, r_1} + \frac{a_4(z_{i+1}) - a_4(z_i)}{h_z}\right] r^4 (z - z_i).$$

Therefore from (1.2.82) and (1.2.83) we obtain that

(1.2.84) $$\sum_{i=0}^{n-1} \int_{0}^{r_1} \int_{z_i}^{z_{i+1}} \frac{1}{r^3} \left[\left(\frac{\partial}{\partial z} (f - \bar{S}_{i,0})\right)^2 + \right.$$

$$\left. + \left(\frac{\partial}{\partial r} (f - \bar{S}_{i,0})\right)^2\right] dz\ dr = O(h_r^2)$$

For f>0 the following integral exists and is bounded

(1.2.85) $$\iint_{G_{i,j}} \frac{1}{r^3} \left[\left(\frac{\partial}{\partial z} (f - \bar{S}_{i,j})\right)^2 + \right.$$

$$\left. + \left(\frac{\partial}{\partial r} (f - \bar{S}_{i,j})\right)^2\right] dz\ dr = o(h_z^2 + h_r^2).$$

From (1.2.84) and (1.2.85) the norm in all the domain Ω can be written

(1.2.86) $$\sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \iint_{G_{i,j}} o(h_z^4 + h_r^4)\ dz\ dr + O(h_r^2) =$$

$$= \frac{1}{h_z h_r}\ o(h_r^2 + h_z^2) + O(h_r^2).$$

Therefore

(1.2.87) $$\left[f(z,r) - \bar{S}_\Delta(z,r)\right] = (h_z^2 + h_r^2)^{-1/2}\ o\left[\sqrt{(h_z^2 + h_r^2}\right].$$

The right side of (1.2.87) tends to zero if h_z and h_r tend to zero.

If $f(z,r) \in W_2^2 \cap H_A$ we can get a better estimations than that obtained in theorem 1.2.4.

THEOREM 1.2.5. if $f(z,r) \in W_2^2 \cap H_A$ there exist $\bar{S}_\Delta(z,r)$ such that the following estimations are true

$$[f(z,r) - \bar{S}_\Delta(z,r)] = O(h_z + h_r)$$

$$||[f(z,r) - \bar{S}_\Delta(z,r)]||_{L_2} = O(h_z^2 + h_r^2)$$

$$||[f(z,r) - \bar{S}_\Delta(z,r)]||_{W_2^1} = O(h_z^2 + h_r^2)$$

PROOF. Because $f(z,r) \in W_2^2$ there exists at least one point (z_c, r_c) on $G_{i,j}$ such that $\left(\frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial z\, \partial r}, \frac{\partial^2 f}{\partial r^2}\right)$ exist and are bounded.

The Taylor's expansion of $f(z,r)$ at the neighborhood of (z_c, r_c) in $G_{i,j}$

(1.2.88) $$f(z,r) = f(z_c, r_c) + \lambda_z h_z \frac{\partial f(z_c, r_c)}{\partial z} + \lambda_r h_r \frac{\partial f(z_c, r_c)}{\partial r} +$$

$+ Q_3(z,r)$ Where $\lambda_z h_z = z - z_c$, $\lambda_r h_r = r - r_c$, $|\lambda_z| \le 1$, $|\lambda_r| \le 1$.

The Peano's form of the remainder in (1.2.88) [7,8]

(1.2.89) $$Q_3(z,r) = \lambda_z^2 h_z^2 \frac{\partial^2 f}{\partial z^2}(z_c, r_c) + \lambda_r^2 h_r^2 \frac{\partial^2 f}{\partial r^2}(z_c, r_c) +$$

$$+ 2\lambda_z h_z \lambda_r h_r \frac{\partial^2 f}{\partial z\, \partial r}(z_c, r_c) + o(h_z^2 + h_r^2).$$

Hence we have that $|Q_3(z,r)| = O(h_z^2 + h_r^2)$.

Let the spline functions form in $G_{i,j}$ be as follows

(1.2.90) $$\bar{S}_\Delta(z,r) = \bar{S}_{i,j}(z,r) := \frac{f_{i,j+1} - f_{i,j}}{r_{j+1}^4 - r_j^4}(r^4 - r_j^4) +$$

$$+ \frac{f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j} + f_{i,j}}{h_z(r_{j+1}^4 - r_j^4)}(r^4 - r_j^4)(z - z_i) +$$

$$+ \frac{f_{i+1,j} - f_{i,j}}{h_z}(z - z_i) + f_{i,j}.$$

where $f_{i,j} := f(z_i, r_j)$. It can be seen that (1.2.90) is continuous in Ω. Formula (1.2.88) can be used to get the functions $f_{i,j}$, $f_{i+1,j}, f_{i,j+1}, f_{i+1,j+1}$ in (1.2.90). Subtract (1.2.87) from (1.2.85) we obtain

(1.2.91) $$|f(z,r) - \overline{S}_\Delta(z,r)| = O(h_z^2 + h_r^2).$$

For the L_2-norm in Ω we have

(1.2.92) $$||f(z,r) - \overline{S}_\Delta(z,r)||_{L_2} = O(h_z^2 + h_r^2).$$

The first partial derivatives of (1.2.88) are

(1.2.93) $$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial z}(z_c, r_c) + Q_3^z(z,r),$$

(1.2.94) $$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial r}(z_c, r_c) + Q_3^r(z,r).$$

The Peano's form of the remainders in (1.2.93)-(1.2.94) are

(1.2.95) $$Q_3^z = \lambda_z h_z \frac{\partial f}{\partial z}(z_c, r_c) + \lambda_r h_r \frac{\partial^2 f}{\partial z \partial r}(z_c, r_c) + o(h_z + h_r),$$

(1.2.96) $$Q_3^r = \lambda_r h_r \frac{\partial f}{\partial r}(z_c, r_c) + \lambda_z h_z \frac{\partial^2 f}{\partial z \partial r}(z_c, r_c) + o(h_z + h_r).$$

It can be seen that

(1.2.97) $$|Q_3^z| = O(h_z + h_r), \qquad |Q_3^r| = O(h_z + h_r).$$

The first partial derivatives of (1.2.90) are

(1.2.98) $$\frac{\partial \overline{S}_\Delta}{\partial z}(z,r) = O(h_z + h_r),$$

(1.2.99) $$\frac{\partial \overline{S}_\Delta}{\partial r}(z,r) = O(h_z + h_r).$$

from formulas (1.2.93)-(1.2.94) and (1.2.98)-(1.2.99) we get

$$(1.2.100) \qquad \left|\frac{\partial f}{\partial z} - \frac{\partial \bar{S}_\Delta}{\partial z}\right| = O(h_z + h_r),$$

$$(1.2.101) \qquad \left|\frac{\partial f}{\partial r} - \frac{\partial \bar{S}_\Delta}{\partial r}\right| = O(h_z + h_r).$$

Since (1.2.98)-(1.2.99) are bounded piecewise continuous in Ω, and $f(z,r) \in W_2^2$ we have

$$(1.2.102) \qquad \left\|\frac{\partial f(z)}{\partial z} - \frac{\partial \bar{S}_\Delta}{\partial z}\right\|_{L_2} = O(h_z + h_r),$$

$$(1.2.103) \qquad \left\|\frac{\partial f(z)}{\partial r} - \frac{\partial \bar{S}_\Delta}{\partial r}\right\|_{L_2} = O(h_z + h_r).$$

Hence, from (1.2.92) and (1.2.102) - (1.2.103) we get

$$(1.2.104) \qquad \| f(z,r) = \bar{S}_\Delta(z,r) \|_{W_2^1} = O(h_z + h_r).$$

Following theorem 1.2.4 we can obtain the norm in the H_A space as follows

$$(1.2.105) \qquad [f(z,r) - \bar{S}_\Delta(z,r)]^2 = \sum_{i=0}^{n-1} \int_0^{r_1} \int_{z_i}^{z_{i+1}} \frac{1}{r^3} \Big[\big(\frac{\partial}{\partial z}(f - \bar{S}_{i,0})\big)^2$$

$$+ \big(\frac{\partial}{\partial r}(f - \bar{S}_{i,0})\big)^2 \Big] dz\, dr + \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \iint_{G_{i,j}} \frac{1}{r^3} \Big[\big(\frac{\partial}{\partial z}(f - \bar{S}_{i,j})\big)^2 +$$

$$+ \big(\frac{\partial}{\partial r}(f - \bar{S}_{i,j})\big)^2 \Big] dz\, dr = O(h_r^2) + O(h_z^2 + h_r^2) = O(h_z^2 + h_r^2).$$

Therefore, from (1.2.35) and (1.2.105) we have

$$(1.2.106) \qquad [f(z,r) - \bar{S}_\Delta(z,r)] = O(h_z + h_r).$$

1.2.6. The convergence of the approximate solutions.

LEMMA 1.2.1. In the One-dimensional problem (1.2.40), for every $u(r) \in H_A$, there exists a constant c_1 such that

$$||u||_{W_2^1} \le c_1[u].$$

PROOF. For the L_2-norm of the first derivative of the function u on the interval (0,R) we have

$$(1.2.107) \qquad ||\frac{du}{dr}||^2_{L_2} = \int_0^R |\frac{du}{dr}|^2 \, dr = \int_0^R r^3 \cdot \frac{1}{r^3} \, |\frac{du}{dr}|^2 \, dr \le$$

$$\le R^3 \int_0^R \frac{1}{r^3} \, |\frac{du}{dr}|^2 = R^3[u,u].$$

Let

$$(1.2.108) \qquad u = \int_0^r \frac{du}{dx} \, dx = \int_0^r x^{3/2} \, (\frac{1}{x^{3/2}} \, \frac{du}{dx}) \, dx \, .$$

Using Cauchy-Bunyakovsky's inequality to the integral (1.2.105)

$$(1.2.109) \qquad u^2 \le \int_0^r x^3 \, dx \int_0^r (\frac{1}{x^{3/2}} \, |\frac{du}{dx}|^2) \, dx \le \frac{r^4}{4} \int_0^R \frac{1}{x^3} \, |\frac{du}{dx}|^2) \, dx.$$

Hence we get

$$(1.2.110) \qquad ||u||^2_{L_2} = \int_0^R u^2 \, dr \le \int_0^R \frac{r^4}{4} \, dr \, [u,u] = \frac{R^5}{20} \, [u]^2 \, .$$

Therefore, from (1.2.107) and (1.2.110) we obtain the norm of the function u in W_2^1

$$(1.2.111) \qquad ||u||_{W_2^1} \le c_1[u],$$

where $c_1 = R \sqrt{R + \frac{R^3}{20}}$.

LEMMA 1.2.2. In the two-dimensional problem (1.1.1), for every $u(z,r) \in H_A$, there exists a constant c_2 such that

$$||u||_{V_2^1} \leq c_2[u].$$

PROOF. For the L_2-norm of the first partial derivatives of the function $u(z,r)$ on Ω we have

$$(1.2.112) \qquad \left|\left|\frac{\partial u}{\partial z}\right|\right|^2_{L_2} \leq R^3 \int_\Omega \frac{1}{r^3} \left|\frac{\partial u}{\partial z}\right|^2 dz\ dr.$$

$$(1.2.113) \qquad \left|\left|\frac{\partial u}{\partial r}\right|\right|^2_{L_2} \leq R^3 \int_\Omega \frac{1}{r^3} \left|\frac{\partial u}{\partial r}\right|^2 dz\ dr.$$

Therefore, from (1.2.112) - (1.2.113) we obtain

$$(1.2.114) \qquad \left|\left|\frac{\partial u}{\partial z}\right|\right|^2_{L_2} + \left|\left|\frac{\partial u}{\partial r}\right|\right|^2_{L_2} \leq R^3\ [u,u].$$

Let

$$(1.2.115) \qquad u(z,r) = \int_0^r \frac{\partial u}{\partial x}(z,x)\ dx = \int_0^r x^{3/2} \frac{1}{x^{3/2}} \frac{\partial u}{\partial x}(z,x)\ dx.$$

Using Cauchy-Bunyakovsky's inequality to the integral (1.2.115)

$$(1.2.116) \qquad u^2(z,r) \leq \int_0^r x^3\ dx \int_0^r \frac{1}{x^3} \left|\frac{\partial u}{\partial x}(z,x)\right|^2 dx \leq$$

$$\leq \frac{r^4}{4} \int_0^R \frac{1}{x^3} \left|\frac{\partial u}{\partial x}(z,x)\right|^2 dx.$$

Integrating the both sides of (1.2.116) over the domain Ω we get

$$(1.2.117) \qquad \int_\Omega u^2(z,r)\ dz\ dr \leq \frac{r^4}{4}\ dr \int_0^R \int_0^Z \frac{1}{x^3} \left|\frac{\partial u}{\partial x}\right|^2 dz\ dx =$$

$$= \frac{R^5}{20} \int_\Omega \frac{1}{r^3} \left|\frac{\partial u}{\partial r}\right|^2 dz\, dr.$$

By analogy one can have

$$(1.2.118) \qquad \int_\Omega u^2(z,r)\, dz\, dr \le \frac{R^3 Z^2}{2} \int_\Omega \frac{1}{x^3} \left|\frac{\partial u}{\partial x}\right|^2 dz\, dx .$$

Therefore from (1.2.117) and (1.2.118) we obtain

$$(1.2.119) \qquad 2 \int_\Omega u^2(z,r)\, dz\, dr \le \max \left(\frac{R^5}{20}, \frac{R^3 Z^2}{2}\right) [u,u].$$

Hence

$$(1.2.120) \qquad ||u||^2 \le \max \left(\frac{R^5}{40}, \frac{R^3 Z^2}{4}\right) [u,u].$$

From (1.2.114) and (1.2.120) we get the norm of the function u in W_2^1.

$$(1.2.121) \qquad ||u||_{W_2^1} \le c_2 [u],$$

where

$$(1.2.122) \qquad c_2 = R^3 \left[1 + \max \left(\frac{R^2}{40}, \frac{Z^2}{4}\right)\right].$$

THEOREM 1.2.6. The approximated solutions $\tilde{S}$ of the one and two-dimensional problems, obtained form the minimization of the functional (1.2.39) over the families of splines, are convergent to the generalized solution $\tilde{u}_o$ of the original problems (1.1.1) and (1.2.40) in the W_2^1-norm

$$||\tilde{u}_o - \tilde{S}||_{W_2^1} \to 0 \text{ if } h \to 0, \text{ where } h = \max|h_r| \text{ and } h = \max|h_z, h_r|.$$

PROOF. The generalized solution $\tilde{u}_o$ satisfy the following equality (see (1.2.36) and (1.2.38)) for every $v \in H_A$

(1.2.123) $[\tilde{u}_o - v]^2 = [\tilde{u}_o - v, \tilde{u}_o - v] = [v,v] - 2[v,\tilde{u}_o] + [\tilde{u}_o,\tilde{u}_o] =$

$= [v,v] - 2(v,\tilde{F}) - [\tilde{u}_o,\tilde{u}_o] + 2[\tilde{u}_o,\tilde{u}_o] = I(v) - [\tilde{u}_o,\tilde{u}_o] + 2(\tilde{u}_o,\tilde{F}) =$

$= I(v) - I(\tilde{u}_o)$.

Because $\tilde{S}$ are minimizing the functional I, for every spline function S_Δ we have

(1.2.124) $I(\tilde{S}) - I(\tilde{u}_o) \le I(S_\Delta) - I(\tilde{u}_o) = [\tilde{u}_o - S_\Delta]^2$.

Therefore from (1.2.123)-(1.2.124) we obtain

(1.2.125) $[\tilde{u}_o - \tilde{S}] \le \inf_{S_\Delta} [\tilde{u}_o - S_\Delta]$

Hence, from lemma 1.2.1 and theorem 1.2.2, respectivly, from lemma 1.2.2. and theorem 1.2.4 we have

(1.2.126) $||\tilde{u}_o - \tilde{S}||_{W_2^1} \le C[\tilde{u}_o \tilde{S}] \le C \inf_{S_\Delta} [\tilde{u}_o - S_\Delta] \to 0$ if $h \to 0$.

Remark. If $\tilde{u}_o$ are the classical solutions of the equivalent problems of (1.1.1) and (1.2.40), it means $\tilde{u}_o \in D(A)$ and at the same time $\tilde{u}_o \in W_2^2$. Therefore, theorem 1.2.3, theorem 1.2.5 can be used with lemma 1.2.1, lemma 1.2.2 and (1.2.125) to obtain the better estimations for $||\tilde{u}_o - \tilde{S}||$:

(1.2.127) $||\tilde{u}_o - \tilde{S}||_{W_2^1} = O(h)$.

1.3.1. Let us look for an approximate solution to the elastic problem (1.1.1) in a cylinder of variable diameter with the boundary conditions (1.2.1)-(1.2.4) on the given domain $\Omega_e := [0,Z]\times[0,R(z)]$ (FIG. 3.1).

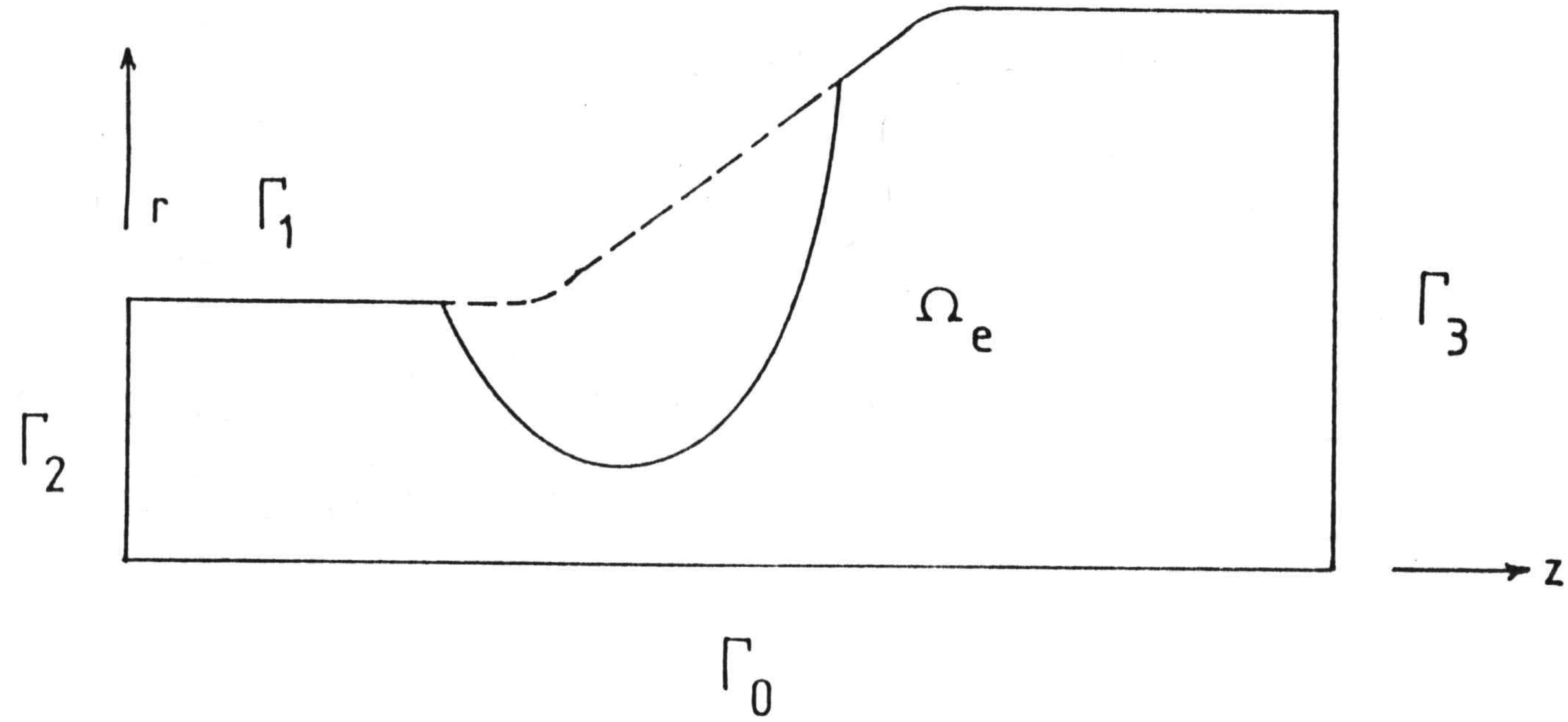

FIGURE 3.1. The domain Ω_e

We shall propagate the spline method introduced in the previous section to the present case too.

1.3.2. The construction of the spline function. The intervals of the zr-plane (FIG. 3.2) are breaked up as follows

(1.3.1) $$o = z_o < z_1 < \ldots < z_i < z_{i+1} < \ldots < z_n = Z ,$$

(1.3.2) $$o = r_o < r_1 < \ldots < r_j < r_{j+1} < \ldots < r_m .$$

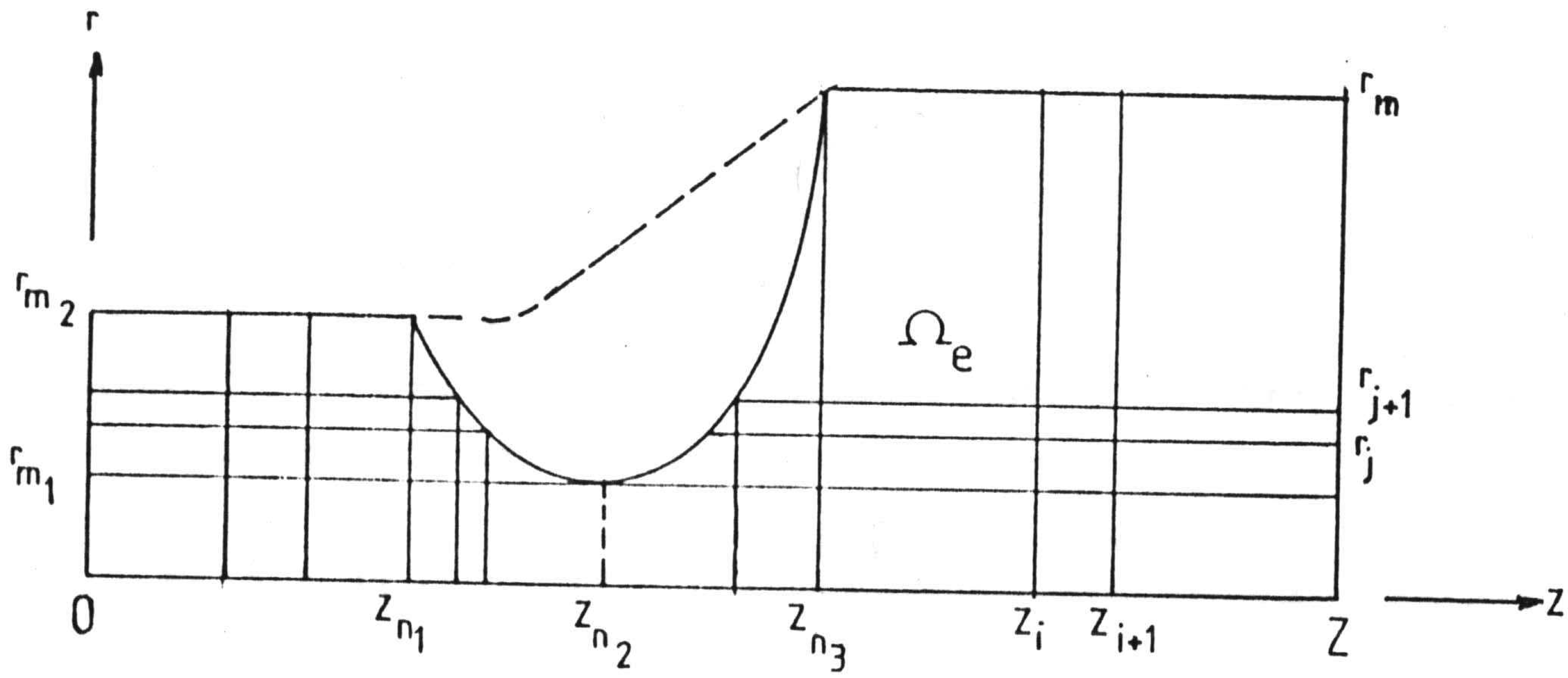

FIGURE 3.2. The partition of the domain Ω_e

It is supposed that: (z_{n_1}, r_{m_2}), (z_{n_2}, r_{m_1}) and (z_{n_3}, r_m) are grid-points, where $o<n_1<n_2<n$ and $o<m_1<m_2<m$. The given function $r = R(z)$ (which is denote the boundary Γ_1) is constant for $z \in [o, z_{n_1}]$, $z \in [z_{n_3}, Z]$, decreasing on the interval (z_{n_1}, z_{n_2}) and increasing on the interval (z_{n_2}, z_{n_3}). Its minimum at the point (z_{n_2}, r_{m_1}). Suppose that

(1.3.3) $\qquad r_{m_2-i} = R(z_{n_1+i})$, $\quad i = \overline{1, n_2 - n_1}$,

(1.3.4) $\qquad r_{m_1+i} = R(z_{n_2+i})$, $\quad i = \overline{1, n_3 - n_2}$.

It follows that

(1.3.5) $\qquad n_2 - n_1 = m_2 - m_1$, $\quad n_3 - n_2 = m - m_1$.

Let us approximate $r^4 = R^4(z)$ (we shall use it later on) on the given intervals by the following piecewise linear functions

(1.3.6) $$R^4(z) \approx \tilde{R}^4(z) := r^4_{j+1} + \frac{r^4_j - r^4_{j+1}}{z_{i+1} - z_i}(z - z_i),$$

$n_1 \le i < n_2$, $z_i < z < z_{i+1}$,

(1.3.7) $$R^4(z) \approx \tilde{R}^4(z) := r^4_j + \frac{r^4_{j+1} - r^4_j}{z_{i+1} - z_i}(z - z_i),$$

$n_2 \le i < n_3$, $z_i < z < z_{i+1}$,

The function $u = F(z)$ on Γ_1 is approximated by the following piecewise linear function

(1.3.8) $$F(z) \approx \tilde{F}(z) := F(z_i) + \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}(z - z_i),$$

$i = \overline{0, n-1}$.

Let Ω_e be divided to the elements $\Omega_{i,j}$: Rectangulars into the domain Ω_e. Trianguairs (with two straight sides and one curved side) at the neighborhood of Γ_1 on the intervals (z_{n_1}, z_{n_2}), (z_{n_1}, z_{n_3}) (FIG. 3.2).

The total number of the elements $\Omega_{i,j}$ are

(1.3.9) $$n \cdot m_1 + n_1 (m_2 - m_1) + (n - n_3)(m - m_1) + \frac{(n_2 - n_1)(n_2 - n_1 + 1)}{2} + \frac{(n_3 - n_2)(n_3 - n_2 + 1)}{2} .$$

The spline functions (1.2.9) have the following forms at the neighborhood of the boundaries Γ_0 and Γ_1 on $\Omega_{i,j}$

(1.3.10) $$S_{i,0}(z,r) = a_{i,0}\, r^4 + b_{i,0}\, r^4(z - z_i), \quad i = \overline{0, n-1} ,$$

(1.3.11) $$S_{i,m_2-1}(z,r) = a_{i,m_2-1}(r^4 - r^4_{m_2}) + b_{i,m_2-1}(r^4 - r^4_{m_2})(z \; z_i) + \tilde{F}(z) , \quad i = \overline{0, n_1 - 1} ,$$

(1.3.12) $$S_{i,j}(z,r) = a_{i,j}(r^4 - \tilde{R}^4(z)) + \tilde{F}(z) ,$$

$i = n_2 - l_1 - 1$, $j = m_1 + l_1$, $l_1 = \overline{0, n_2 - n_1 - 1}$, (1.3.13) $\qquad S_{i,j}(z,r) = a_{i,j}\left(r^4 - \tilde{R}^4(z)\right) + \tilde{F}(z)$, $i = n_2 - l_2$,

$j = m_1 + l_2$, $l_2 = \overline{0, n_3 - n_2 - 1}$,

(1.3.14) $\qquad S_{i,m-1}(z,r) = a_{i,m-1}(r^4 - r_m^4) + b_{i,m-1}(r^4 - r_m^4)(z - z_i) +$

$+ \tilde{F}(z)$, $i = \overline{n_3, n - 1}$.

THEOREM 1.3.1. The continuous spline functions (1.2.9) exist in $\bar{\Omega}_e$ and satisfy the main boundary conditions (1.2.1) - (1.2.2).

PROOF. The conditions that the spline functions (1.2.9) and (1.3.10) - (1.3.14) be continuous in $\bar{\Omega}_e$ and satisfy the main boundary conditions are expressed by the following system of equations

(1.3.15) $\qquad a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0$, $i = \overline{0, n - 2}$,

$j = \overline{0, m_1 - 1}$,

(1.3.16) $\qquad c_{i,j}(z_{i+1} - z_i) + d_{i,j} - d_{i+1,j} = 0$, $i = \overline{0, n - 2}$,

$j = \overline{1, m_1 - 1}$,

(1.3.17) $\qquad a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0$, $i = \overline{0, n_1 - 1}$,

$j = \overline{m_1, m_2 - 1}$,

(1.3.18) $\qquad c_{i,j}(z_{i+1} - z_i) + d_{i,j} - d_{i+1,j} = 0$, $i = \overline{0, n_1 - 1}$,

$j = \overline{m_1, m_2 - 2}$,

(1.3.19) $a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0, \quad i = n_1 + l_1,$

$j = m_1 + k_1, \quad l_1 = \overline{0, n_2 - n_1 - 3}, \quad k_1 = \overline{0, n_2 - n_1 - l_1 - 3},$

(1.3.20) $c_{i,j}(z_{i+1} - z_i) + d_{i,j} - d_{i+1,j} = 0, \quad i = n_1 + l_1,$

$j = m_1 + k_1, \quad l_1 = \overline{0, n_2 - n_1 - 3}, \quad k_1 = \overline{0, n_2 - n_1 - l_1 - 3},$

(1.3.21) $a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0, \quad i = n_2 + l_1 - 2,$

$j = m_1 + l_1, \quad l_1 = \overline{0, n_2 - n_1 - 2},$

(1.3.22) $c_{i,j}(z_{i+1} - z_i) + d_{i,j} - a_{i+1,j}\left(r_j^4 - r_{j+1}^4\right) = F(z_{i+1}),$

$i = n_2 + l_1 - 2, \quad j = m_1 + l_1, \quad l_1 = \overline{0, n_2 - n_1 - 2},$

(1.3.23) $a_{i,j} - a_{i+1,j} = 0, \quad i = n_2 + l_2, \quad j = m_1 + l_1,$

$l_1 = \overline{0, n_3 - n_2 - 1},$

(1.3.24) $a_{i+1,j}\left(r_{j+1}^4 - r_j\right) + d_{i+1,j} = F(z_{i+1}), i = n_2 + l_2,$

$j = m_1 + l_2, \quad l_2 = \overline{0, n_3 - n_2 - 2},$

(1.3.25) $a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0, \quad i = n_2 + l_2 + k_2 + 1,$

$j = m_1 + k_2, \quad l_2 = \overline{0, n_3 - n_2 - k_2 - 2}, \quad k_2 = \overline{0, n_3 - n_2 - 2},$

(1.3.26) $c_{i,j}(z_{i+1} - z_i) + d_{i,j} - d_{i+1,j} = 0, \quad i = n_2 + l_2 + k_2 + 1,$

$j = m_1 + k_2, \quad l_2 = \overline{0, n_3 - n_2 - k_2 - 2}, \quad k_2 = \overline{0, n_3 - n_2 - 2},$

(1.3.27) $a_{i,j} + b_{i,j}(z_{i+1} - z_i) - a_{i+1,j} = 0, \quad i = \overline{n_3 + n - 2},$

$j = \overline{m_1, m - 1}$,

(1.3.28) $\quad c_{i,j}(z_{i+1} - z_i) + d_{i,j} - d_{i+1,j} = 0, \quad i = \overline{n_3, n - 2}$,

$j = \overline{m_1, m - 1}$,

(1.3.29) $\quad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} - d_{i+1,j} = 0, \quad i = \overline{0, n - 1}$,

$j = \overline{0, m_1 - 2}$,

(1.3.30) $\quad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - c_{i+1,j} = 0, \quad i = \overline{0, n - 1}$,

$j = \overline{0, m_1 - 2}$,

(1.3.31) $\quad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} - d_{i+1,j} = 0, \quad i = \overline{0, n_1 - 1}$,

$j = \overline{m_1 - 1, m_2 - 3}$,

(1.3.32) $\quad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - c_{i+1,j} = 0, \quad i = \overline{0, n_1 - 1}$,

$j = \overline{m_1 - 1, m_2 - 3}$,

(1.3.33) $\quad a_{i,m_2-2}\left(r^4_{m_2-1} - r^4_{m_2-2}\right) + d_{i,m_2-2} - a_{i,m_2-1}\left(r^4_{m_2-1} - r^4_{m_2}\right) =$

$= F(z_i), \quad i = \overline{0, n_1 - 1}$,

(1.3.34) $\quad b_{i,m_2-2}\left(r^4_{m_2-1} - r^4_{m_2-2}\right) + c_{i,m_2-2} - b_{i,m_1-1}\left(r^4_{m_2-1} - r^4_{m_1}\right) =$

$$= \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}, \quad i = \overline{0, n_1 - 1},$$

(1.3.35) $\quad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} - d_{i+1,j} = 0, \quad i = n_1 + 1_1$,

$j = m_1 + k_1 - 1,\ k_1 = 0, n_2 - n_1 - l_1 - 2,\ l_1 = \overline{0, n_2 - n_1 - 2},$

$$(1.3.36) \qquad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - c_{i,j+1} = 0, \quad i = n_1 + l_1,$$

$j = m_1 + k_1 - 1,\ k_1 = 0, n_2 - n_1 - l_1 - 2,\ l_1 = \overline{0, n_2 - n_1 - 2},$

$$(1.3.37) \qquad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - a_{i+1,j} \frac{r^4_{j+1} - r^4_{j+2}}{z_{i+1} - z_i} =$$

$$= \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}, \ i = n_2 + l_1 - 1,\ j = m_1 + l_1 - 1,\ l_1 = \overline{0, n_2 - n_1},$$

$$(1.3.38) \qquad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} - a_{i+1,j}\,(r^4_{j+1} - r^4_{j+2}) = F(z_i),$$

$i = n_2 + l_1 - 1, \quad j = m_1 + l_1 - 1, \quad l_1 = \overline{0, n_2 - n_1},$

$$(1.3.39) \qquad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - c_{i+1,j} = 0, \quad i = n_2 + l_2 + k_2 + 1,$$

$j = m_1 + k_2 - 1,\ l_2 = \overline{0, n_3 - n_2 - k_2 - 2},\ k_2 = \overline{0, n_3 - n_2 - 2},$

$$(1.3.40) \qquad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} - d_{i+1,j} = 0, \quad i = n_2 + l_2 + k_2 + 1,$$

$j = m_1 + k_2 - 1,\ l_2 = \overline{0, n_3 - n_2 - k_2 - 2},\ k_2 = \overline{0, n_3 - n_2 - 2},$

$$(1.3.41) \qquad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} = F(z_i), \quad i = n_2 + l_2,$$

$j = m_1 + l_2 - 1, \quad l_2 = \overline{0, n_3 - n_2 - 1},$

$$(1.3.42) \qquad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - a_{i,j} \frac{r^4_{j+2} - r^4_{j+1}}{z_{i+1} - z_i} =$$

$$= \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}, \ i = n_2 + l_2,\ j = m_1 + l_2 - 1,\ l_2 = \overline{0, n_3 - n_2 - 1},$$

(1.3.43) $\quad b_{i,j}(r^4_{j+1} - r^4_j) + c_{i,j} - c_{i,j+1} = 0, \; i = \overline{n_3, n-1},$

$j = \overline{m_1 - 1, m-2},$

(1.3.44) $\quad a_{i,j}(r^4_{j+1} - r^4_j) + d_{i,j} - d_{i,j+1} = 0, \; i = \overline{n_3, n-1},$

$j = \overline{m_1 - 1, m-2},$

(1.3.45) $\quad a_{i,m-2}(r^4_{m+1} - r^4_{m-2}) + d_{i,m-2} - a_{i,m-1}(r^4_{m-1} - r^4_m) = F(z_i),$

$i = \overline{n_3, n-1},$

(1.3.46) $\quad b_{i,m-2}(r^4_{m+1} - r^4_{m-2}) + c_{i,m-2} - b_{i,m-1}(r^4_{m-1} - r^4_m) =$

$$= \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}, \quad i = \overline{n_3, n-1},$$

The system (1.2.15) - (1.3.46) has

(1.3.47) $\quad 2(n_2 - n_1)^2 + 2(n_3 - n_2)^2 + 4n_1(m_2 - m_1) - n(1 - 4m) -$

$- 4n_3(m - m_1) - m - m_1 - n_1,$

equations with

(1.3.48) $\quad 2(n_2 - n_1)^2 + 2(n_3 - n_2)^2 + 4n(m - 1) - 4n_1(m_2 - m_1) +$

$+ 4n_3(m_1 - m) - n_1 + n_3,$

coefficients. The analysis shows that there exist

(1.3.49) $\quad (n_2 - n_1)^2 + (n_3 - n_2)^2 + n_1(m_2 - m_1) - m(n + 1) - n_3(m - m_1) -$

$$- m_1 + 2n_1 - n_3 - \frac{(n_2 - n_1)(n_2 - n_1 - 1)}{2} - \frac{(n_3 - n_3)(n_3 - n_2 - 1)}{2}$$

dependent equations in the system. By deleting those dependent equations from the system (using the equations (1.3.29), (1.3.33) for $i = 0$ only, and neglecting the equations (1.3.24), (1.3.31), (1.3.35), (1.3.38), (1.3.40), (1.3.44) and (1.3.45)) we get the $\tilde{M}$

system of independent equations. The number of the independent equations are

$$(1.3.50) \qquad \tilde{N} = (n_2 - n_1)^2 + (n_3 - n_2)^2 + 3n_1(m_2 - m_1) - n(1 - 3m) -$$

$$- 3n_3(m - m_1) - 3n_1 + n_3 - 2m + \frac{(n_2 - n_1)(n_2 - n_1 - 1)}{2} +$$

$$+ \frac{(n_3 - n_2)(n_3 - n_2 - 1)}{2} .$$

with the number of the coefficients (1.3.48).

By induction we found that the system $\tilde{M}$ is consistent [3] and its solution exists but not unique. Hence we have now a family of continuous spline functions in $\bar{\Omega}_e$.

1.3.3. Let the spline functions (1.2.9), (1.3.10) - (1.3.14) have the similar properties described in 1.2.5 for the spline functions (1.2.9) - (1.2.11). In present section we are going to prove two theorems for the density of the spline functions (1.2.9), (1.3.10) - (1.3.14) in the domain Ω_e.

THEOREM 1.3.2. If $f(z,r) \in W_2^2 \cap H_A$ the set of splines $S_\Delta(z,r)$ is dense in a limit sense in H_A with better estimation. That is there exists $\bar{S}_\Delta$ such that.

$$[f(z,r) - \bar{S}_\Delta(z,r)] = O(h_z + h_r),$$

in mean time for $f(z,r)$ and $\bar{S}_\Delta(z,r)$ are true the following

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{L_2} = O(h_z^2 + h_r^2),$$

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{W_2^1} = O(h_z + h_r),$$

where $h_z = \max |z_{i+1} - z_i|$, $h_r = \max |r_{j+1} - r_j|$.

PROOF. The estimates of the approximation by the spline functions (1.2.68) in the rectangular elements in Ω_e are introduced in theorem 1.2.5. Therefore let us look for the estimates of the approximation in the triangular elements in Ω_e.

Because $f(z,r) \in W_z^2$ there exists at least one point on $\Omega_{i,j}$ such that $\left(\frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial z \partial r}, \frac{\partial^2 f}{\partial r^2}\right)$ exist and are bounded. The Taylor's expansion of $f(z,r)$ at the neighborhood of (z_{i+1}, r_j) in $\Omega_{i,j}$ (the left curved sides triangular elements with the points (z_i, r_j), (z_{i+1}, r_j), (z_{i+1}, r_{j+1}))

$$f(z,r) = f_{i+1,j} + \lambda_z h_z \left.\frac{\partial f}{\partial z}\right|_{i+1,j} + \lambda_r h_r \left.\frac{\partial f}{\partial z}\right|_{i+1,j} + Q_4(z,r), \tag{1.3.51}$$

where

$$z = z_{i+1} + \lambda_z h_z, \quad (-1 \le \lambda_z \le 0), \tag{1.3.52}$$

$$r = r_j + \lambda_r h_r, \quad (0 \le \lambda_r \le 1). \tag{1.3.53}$$

The Peano's form of the remainder in (1.3.50) [7,8] is

$$Q_4(z,r) = \lambda_z^2 h_z^2 \left.\frac{\partial^2 f}{\partial z^2}\right|_{i+1,j} + \lambda_r^2 h_r^2 \left.\frac{\partial^2 f}{\partial r^2}\right|_{i+1,j} + \tag{1.3.54}$$

$$+ 2\lambda_z h_z \lambda_r h_r \left.\frac{\partial^2 f}{\partial z \partial r}\right|_{i+1,j} + o\left(\lambda_z^2 + h_r^2\right).$$

From (1.3.54) it can be seen that

$$|Q_4(z,r)| = O\left(h_z^2 + h_r^2\right). \tag{1.3.55}$$

For the spline functions in the left curved sides triangular elements we let

$$\bar{S}_\Delta(z,r) = \bar{S}_{i,j}(z,r) := \frac{f_{i+1,j+1} - f_{i+1,j}}{r_{j+1}^4 - r_j^4}\left[r^4 - r_j^4 - \right. \tag{1.3.56}$$

$$\left. - \frac{r_{j+1}^4 - r_j^4}{z_{i+1} - z_i}(z - z_i)\right] + F(z_i) + \frac{F(z_{i+1}) - F(z_i)}{z_{i+1} - z_i}(z - z_i),$$

where $f_{i,j} := f(z_i, r_j)$ and $F(z) := f(z, R(z))$.

It is obvious that the spline functions given by (1.2.68) and (1.3.56) are cotinuous in Ω_e. From formula (1.3.51) we get the functions $f_{i+1,j+1}$, $f_{i+1,j}$, in (1.3.56). Subtracting (1.3.51) from (1.3.56) we obtain that

(1.3.57) $$f(z,r) - \bar{S}_{i,j}(z,r) = Q_4(z,r) - \lambda_r Q_4 \left(z_{i+1}, r_{j+1}\right) +$$

$$+ \lambda_z Q_4 \left(z_i, r_j\right) + O\left(h_z^2 + h_r^2\right).$$

Therefore

(1.3.58) $$\left|f(z,r) - \bar{S}_{\Delta}(z,r)\right| = O\left(h_z^2 + h_r^2\right),$$

For the L_2-norm on Ω_e we get

(1.3.59) $$\left|\left|f(z,r) - \bar{S}_{\Delta}(z,r)\right|\right|_{L_2} = O\left(h_z^2 + h_r^2\right),$$

Notice because $f(z,r) \in W_2^2$ it follows that $f(z,r) \in C(\Omega_e)$, and so it is true that

(1.3.60) $$\left|\left|f(z,r) - \bar{S}_{\Delta}(z,r)\right|\right|_{C(\Omega_e)} = O\left(h_z^2 + h_r^2\right),$$

The first partial derivatives of (1.3.51) are

(1.3.61) $$\frac{\partial f}{\partial z}(z,r) = \left.\frac{\partial f}{\partial z}\right|_{i+1,j} + Q_4^z(z,r),$$

(1.3.62) $$\frac{\partial f}{\partial r}(z,r) = \left.\frac{\partial f}{\partial r}\right|_{i+1,j} + Q_4^z(z,r).$$

The Peano's form of the remainders in (1.3.61) - (1.3.62) are

(1.3.63) $$Q_4^z(z,r) =$$

$$= (z - z_{i+1}) \left.\frac{\partial^2 f}{\partial z^2}\right|_{i+1,j} + (r - r_j) \left.\frac{\partial^2 f}{\partial z \partial r}\right|_{i+1,j} +$$

$$+ o\left(|z - z_{i+1}|\right) + o\left(|r - r_j|\right),$$

(1.3.64) $$Q_4^r(z,r) = (z - z_{i+1}) + $$

$$+ \left.\frac{\partial^2 f}{\partial z \partial r}\right|_{i+1,j} + (r - r_{i+1}) \left.\frac{\partial^2 f}{\partial r^2}\right|_{i+1,j} + o(|z - z_{i+1}|) + o(|r - r_j|).$$

It can be seen that

(1.3.65) $$|Q_4^z(z,r)| = O(h_z + h_r), \quad |Q_4^r(z,r)| = O(h_z + h_r).$$

The first partial derivatives of (1.3.56) are

(1.3.66) $$\frac{\partial \bar{S}_{i,j}}{\partial z}(z,r) = \frac{1}{h_z}\left[h_z \left.\frac{\partial f}{\partial z}\right|_{i+1,j} - Q_4(z_i, r_j)\right],$$

(1.3.67) $$\frac{\partial \bar{S}_{i,j}}{\partial z}(z,r) =$$

$$= (1 + O(h_r))\, h_r \left[\left.\frac{\partial f}{\partial z}\right|_{i+1,j} + \frac{1}{h_r} Q_4(z_{i+1}, r_{j+1})\right].$$

From formulas (1.3.61) - (1.3.62) and (1.3.66) - (1.3.67) we get

1.3.68) $$\left|\frac{\partial f}{\partial z} - \frac{\partial \bar{S}_\Delta}{\partial z}\right| = \left|Q_4^z(z,r) + \frac{1}{h_z} Q_4(z_i, r_j)\right| = O(h_z + h_r),$$

1.3.69) $$\left|\frac{\partial f}{\partial r} - \frac{\partial \bar{S}_\Delta}{\partial r}\right| = \left|Q_4^r(z,r) + \frac{1}{h_r} Q_4(z_{i+1}, r_{j+1}) + O(h_r)\right| =$$

$$= O(h_z + h_r).$$

Since (1.3.66) - (1.3.67) are bounded, piecewise continuous, on Ω_e, and $f(z,r) \in W_2^2$ we have

(1.3.70) $$\left\|\frac{\partial f}{\partial z} - \frac{\partial \bar{S}_\Delta}{\partial z}\right\|_{L_2} = O(h_z + h_r),$$

(1.3.71) $$\left\|\frac{\partial f}{\partial r} - \frac{\partial \bar{S}_\Delta}{\partial r}\right\|_{L_2} = O(h_z + h_r),$$

Therefore, from (1.3.59) and (1.3.70) - (1.3.71) we obtain

$$(1.3.72) \qquad \| f(z,r) - \bar{S}_{\Delta}(z,r) \|_{W_2^1} = O(h_z + h_r).$$

Similar results can be obtained for the right curved side triangular elements too. Therefore, for all the triangular elements $\Omega_{i,j}$ on Ω_e we get that

$$(1.3.73) \qquad \iint_{\Omega_{i,j}} \frac{1}{r^3} \left[\left(\frac{\partial}{\partial z} (f(z,r) - \bar{S}_{i,j}(z,r)\right)^2 + \frac{\partial}{\partial r} (f(z,r) - \bar{S}_{i,j}(z,r))^2\right] dz\, dr = O(h_z^2 + h_r^2).$$

Now, from theorem 1.2.5 and (1.3.73) we can write the norm in the H_A space on all the domain Ω_e as follows

$$(1.3.74) \qquad [f(z,r) - \bar{S}_{\Delta}(z,r)]^2 = \sum_{i=0}^{n-1} \sum_{j=1}^{m_i-1} \iint_{\Omega_{i,j}} \frac{1}{r^3} \left[\left(\frac{\partial}{\partial z}(f(z,r) - \bar{S}_{i,j}(z,r))\right)^2 + \left(\frac{\partial}{\partial r} (f(z,r) - \bar{S}_{i,j}(z,r))\right)^2\right] dz\, dr +$$

$$+ \sum_{i=0}^{n-1} \int_0^{r_1} \int_{z_i}^{z_{i+1}} \frac{1}{r^3} \left[\left(\frac{\partial}{\partial z}(f(z,r) - \bar{S}_{i,0}(z,r))\right)^2 + \left(\frac{\partial}{\partial r} (f(z,r) - \bar{S}_{i,0}(z,r))\right)^2\right] dz\, dr = O(h_z^2 + h_r^2).$$

where $m_i := \left[\frac{R(z_i)}{h_r}\right] + 1.$

Therefore, from (1.2.35) the norm in H_A have the following estimate

$$(1.3.75) \qquad [f(z,r) - \bar{S}_{\Delta}(z,r)] = O(h_z + h_r).$$

THEOREM 1.3.3. The set of the spline functions $S_\Delta(z,r)$ (1.2.9) on Ω_e in a limit sense are dense in H_A. That is for every $f(z,r) \in H_A$ there exists $\bar{S}_\Delta(z,r)$ such that

$$|f(z,r) - \bar{S}_\Delta(z,r)| \to 0 \quad \text{if } h_z, h_r \to 0, \quad (n,m \to \infty),$$

in mean time for $f(z,r)$ and $\bar{S}_\Delta(z,r)$ the following are true

$$\|f(z,r) - \bar{S}_\Delta(z,r)\|_{L_2} \to 0 \quad \text{if } h_z, h_r \to 0, \quad (n,m \to \infty),$$

$$\|f(z,r) - \bar{S}_\Delta(z,r)\|_{W_2^1} \to 0 \quad \text{if } h_z, h_r \to 0, \quad (n,m \to \infty),$$

where $h_z = \max |z_{i+1} - z_i|$, $h_r = \max |r_{j+1} - r_j|$.

PROOF. The estimates of the approximation by the spline functions (1.2.68) in the rectangular elements on Ω_e are introduced in theorem 1.2.4. Let us look for the estimates of the approximation in the triangular elements on Ω_e.

For the spline functions in the left curved side triangular elements we let it have the form (1.3.56). From lemma 1.3.2 it follows that $f(z,r) \in W_2^1$. Therefore there exists at least one point (z_c, r_c) on the triangular elements $\Omega_{i,j}$ such that $\frac{\partial f}{\partial z}(z_c, r_c)$ and $\frac{\partial f}{\partial r}(z_c, r_c)$ exist and are bounded. The Taylor's expansion of $f(z,r)$ at (z_c, r_c) is

$$f(z,r) = f(z_c, r_c) + (z - z_c)\frac{\partial f}{\partial z}(z_c, r_c) + \tag{1.3.76}$$

$$(r - r_c)\frac{\partial f}{\partial r}(z_c, r_c) + o\left[\sqrt{(z - z_c)^2 + (r - r_c)^2}\right].$$

where $o\left[\sqrt{(z - z_c)^2 + (r - r_c)^2}\right] = o(h_z + h_r)$.

Subtracting (1.3.56) from (1.3.76) we get

$$f(z,r) - \bar{S}_\Delta(z,r) = o(h_z + h_r). \tag{1.3.77}$$

For the L_2-norm on Ω_e we have

(1.3.78) $$||f(z,r) - \bar{S}_\Delta(z,r)||_{L_2} = o(h_z + h_r).$$

From the first derivative of the spline functions (1.3.56), the first derivatives of the Taylor's expansion (1.3.76), and since $\partial\bar{S}_\Delta/\partial z$, $\partial\bar{S}_\Delta/\partial r$ are bounded, piecewise continuous on Ω_e, and $f(z,r) \in W_2^1$ we get the following for the L_2-norm

(1.3.79) $$\left|\left|\frac{\partial f(z,r)}{\partial z} - \frac{\partial\bar{S}_\Delta(z,r)}{\partial z}\right|\right|_{L_2(\Omega_e)} =$$

$$= (h_z^2 + h_r^2)^{-1/2}\, o\left[\sqrt{h_z^2 + h_r^2}\right],$$

(1.3.80) $$\left|\left|\frac{\partial f(z,r)}{\partial r} - \frac{\partial\bar{S}_\Delta(z,r)}{\partial r}\right|\right|_{L_2(\Omega_e)} =$$

$$= (h_z^2 + h_r^2)^{-1/2}\, o\left[\sqrt{h_z^2 + h_r^2}\right],$$

Therefore from (1.3.78) and (1.3.79-1.3.80) we have

(1.3.81) $$||f(z,r) - \bar{S}_\Delta(z,r)||_{W_2^1} = (h_z^2 + h_r^2)^{-1/2} o\left[\sqrt{h_z^2 + h_r^2}\right].$$

The right side of (1.3.81) tends to zero if h_z, h_r tend to zero.

Similar results can be obtained for the right curved side triangular elements too.

From theorem 1.2.4 and (1.3.81) we can write the norm in the H_A space on Ω_e as follows

(1.3.82) $$[f(z,r) - \bar{S}_\Delta(z,r)]^2 = \sum_{i=0}^{n-1} \sum_{j=1}^{m_i-1} \iint_{\Omega_{i,j}} \frac{1}{r^3} \left[\left[\frac{\partial}{\partial z}(f(z,r) -\right.\right.$$

$$- \bar{S}_{i,j}(z,r)\Big)\Big]^2 + \left(\frac{\partial}{\partial r}\big(f(z,r) - \bar{S}_{i,j}(z,r)\big)\right)^2\Bigg] dz\, dr +$$

$$+ \sum_{i=0}^{n-1} \int_0^{r_1} \int_{z_i}^{z_{i+1}} \frac{1}{r^3} \left[\left(\frac{\partial}{\partial z}\big(f(z,r) - \bar{S}_{i,0}(z,r)\big)\right)^2 +\right.$$

$$\left. + \left(\frac{\partial}{\partial r}\big(f(z,r) - \bar{S}_{i,0}(z,r)\big)\right)^2\right] dz\, dr = \frac{I}{h_z h_r}\, o(h_z^2 + h_r^2) + o(h_r^2).$$

where $m_i := \left[\frac{R(z_i)}{h_r}\right] + 1$. Hence we get that

$$[f(z,r) - \bar{S}_\Delta(z,r)] = (h_z^2 + h_r^2)^{-1/2} o\left(\sqrt{h_z^2 + h_r^2}\right). \tag{1.3.83}$$

The right side of (1.3.83) tends to zero if h_z, h_r tend to zero.

1.3.4. Definition for the partition of quadrilateral domain with one side curved. From this definition we can consider the partition of the domain Ω_e which is needed for the integration of the spline functions.

Let

$$(z^k, r^k) \in \Gamma_1:\ R'(z^k) = 0,\ R''(z^k) > 0,\ k = \overline{1,N}, \tag{1.3.84}$$

$$(\bar{z}^k, \bar{r}^k) \in \Gamma_1:\ R'(\bar{z}^k) = 0,\ R''(\bar{z}^k) < 0,\ k = \overline{1,N}, \tag{1.3.84}$$

where $0 = z^0 < z^1 < \ldots < z^N < z^{N+1} = Z$, $z^k < \bar{z}^k < z^{k+1}$.

Let us consider the following

a) If $R'(z) > 0$ (or $R'(z) \le 0$) for $z \in [0,Z]$ then $N = 1$ and $\bar{z}^0 = 0$ (or $\bar{z}^0 = Z$).

b) If $R'(0) < 0$ then $\bar{z}^0 = 0$.

c) If $R'(Z) > 0$ then $\bar{z}^N = Z$.

d) If $R'(0) = 0$ and $\max R(z) = R_1$ for $z \in [0,z^1]$ then $\bar{z}^0 = 0$.

e) If $R'(Z) = 0$ and $\max R(z) = R_2$ for $z \in [z^N, Z]$ then $\bar{z}^N = Z$.

Let the inverse of the boundary functions as follows

$$\text{(1.3.86)} \qquad \operatorname{inv} R^k(r) = \operatorname{inv}\big(R(z)\big), \quad z^k \le z < \bar{z}^k,$$

$$\text{(1.3.87)} \qquad \operatorname{inv} \tilde{R}^k(r) = \operatorname{inv}\big(R(z)\big), \quad \bar{z}^k \le z < z^{k+1}.$$

Let us define the following functions:

$$\text{(1.3.88)} \qquad Z^0(r) = \begin{cases} 0 & , 0 \le r \le r^0. \\ \operatorname{inv} R^0(r), & r^0 < r \le \bar{r}^0. \end{cases}$$

$$\text{(1.3.89)} \qquad \bar{Z}^0(r) = \begin{cases} \operatorname{inv} \tilde{R}^0(r) & , r^0 > r \ge r^1. \\ \operatorname{inv} \tilde{R}^{m_{i-1}}(r), & r^{m_{i-1}} > r \ge r^{m_i}. \\ Z & , \min\limits_j r^j > r \ge 0. \end{cases}$$

where $m_0 = 1$ and $m_i = \min \{j \mid j > m_{i-1},\ r^j < r^{m_{i-1}}\}$.

Let

$$\text{(1.3.90)} \qquad Z^k(r) = \operatorname{inv} R^k(r), \quad r^k \le r < \bar{r}^k.$$

$$\text{(1.3.91)} \qquad \bar{Z}^k(r) = \begin{cases} \operatorname{inv} \tilde{R}^k(r), & \bar{r}^k \ge r \ge \max\big(r^k, r^{k+1}\big). \\ \operatorname{inv} \tilde{R}^{m^k_{i-1}} & , r^{m^k_{i-1}} > r \ge \max\big(r^k, r^{m^k_i}\big). \\ Z & , r^{k+1} > r \ge r^k \text{ if } r^k = \min\big(r^j \mid j \ge k\big). \end{cases}$$

where $m^k_0 = k + 1$, $m^k_i = \min \Big(j \mid j > m^k_{k-1},\ r^j < r^{m^k_{i-1}}\Big)$.

Therefore, it is true that $\Omega_e = \bigcup\limits_{k=0}^{N} \Omega^k_e$, where the elements Ω^k_e are defined as follows

$$\text{(1.3.92)} \qquad \Omega^0_e = \big[0, \bar{r}^0\big] \times \big[Z^0(r), \bar{Z}^0(r)\big],$$

(1.3.93) $\Omega_e^k = [r^k, \bar{r}^k] \times [Z^k(r), \bar{Z}^k(r)]$, $k = \overline{1,N}$.

It can be seen that if $(z^*, r^*) \in \Omega_e^k$ then for every $z < z^*$

(1.3.94) $(z, r^*) \in \Omega_e^k$, $k = \overline{0,N}$.

1.3.5. The convergence of the approximate solutions.

LEMMA 1.3.1. In the two-dimensional problem (1.1.1) on Ω_e, for every $u(z,r) \in H_A$ there exists a constant c_o such that

$$||u||_{W_2^1} \le c_o[u].$$

PROOF. For the L_2-norm of the first derivatives of the function $u(z,r)$ on Ω_e we have the same result obtained in lemma 1.2.2, i.e.

(1.3.95) $$||\frac{\partial u}{\partial z}||_{L_2}^2 + ||\frac{\partial u}{\partial r}||_{L_2}^2 \le \bar{R}^3[u,u],$$

where $\bar{R} = \max_z R(z)$.

To get the L_2-norm of the function $u(z,r)$ on Ω_e, let

(1.3.96) $$u(z,r) \int_0^r \frac{\partial u(z,r)}{\partial x} dx = \int_0^r x^{3/2} \frac{1}{x^{3/2}} \frac{\partial u(z,r)}{\partial x} dx.$$

By using Cauchy-Bunyakovsky's inequality to the square of the integral (1.3.96) we get

(1.3.97) $$|(z,r)|^2 \le \int_0^r x^3 dx \int_0^r \frac{1}{x^3} |\frac{\partial u(z,r)}{\partial x}|^2 dx \le$$

$$\le \frac{r^4}{4} \int_0^{R(z)} \frac{1}{x^3} |\frac{\partial u(z,r)}{\partial x}|^2 dx.$$

Integrating the both sides of (1.3.97) over the domain Ω_e we have

(1.3.98) $$\int_{\Omega_e} |u(z,r)|^2 dzdr = ||u||^2_{L_2} \leq$$

$$\leq \int_0^Z \left[\int_0^{R(Z)} \frac{r^4}{4} dr \int_0^{R(Z)} \frac{1}{x^3} \left| \frac{\partial u(z,x)}{\partial x} \right|^2 dx \right] dz \leq$$

$$\leq \int_0^{\bar{R}} \frac{r^4}{4} dr \int_0^Z \left[\int_0^{R(Z)} \frac{1}{r^3} \left| \frac{\partial u(z,x)}{\partial x} \right|^2 dr \right] dz = \frac{\bar{R}^5}{20} \int_{\Omega_e} \frac{1}{r^3} \left| \frac{\partial u(z,r)}{\partial r} \right|^2 dzdr$$

Let

(1.3.99) $$u(z,r) = \int_{z^k(r)}^{z} \frac{\partial u(y,r)}{\partial y} dy, \quad (z,r) \in \Omega_e^k,$$

where $u(z,r) = 0$ if $(z,r) \in \Gamma_1 \cup \Gamma_2$.
By using Cauchy-Bunyakovsky's inequality to the square of the integral (1.3.99) we obtain

(1.3.100) $$|u(z,r)|^2 \leq \int_{z^k(r)}^{z} dy \int_{z^k(r)}^{z} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy \leq$$

$$\leq (z - z^k(r)) \cdot \int_{z^k(r)}^{\bar{z}^k(r)} \left|\frac{\partial u(y,r)}{\partial y}\right| dy.$$

Let us integrate the left hand side of (1.3.100) over Ω_e

(1.3.101) $$\int_{\Omega_e} |u(z,r)|^2 dzdr = \sum_{k=o}^{n} \int_{\Omega_e^k} |u(z,r)|^2 dzdr.$$

Integrating the both sides of (1.3.100) over Ω_e^k we get

(1.3.102) $$\int_{\Omega_e^k} |u(z,r)|^2 dzdr \le$$

$$\le \int_{r^k}^{\bar r^k}\left[\int_{z^k(r)}^{\bar z^k(r)} (z - z^k(r))dz \int_{z^k(r)}^{\bar z^k(r)} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy\right]dr =$$

$$= \int_{r^k}^{\bar r^k}\left[\frac{(\bar z^k(r) - z^k(r))^2}{2} r^3 \int_{z^k(r)}^{\bar z^k(r)} \frac{1}{r^3} \left|\frac{\partial u(y,r)}{\partial y}\right|^2 dy\right]dr \le$$

$$\le \frac{Z^2\bar R^3}{2}\int_{r^k}^{\bar r^k}\int_{z^k(r)}^{\bar z^k(r)} \frac{1}{r^3}\left|\frac{\partial u(y,r)}{\partial z}\right|^2 dzdr =$$

$$= \frac{Z^2\bar R^3}{2}\int_{\Omega_e^k}\frac{1}{r^3}\left|\frac{\partial u(y,r)}{\partial z}\right|^2 dzdr.$$

Therefore from (1.3.102) we have that

(1.3.103) $$||u||^2_{L_2} \le \frac{Z^2\bar R^3}{2}\sum_{k=o}^{n}\int_{\Omega_e^k}\frac{1}{r^3}\left|\frac{\partial u(y,r)}{\partial z}\right|^2 dzdr.$$

Now from (1.3.98) and (1.3.103) we get the L_2-norm of the function $u(z,r)$ on Ω_e

(1.3.104) $$||u||^2_{L_2} \le \frac{1}{2}\max\left(\frac{\bar R^5}{2}, \frac{Z^2\bar R^3}{2}\right)\int_{\Omega_e}\frac{1}{r^3}\left[\left|\frac{\partial u(z,r)}{\partial z}\right|^2 + \right.$$

$$\left. + \left|\frac{\partial u(z,r)}{\partial z}\right|^2 dzdr.\right.$$

Finally from (1.3.95) and (1.3.104) we get the W_2^1-norm of $u(z,r)$ on Ω_e

(1.3.105) $$||u||^2_{W_2^1} \le c_o[u],$$

where

(1.3.106) $$c_o = \sqrt{\bar{R}^3 + \max\left(\frac{\bar{R}^3}{40}, \frac{Z^2\bar{R}^3}{4}\right)} > 0.$$

THEOREM 1.3.2. The approximated solution $\tilde{S}$ obtained from the minimization of the functional (1.2.39) over the family of splines (1.2.9) on Ω_e is convergent to the generalized solution $\tilde{u}_o$ of the original problem (1.1.1) in the W_2^1-norm

$$||\tilde{u}_o - \tilde{S}||_{W_2^1} \to 0 \text{ if } h_z, h_r \to 0,$$

where $h_z = \max |z_{i+1} - z_i|$, $h_r = \max |r_{j+1} - r_j|$.

PROOF. In theorem 1.2.6 we proved the following inequality

(1.3.107) $$[\tilde{u}_o - \tilde{S}] \le \inf_{S_\Delta} [\tilde{u}_o - S_\Delta].$$

Hence from theorem 1.3.3, lemma 1.3.1 and (1.3.107) we obtain that

(1.3.108) $$||[\tilde{u}_o - \tilde{S}||_{W_2^1} \le C[\tilde{u}_o - \tilde{S}] \le C \inf_{S_\Delta} [\tilde{u}_o - S_\Delta] \to 0 \text{ if } h_z, h_r \to 0.$$

Remark. If $\tilde{u}_o$ is the classical solution of the equivalent problem of (1.1.1) on Ω_e, it mean $\tilde{u}_o \in D(A)$ and at the same time $\tilde{u}_o \in W_2^2$. Therefore, theorem 1.3.2 can be used with lemma 1.3.1 and (1.3.107) to obtain the better estimation for $||\tilde{u}_o - \tilde{S}||$:

(1.3.109) $$||\tilde{u}_o - \tilde{S}||_{W_2^1} = O(h_z + h_r).$$

CHAPTER II

Numerical Solution For The Plastic Problem

2.0. Comments and literatures. In a part of the previous chapter we investigated the solution of the elastic problem in a cylinder subjected to a torque applied at both ends. The material of the cylinder of variable diameter is supposed to be only elastic. However, as the torque increases the stress function q detined by

$$q = \frac{1}{r^2} |\text{grad } u|$$

increases, particularly at the neighborhood of Γ_1, and a small plastic domain can forms (see [1], [36], [37]). Ω is thus divided into two domains, the elastic domain Ω_e and the plastic domain Ω_p with unknown free boundary separating them. In this chapter we shall consider the solution of the plastic problem. For the solution of the non-linear first order hyperbolic equation by finite difference method we shall use the common explicit and implicit schemes. In this matter we refer to the books [14], [32], [33].

2.1. Introduction. The plastic problem we shall consider is a nonlinear first order equation with a given boundary conditions, and so it is a Cauchy problem. The problem is summarized as follows [1]: Find the function u which must satisfy the nonlinear hyperbolic equation

(2.1.1) $$r^4 q^2 := \left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = r^4 k^2 , \text{ on } \Omega_p ,$$

(where q is a stress function [1]),

with the boundary conditions:

(2.1.2) $\quad u(z,R_1(z)) = F(z)$, $\quad$ on Γ_1 ,

(2.1.3) $\quad \frac{\partial u}{\partial z}(z,r)\Big|_{z=0} = \phi_1(r)$, $\quad$ on Γ_2 ,

(2.1.4) $\quad \frac{\partial u}{\partial z}(z,r)\Big|_{z=Z} = \phi_2(r)$, $\quad$ on Γ_3 ,

where $\Gamma_i (i = \overline{1,3})$ are as shown in FIG. 2.1

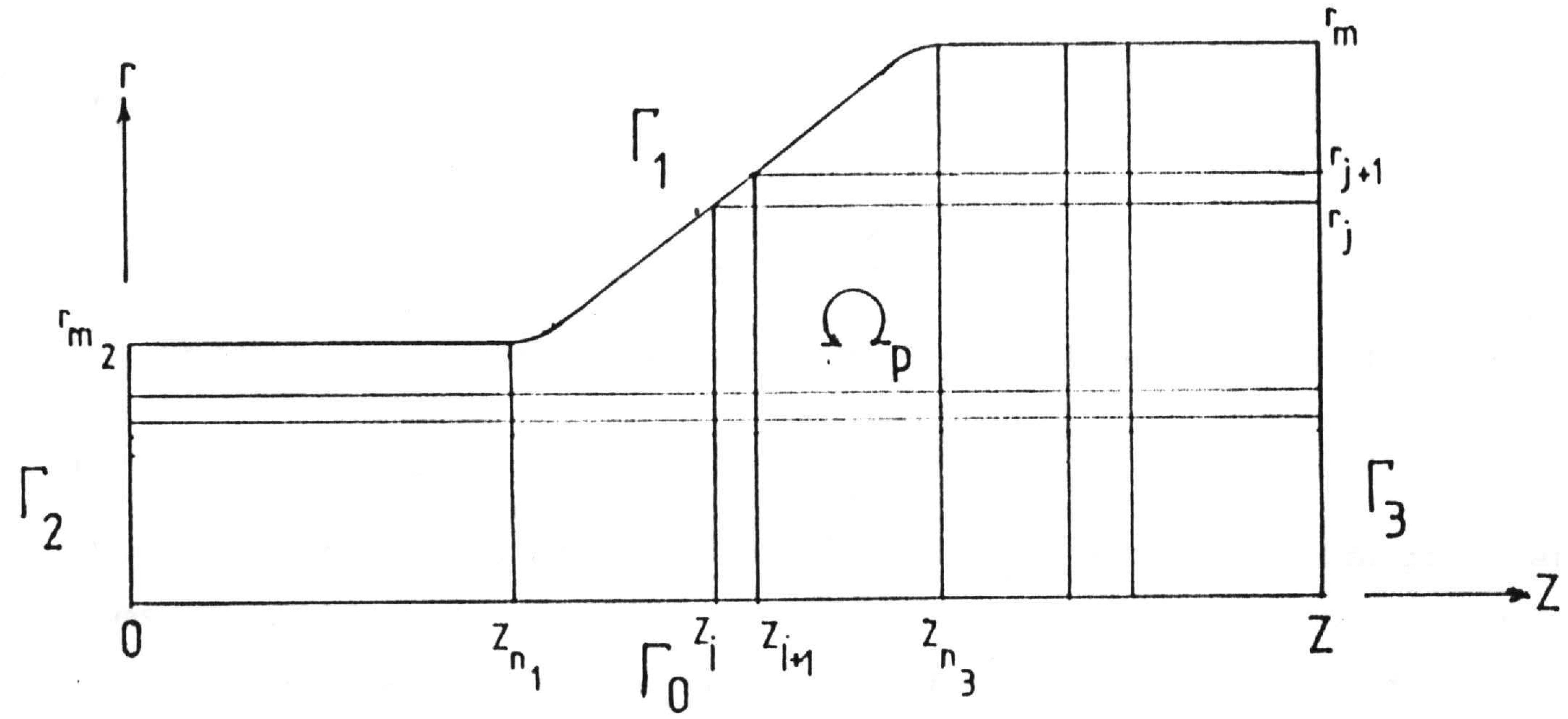

FIGURE 2.1. THE DOMAIN Ω_p

We remark that the conditions (2.1.2)-(2.1.4) are more general than those used in [1]. Notice that the above plastic problem formulation did not involve the boundary Γ_o for two reasons. The first reason because (2.1.1) is the first order hyperbolic equation so theoreticaly it is enough to satisfy the three boundary conditions (2.1.2)-(2.1.4). The second reason that physicaly if the plastic domain Ω_p will occupy all the domain of the cylinder [1] including the boundary Γ_o it means that the rotation of the torque T is too great and no solution exists.

We are going to find an approximate solution for the problem by using finite difference method.

2.2. The construction of the finite difference method.

Let the domain Ω_p $[0,Z]\times[0,R_1(z)]$ be divided by a grid-points similar to those used for the division of the domain Ω_e in chapter I (see FIG.2.1). That is let

(2.2.1) $$0 = z_0 < z_1 < \ldots < z_i < z_{i+1} < \ldots < z_n = Z,$$

(2.2.2) $$0 = r_0 < r_1 < \ldots < r_j < r_{j+1} < \ldots < r_m ,$$

and $(z_{n_1}, r_{m_2}), (z_{n_3}, r_m)$ are grid-points, where

(2.2.3) $$0 < n_1 < n_3 < n , \qquad 0 < m_2 < m.$$

The given function $r = R_1(z)$ is constant for $z \in [0, z_{n_1}]$ and $z \in [z_{n_3}, Z]$. Increasing on the interval (z_{n_1}, z_{n_3}). Suppose that

(2.2.4) $$r_{m_2+i} = R_1(z_{n_1+i}) , \quad i = \overline{1, n_3 - n_1} .$$

Obviosly the elementary domains on Ω_p can be rectangles and triangles as well:

1. Rectangulars, inside Ω_p, at the neighborhood of Γ_1 for the intervals $(0, z_{n_1})$, (z_{n_3}, Z), and at the neighborhood of Γ_2 and Γ_3.

2. Triangulars only at the neighborhood of Γ_1 in the interval (z_{n_1}, z_{n_3}).

Now from the finite difference method let us first construct an explicit scheme for the equation (2.1.1) with the boundary conditions (2.1.2)-(2.1.4) on the predescribed elementary domains separately. First for the rectangular elements let

(2.2.5) $$\left.\frac{\partial u}{\partial r}\right|_{z_i, r_j} \cong \frac{u_{i,j} - u_{i,j+1}}{h_{r_j}} ,$$

(2.2.6) $$\left.\frac{\partial u}{\partial z}\right|_{z_i, r_j} \cong \frac{u_{i+1,j+1} - u_{i-1,j+1}}{h_{z_{i+1}} + h_{z_i}} ,$$

where

(2.2.7) $h_{r_j} = r_j - r_{j-1}$, $h_{z_i} = z_i - z_{i-1}$.

Therefore from (2.2.5),(2.2.6), the equation (2.1.1) can be written approximatly as follows:

(2.2.8)

$$u_{i,j} = u_{i,j+1} - h_{r_{j+1}} \sqrt{\left(\frac{r_j + r_{j+1}}{2}\right)^4 k^2 - \left(\frac{u_{i+1,j+1} - u_{i-1,j+1}}{h_{z_{i+1}} + h_{z_i}}\right)^2} .$$

The scheme (2.28) satisfy the boundary conditions (2.1.2)-(2.1.4) as follows:

(2.2.9) $u_{i,m_2-1} = F(z_i) -$

$$- h_{r_{m_2}} \sqrt{\left(\frac{r_{m_2-1} + r_{m_2}}{2}\right)^4 k^2 - \left(\frac{F(z_{i+1}) - F(z_{i-1})}{h_{z_{i+1}} + h_{z_i}}\right)^2}, \quad i = \overline{0, n_1 - 1},$$

(2.2.10) $u_{i,m-1} = F(z_i) -$

$$- h_{r_m} \sqrt{\left(\frac{r_{m-1} + r_m}{2}\right)^4 k^2 - \left(\frac{F(z_{i+1}) - F(z_{i-1})}{h_{z_{i+1}} + h_{z_i}}\right)^2}, \quad i = \overline{n_3, n},$$

(2.2.11)

$$u_{0,j} = u_{0,j+1} - h_{r_{j+1}} \sqrt{\left(\frac{r_j + r_{j+1}}{2}\right)^4 k^2 - \left(\phi_1\left(\frac{r_j + r_{j+1}}{2}\right)\right)^2} , j = m_2-1, m_2-2, ..$$

(2.2.12)

$$u_{n,j} = u_{n,j+1} - h_{r_{j+1}} \sqrt{\left(\frac{r_j + r_{j+1}}{2}\right)^4 k^2 - \left(\phi_2\left(\frac{r_j + r_{j+1}}{2}\right)\right)^2} , j = m - 1, m - 2, .$$

In the triangular elements let us look for an explicit scheme to find the solutions $u_{i,j-1}$ ($i = n_1 + \ell$, $j = m_2 + \ell - 1$, $\ell = \overline{1, n_3 - n_1}$). It is seen that for these solutions exist two cases.

Case 1. Let

(2.2.13) $F(z_i) = u(z_i, r_j)$,

on the curved part of Γ_1. Suppose that we know the solution in $r = r_j$. Hence we can write the following relation

(2.2.14) $$\frac{F(z_i)-F(z_{i-1})}{\sqrt{h_{z_i}^2 + h_{r_j}^2}} = \frac{u(z_i, r_j)-u(z_{i-1}, r_{j-1})}{\sqrt{h_{z_i}^2 + h_{r_j}^2}} \cong (\underline{S}\ \nabla)u,$$

where $$\underline{S} = \left[\frac{h_z}{\sqrt{h_z^2 + h_r^2}} , \frac{h_r}{\sqrt{h_z^2 + h_r^2}} \right], \qquad |\underline{S}| = 1.$$

Hence

(2.2.15) $$(\underline{S}\ \nabla)u = \frac{h_{z_i}}{\sqrt{h_z^2 + h_r^2}} \frac{\partial u}{\partial z} + \frac{h_r}{\sqrt{h_z^2 + h_r^2}} \frac{\partial u}{\partial r}.$$

Therefore on the interval (z_{i-1}, z_i) we get that

(2.2.16) $$\frac{\partial u}{\partial z}\Big|_- \cong \frac{F(z_i)-F(z_{i-1})}{h_{z_i}} - \frac{h_{r_j}}{h_{z_i}} \frac{\partial u}{\partial r},$$

and on the interval (z_i, z_{i+1}) we get that

(2.2.17) $$\frac{\partial u}{\partial z}\Big|_+ \cong \frac{u(z_{i+1}, r_j)-u(z_i, r_j)}{h_{z_{i+1}}} = \frac{u(z_{i+1}, r_j)-F(z_i)}{h_{z_{i+1}}}.$$

From (2.2.16),(2.2.17) we get

$$\frac{\partial u}{\partial z}\Big|_{z_i} \cong \frac{1}{2}\left(\frac{\partial u}{\partial z}\Big|_+ + \frac{\partial u}{\partial z}\Big|_-\right) \cong \frac{u(z_{i+1}, r_j)-F(z_i)}{2h_{z_{i+1}}} +$$

$$+ \frac{F(z_i)-F(z_{i-1})}{2h_{z_i}} - \frac{h_{r_j}}{2h_{z_i}} \frac{\partial u}{\partial r}.$$

Let

(2.2.18) $$F^* = \frac{u(z_{i+1}, r_j)-F(z_i)}{2h_{z_{i+1}}} + \frac{F(z_i)-F(z_{i-1})}{2h_{z_i}},$$

and so

(2.2.19) $$\frac{\partial u}{\partial z}\Big|_{z_i} \cong F^* - \frac{h_{r_j}}{2h_{z_i}} \frac{\partial u}{\partial r}.$$

Therefore from (2.2.19), the equation (2.1.1) can be written as follows

(2.2.20) $$\left(\frac{\partial u}{\partial r}\right)^2 - \frac{4h_{z_i} h_{r_j}}{h_{r_j}^2 + 4h_{z_i}^2} F^* \frac{\partial u}{\partial r} -$$

$$- \frac{4h_{z_i}^2}{h_{r_j}^2 + 4h_{z_i}^2} \left[k^2\left(\frac{r_j + r_{j-1}}{2}\right)^4 - F^{*2}\right] = 0.$$

Hence

(2.2.21) $$\frac{\partial u}{\partial r} \cong \frac{2h_{z_i}}{h_{r_j}^2 + 4h_{z_i}^2} \left[h_{r_j} F^* + \right.$$

$$\left. + \sqrt{(h_{r_j}^2 + 4h_{z_i}^2)\left(\frac{r_j + r_{j-1}}{2}\right)^4 k^2 - 4h_{z_i}^2 F^{*2}}\right].$$

Let us write that

(2.2.22) $$\frac{\partial u}{\partial r} \cong \frac{u_{i,j} - u_{i,j-1}}{h_{r_j}}.$$

Therefore

(2.2.23) $$u_{i,j-1} = u_{i,j} - \frac{2h_{z_i} h_{r_i}}{h_{r_j}^2 + 4h_{z_i}^2} \left[h_{r_j} F^* + \right.$$

$$\left. + \sqrt{(h_{r_j}^2 + 4h_{z_i}^2)\left(\frac{r_j + r_{j-1}}{2}\right)^4 k^2 - 4h_{z_i}^2 F^{*2}}\right].$$

Case 2. The only difference in this case from the first that now for $i = n_3$ we have

$$(2.2.24)\qquad \frac{\partial u}{\partial z}\Big|_{+} = \frac{F(z_{i+1})-F(z_i)}{h_{z_{i+1}}}.$$

Therefore (2.2.18) become

$$(2.2.25)\qquad F^{*} = \frac{F(z_{i+1},r_j)-F(z_i)}{2h_{z_{i+1}}} + \frac{F(z_i)-F(z_{i-1})}{2h_{z_i}}.$$

2.3. The accuracy of the approximate solution.

THEOREM 2.3.1. The accuracy of the approximate solution obtained by the explicit scheme (2.2.8) has the following estimation

$$\psi = O(h_r + h_z^2).$$

PROOF. From the equation (2.2.8). we can write the following

$$(2.3.1)\qquad \left(\frac{u_{i,j+1} - u_{i,j})}{h_{r_{j+1}}}\right)^2 + \left(\frac{u_{i+1,j+1} - u_{i-1,j+1}}{h_{z_i} + h_{z_{i+1}}}\right)^2 =$$

$$= \left(\frac{r_j + r_{j+1}}{2}\right)^4 k^2 + \psi,\quad i = \overline{0,Z_n-1},\quad j = \overline{0,r_m-1},$$

where ψ is the error of the approximation. Let

$$(2.3.2)\qquad u^{*} = u\left(z_i, r_j + \frac{h_{r_{j+1}}}{2}\right).$$

The Taylor's expansions for the functions $u_{i,j}$, $u_{i,j+1}$, $u_{i-1,j+1}$, $u_{i+1,j+1}$, can be written as follows

$$(2.3.3)\qquad u_{i,j} = u^{*} - \frac{h_{r_{j+1}}}{2}\,\frac{\partial u}{\partial r}\Big|^{*} + \frac{h_{r_{j+1}}^2}{8}\,\frac{\partial^2 u}{\partial r^2}\Big|^{*} + O(h_r^3),$$

$$(2.3.4) \qquad u_{i,j+1} = u^* + \frac{h_{r_{j+1}}}{2} \frac{\partial u}{\partial r}\Big|^* + \frac{h^2_{r_{j+1}}}{8} \frac{\partial^2 u}{\partial r^2}\Big|^* + O(h^3_r),$$

$$(2.3.5) \qquad u_{i+1,j+1} = u^* + \frac{h_{r_{j+1}}}{2} \frac{\partial u}{\partial r}\Big|^* + h_{z_{i+1}} \frac{\partial u}{\partial z}\Big|^* +$$

$$+ \frac{h^2_{r_{j+1}}}{8} \frac{\partial^2 u}{\partial r^2}\Big|^* + h_{r_{j+1}} h_{z_{i+1}} \frac{\partial^2 u}{\partial r\,\partial z}\Big|^* + \frac{h^2_{z_{j+1}}}{2} \frac{\partial^2 u}{\partial z^2}\Big|^* +$$

$$+ O(h^3_r + h^3_z),$$

$$(2.3.6) \qquad u_{i-1,j+1} = u^* + \frac{h_{r_{j+1}}}{2} \frac{\partial u}{\partial r}\Big|^* - h_{z_i} \frac{\partial u}{\partial z}\Big|^* +$$

$$+ \frac{h^2_{r_{j+1}}}{8} \frac{\partial^2 u}{\partial r^2}\Big|^* - h_{r_{j+1}} h_{z_i} \frac{\partial^2 u}{\partial r\,\partial z}\Big|^* + \frac{h^2_{z_i}}{2} \frac{\partial^2 u}{\partial z^2}\Big|^* + O(h^3_r + h^3_z).$$

By substituting (2.3.3)-(2.3.6) in (2.3.1) we get that

$$(2.3.7) \qquad \frac{\partial u}{\partial r}\Big|^{*2} + O(h^2_r) + \frac{\partial u}{\partial z}\Big|^{*2} + O(h_r + h^2_z) = r^{*4} K^2 + \psi .$$

Therefore, from (2.1.1) and (2.3.7) we have that

$$(2.3.8) \qquad \psi = O(h_r + h^2_z).$$

Remark. In practise the domain Ω_p must be small [1], therefore it is not need to consider the convergence of the approximate solution of the problem. But we can say that from the theory of the stability of hyperbolic equations it is known that the scheme is stabile if $h_r = O(h_z)$ and this condition easy can be satisfied by the scheme (2.2.8).

2.4. In section 2.2 the difference scheme was an explicit one. It can be constructed also the following implicit scheme for the equation (2.1.1)

(2.4.1) $$u_{i,j} = u_{i,j+1} -$$

$$- h_{r_{j+1}} \sqrt{\left(\frac{r_j + r_{j+1}}{2}\right)^4 k^2 - \left(\frac{u_{i+1,j+1} - u_{i-1,j+1} + u_{i+1,j} - u_{i-1,j}}{2(h_{z_{i-1}} + h_{z_i})}\right)^2}.$$

The implicit scheme (2.4.1) have two advantages with respect to the explicit scheme (2.2.8). First the stability of the implicit scheme is better than those of the explicit scheme. Second the implicit scheme is more accurate than the explicit scheme.

THEOREM 2.4.1. The accuracy of the approximate solution obtained by implicit scheme (2.4.1) has the following estimation

(2.4.2) $$\psi = O(h_z^2 + h_r^2).$$

The proof of this theorem is similar to that one of theorem 2.3.1.

Remark. In the explicit scheme if we have the solution in a given row, the next row for every equation can be solved independently from the other. The first disadvantage of the implicit scheme that in a given row one must solve a system of nonlinear equations. For the solution of the system (2.4.1) one of the well known methods from the theory of nonlinear systems of algebric equations can be used. The second disadvantage of the implicit scheme that the explicit scheme can be better applied for problems of varied boundaries as the case in the free boundary problems.

Notice that in the solution of the elastic-plastic problems, which will be consider in next chapters, we can use the obtained solution of the plastic problem as a given one, while it is not depend upon the solution of the elastic problem.

CHAPTER III

Interation Process For The One-Dimensional Elastic Platic Problem

3.0. Comments and literature. The classical one-dimensional elastic-plastic problem can be alternatively reformulated to the following variational inequality [1]: Find $u \in H_A$ such that

$$a(u, v-u) \geq 0, \text{ for all } v \in H_A$$

where

$$H_A = (v \in c^1(0,1); \quad v(0) = 0, \quad v(1) = 1, \quad v \geq \phi)$$

and the obstcal ϕ be such that

$$\text{grad } \phi = \frac{\partial \phi}{\partial r} = kr^2, \quad 0 < r < 1, \quad \phi(1) = 1,$$

so that

$$\phi = kr^3/3 + (1 - k/3), \quad 0 < r < 1.$$

Let a be the bilinear functionnal

$$a(u,v) = \int_0^1 \frac{1}{r^3} u_r(v_r - u_r)dr + \int_\tau^1 \frac{1}{r^3} u_r(v_r - u_r)dr.$$

Integrating by parts and since $v(0) = u(0) = 0$, $v(1) = u(1) = 1$, $Au = 0$ in $\Omega_e = (0,\tau)$, and $u = \phi$ in $\Omega_p = (\tau,1)$, it is obtained

$$a(u, v - u) = \int_\tau^1 (v - \phi) \, Au \, dr.$$

One can see that $v \geq \phi$ and $Au = k/r^2 > 0$ in Ω_p, therefore u is the solution of the above variational inequality. In [1] the classical two-dimensional elastic-plastic problem also was reformulated as a variational inequality problem. The free boundary problems were reformulated almost as a variational inequalities regarding their mechanical and physical situtions. We refer to the books and works of [15], [16], [17], [18], [19],

[20], [21], [22].

There is a connection between the variational inequalities and the minimization problems introduced in chapter I. This connection is given by the following result: An element u is the solution of the abstract minimization problem if and only if

$$u \in U \text{ and } v \in U, \quad a(u, v - u) \geq f(v - u).$$

Methods of numerical approximation of solutions of variational inequalities are treated in [23], [24], [25], [26].

3.1. Introduction. In this chapter we want to present iteration process ideas for solving the one-dimensional elastic-plastic problem although its exact solution is known. Because we intend from presenting these ideas that they will be useful for solving the two-dimensional problems too as it will be seen in the next chapters.

3.2. The elastic-plastic problem for the one-dimensional case can be summarized as follows:

The solutions, of the elastic problem u^e, and of the plastic problem u^p, satisfy the following equations

$$Au^e := -\frac{d}{dr}\left(\frac{1}{r^3}\frac{du^e}{dr}\right) = 0, \qquad r \in (0, \bar{r}), \tag{3.2.1}$$

$$q := \frac{1}{r^2}\left|\frac{du^p}{dr}\right| = k, \qquad r \in (\bar{r}, R), \tag{3.2.2}$$

with the boundary conditions

$$u^e(0) = 0, \tag{3.2.3}$$

$$u^p(R) = T, \tag{3.2.4}$$

$$u^e(\bar{r}) = u^p(\bar{r}), \tag{3.2.5}$$

$$\frac{du^e}{dr}(\bar{r}) = \frac{du^p}{dr}(\bar{r}). \tag{3.2.6}$$

where $\bar{r} \in (0, R)$ is the unknown free boundary of the problem.

Now from the equations (3.2.1), (3.2.2) and the boundary conditions (3.2.3)-(3.2.6), one can obtain the exact solutions of

u^e, u^p and $\bar{r}$ [1]:

$$(3.2.7) \qquad u^e = \frac{k}{4\bar{r}} r^4 ,$$

$$(3.2.8) \qquad u^p = T - \frac{k}{3} (R^3 - r^3),$$

$$(3.2.9) \qquad \bar{r} = \sqrt[3]{4R^3 - \frac{12}{k} T} .$$

Notice that the solution u^p is not depends upon $\bar{r}$. The point $\bar{r}$ depends upon the value of the coefficient k. The only possibilty of the existence of the solution $\bar{r}$ that k has the following values

$$\frac{3T}{R^3} < k < \frac{4T}{R^3} .$$

3.3. The first iteration process idea and its convergence. Suppose that the free boundary $\bar{r}$ is approximated by $\tilde{r}$ and it is true for $\tilde{r}$ the following

$$(3.3.1) \qquad \tilde{r} = \bar{r} + \delta ,$$

where δ is a small number.

Let us construct from (3.2.7), (3.2.8)an approximate solution for the elastic problem by using the condition (3.2.5) at $\tilde{r}$ point:

$$(3.3.2) \qquad \tilde{u}^e (\tilde{r}) = u^p (\tilde{r})$$

Hence it is obtained that

$$(3.3.3) \qquad \tilde{u}^e(r) = \frac{r^4}{\tilde{r}^4} \left(T - \frac{k}{3} (R^3 - \tilde{r}^3)\right).$$

Now the correction of the free boundary can be done from the difference between the first derivatives of the functions $\tilde{u}^e$ and u^p at the point $\tilde{r}$ as follows: Let

$$(3.3.4) \qquad \zeta = \left[\frac{du^p}{dr} (\tilde{r}) - \frac{d\tilde{u}^e}{dr} (\tilde{r})\right] =$$

$$= k\tilde{r}^2 - \frac{4}{\tilde{r}} \left[T - \frac{k}{3} (R^3 - \tilde{r}^3)\right] = k\tilde{r}^2 - \frac{4}{\tilde{r}} \left[- \frac{k\bar{r}^3}{12} + \frac{k\tilde{r}^3}{3}\right] =$$

$$= \frac{k(\bar{r}^3 - \tilde{r}^3)}{3\tilde{r}} = - k\tilde{r}\delta + k\delta^2 + O(\delta^3).$$

From (3.3.4) we can write that

$$\delta \cong -\frac{1}{k\tilde{r}}\zeta. \tag{3.3.5}$$

Let

$$\tilde{\delta} = -\frac{1}{k\tilde{r}}\left(\frac{du^p}{dr} - \frac{d\tilde{u}^e}{dr}\right)_{\tilde{r}}. \tag{3.3.6}$$

The formula (3.3.6) can be used as an approximation of δ, as well as it can be applied for the iteration process: Let $\tilde{r}_j$ be the j-th approximation of $\bar{r}$ and so

$$\tilde{u}^e(\tilde{r}_j) = u^p(\tilde{r}_j), \quad \tilde{\delta}_j = -\frac{1}{k\tilde{r}_j}\left(\frac{du^p}{dr} - \frac{d\tilde{u}^e}{dr}\right)_{\tilde{r}_j}, \tag{3.3.7}$$

The next approximation $\tilde{r}_{j+1}$ can be written as follows

$$\tilde{r}_{j+1} = \tilde{r}_j - \tilde{\delta}_j. \tag{3.3.8}$$

Let us consider the convergence of this iteration process by the following theorem.

THEOREM 3.3.1. The iteration process (3.3.7) and (3.3.8) is convergent and the following estimation is satisfied

$$|\tilde{r}_{j+1} - \bar{r}| = O\left(|\tilde{r}_j - \bar{r}|^2\right).$$

PROOF. From (3.3.1), (3.3.4), (3.3.7), (3.3.8) we have

$$\delta_{j+1} = \tilde{r}_{j+1} - \bar{r} = \tilde{r}_j + \frac{1}{k\tilde{r}_j}\left(\frac{du^p}{dr} - \frac{d\tilde{u}^e}{dr}\right)_{\tilde{r}_j} - \bar{r} = \delta_j + \tag{3.3.10}$$

$$+ \frac{1}{k\tilde{r}_j}\left[-k\tilde{r}_j\delta_j + k\delta_j^2 + O(\delta_j^3)\right] = \frac{\delta_j^2}{\tilde{r}_j} + O(\delta_j^3).$$

Thus

$$|\tilde{r}_{j+1} - \bar{r}| = O\left(|\tilde{r}_j - \bar{r}|^2\right), \qquad j = 0,1\ldots.. \tag{3.3.11}$$

3.4 The second iteration process idea. Let $\tilde{r}$ be the approximated boundary between the elastic and the plastic domain, and

(3.4.1) $\qquad \tilde{u}^e(\tilde{r}) = u^p(\tilde{r}), \ \tilde{r} \neq \bar{r}.$

It is easy to show that there exists the second intersection point $\tilde{\tilde{r}}$ of the curves $\tilde{u}^e(r)$ and $u^p(r)$ where

(3.4.2) $\qquad \tilde{u}^e(\tilde{\tilde{r}}) = u^p(\tilde{\tilde{r}}),$

and $\qquad \tilde{r} < \bar{r} < \tilde{\tilde{r}} \qquad$ or $\qquad \tilde{\tilde{r}} < \bar{r} < \tilde{r}.$

Suppose that

(3.4.3) $\qquad \tilde{r} - \bar{r} = \delta,$

and $|\delta|$ is small number. Now if

(3.4.4) $\qquad \bar{r} - \tilde{\tilde{r}} = \varepsilon,$

it can be shown from (3.3.3) and (3.4.2) that

(3.4.5) $\qquad \varepsilon = \delta - \dfrac{8}{3\tilde{r}}\delta^2 + O(\delta^3).$

Clearly for given $\tilde{r}_j$ the function $\tilde{u}^e(r)$ can be obtained from (3.3.3). After that the equation (3.4.2) need be solved for the other intersection point $\tilde{\tilde{r}}_j$. After that the next approximation can be written as follows

(3.4.6) $\qquad \tilde{r}_{j+1} = \dfrac{\tilde{r}_j + \tilde{\tilde{r}}_j}{2}.$

To consider the convergence of the iteration process one can obtain from (3.4.3)-(3.4.6) that

(3.4.7) $\qquad \delta_{j+1} = \tilde{r}_{j+1} - \bar{r} = \dfrac{\tilde{r}_j + \tilde{\tilde{r}}_j}{2} - \bar{r} = \dfrac{\delta_j + \varepsilon_j}{2} = O(\delta_j^2).$

This result can be formulated in the following theorem.

THEOREM. 3.4.1. The iteration process (3.4.6) is convergent and the following estimation is true

$$|\tilde{r}_{j+1} - \bar{r}| = O\left(|\tilde{r}_j - \bar{r}|\right).$$

Chapter IV

Two-Dimensional Free Boundary
Problem In Rectangular Domain

4.0. Comments and literatures. For the Fourier trigonometric series we shall use in this chapter let the boundary condition function (see 4.1.4) $\bar{f}(z)$ be a piecewise continuous with period Z. This function can be represented in the form of a sum of infinite number of harmonics

$$u_n = a_n \cos \frac{n\pi z}{Z} + b_n \sin \frac{n\pi z}{Z} , \quad n = 0,1,2,\ldots$$

of the same period Z. Thus we come to the Fourier series

$$\bar{f}(z) = \frac{a_o}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi z}{Z} + b_n \sin \frac{n\pi z}{Z}\right].$$

where the coefficients of the series have the following values

$$a_o = \frac{1}{Z} \int_o^Z \bar{f}(z)\, dz$$

$$\left.\begin{matrix} a_n \\ b_n \end{matrix}\right\} = \frac{1}{Z} \int_o^Z f(z) \left\{\begin{matrix} \cos \frac{n\pi z}{Z} \\ \sin \frac{n\pi z}{Z} \end{matrix}\right. dz , \quad n = 0,1,2,\ldots$$

The Fourier series of an even functions contain only cosines of arcs and ifit has an odd functions then it contains only sines of arcs. For the literature on Fourier series we refer to [27], [28].

4.1. Introduction. In this chapter we are going to find a solution for a two-dimensional free boundary problem in a rectangular domain. First, using an analytic represention to get the unknown function of the problem and second applying the idea introduced in chapt. III to obtain an iteration process for the unknown free boundary.

The problem can be summarized as follows: Let the function $u(z,r)$ must satisfy the elliptic differential equation (see chapt. I)

$$(4.1.1) \qquad Au := -\frac{\partial}{\partial u}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial u}{\partial z}\right) = 0,$$

on the elastic domain $\Omega_e \subset \Omega$ (FIG. 4.1) together with the boundary conditions:

$$(4.1.2) \qquad u(z,o) = 0,$$

$$(4.1.3) \qquad \left.\frac{\partial u}{\partial z}\right|_{z=o} = 0, \qquad \left.\frac{\partial u}{\partial z}\right|_{z=Z} = 0.$$

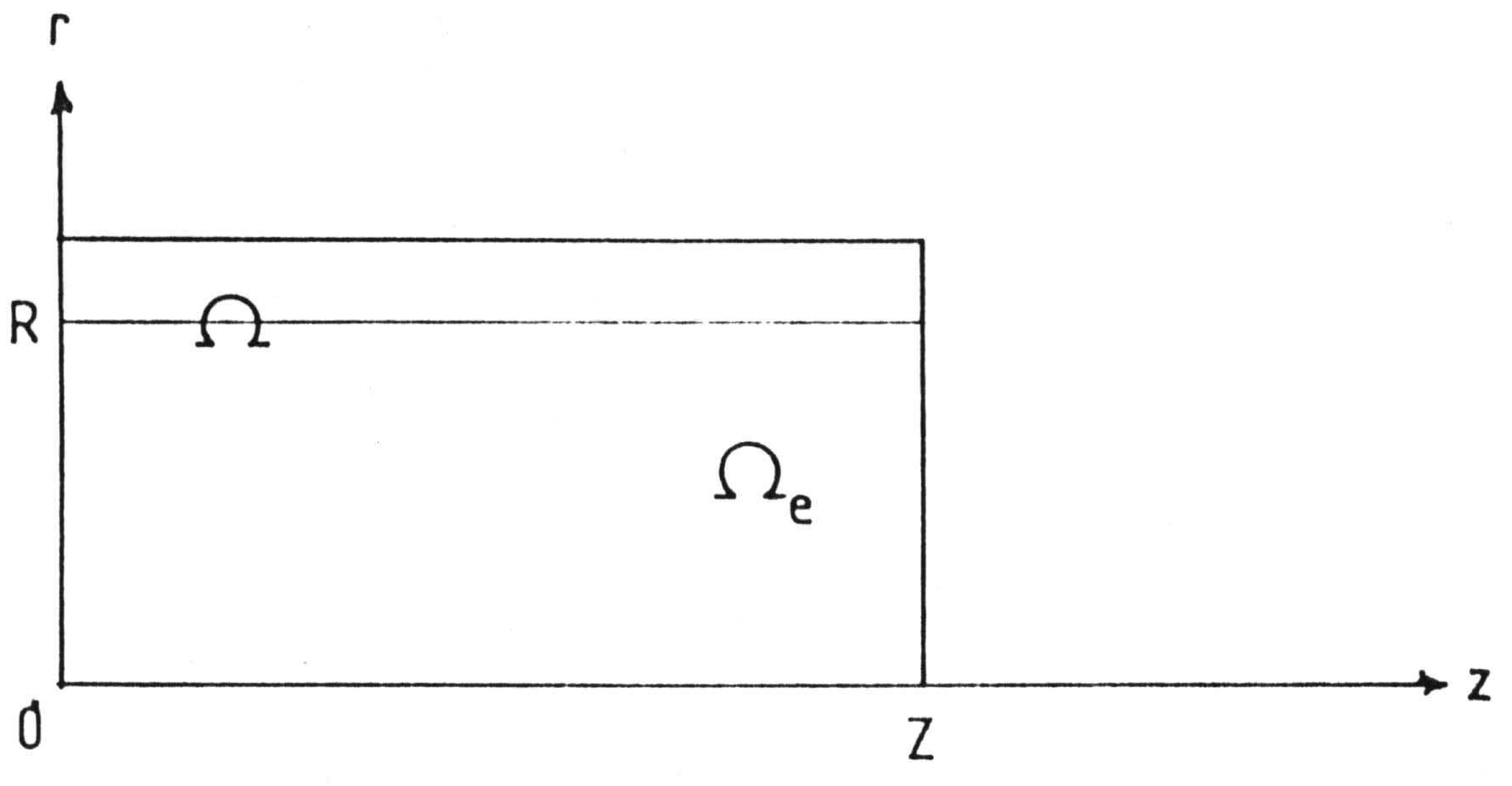

FIGURE 4.1. The domain $\Omega_e \subset \Omega$

Moreover across $r = R =$ const. the following conditions need to be satisfied:

(4.1.4) $u(z,R) = u^P(z,R) = \bar{f}(z)$,

(4.1.5) $\nabla u(z,r)\big|_{r=R} = \nabla u^P(z,r)\big|_{r=R}$,

where $u^P \in W_2^2$ denote a given function on Ω.

Notice the solution of equation (4.1.1) on the given domain Ω_e with the given boundary condition $u(z,R) = \bar{f}(z)$ was obtained by spline function method in chapt. I, but the only difference that in the present case we supposed $\phi_1(r) = \phi_2(r) = 0$.

4.2. An analytic solution with the use of Fourier infinity series method. Suppose that

(4.2.1) $u(z,r) = Z^*(z)\, R^*(r)$.

By substituting (4.2.1) into (4.1.1) we get

(4.2.2) $$\frac{1}{Z^*}\,\frac{d^2Z^*}{d\,z^2} = -\,\frac{1}{R^*}\,\frac{d^2R^*}{d\,r^2} + \frac{3}{rR^*}\,\frac{dR^*}{dr}\,,$$

in which the left side is function of z and the right side is function of r only. So this formula leads to the following two ordinary differential equations for the functions Z^* and R^*

(4.2.3) $$\frac{d^2Z^*}{dz^2} + \lambda Z^* = 0,$$

(4.2.4) $$\frac{d^2R^*}{dr^2} - \frac{3}{rR^*} - \lambda R^* = 0,$$

where λ is an arbitrary constant. The boundary conditions for Z^* and R^* can be determined from (4.1.2) and (4.1.3) as follows

(4.2.5) $$\frac{dZ^*}{dz}(0) = \frac{dZ^*}{dz}(Z) = 0,$$

(4,2,6) $R^*(0) = 0$.

For the solution of the equations (4.2.3) and (4.2.4) we have **three** cases:

(i) If $\lambda < 0$ the solution of Z^* not exists.

(ii) If $\lambda = 0$ the following solutions are obtained.

(4.2.7) $\quad Z^* = \text{const}, \quad R^* = b_o r^4, \quad b_o = \text{const}.$

(iii) If $\lambda > 0$ we get the following solution for the eigenvalue problem (4.2.3) with the condition (4.2.5)

(4.2.8) $$Z_k^* = c_k \cos\sqrt{\lambda_k}\, z,$$

where $$\lambda_k = \frac{\pi^2 k^2}{Z^2}, \quad k = 1,2,\ldots.$$

It is known that the solution of the modified Bessel's equation

(4.2.9) $$\frac{d^2w}{dz^2} + \frac{1-2a}{z}\frac{dw}{dz} + \left[(bcz^{c-1})^2 + \frac{a^2 - m^2 c^2}{z^2}\right]w = 0,$$

with the boundary condition (4.2.6) can be written as follows [8]

(4.2.10) $$w = z^a J_m(b z^c),$$

where J_m is the first kind of Bessel's function. For the solution of the equation (4.2.4) let $a = 2$, $c = 1$, $b^2 = -\lambda_k$, $m = 2$ in (4.2.9). Therefore from (4.2.10) we get

(4.2.11) $$R_k^* = c_k r^2 J_2(i\sqrt{\lambda_k}\, r),$$

or by using the represention of the Bessel's function with complex argument we get

(4.2.12) $$R_k^* = c_k r^2 I_2(\sqrt{\lambda_k}\, r),$$

where I_2 is the modified Bessel's function:

(4.2.13) $$I_m(z) = i^{-m} J_m(iz).$$

Now from using the Fourier series for the solutions (4.2.7), (4.2.8) and (4.2.12) we get the solution of equation (4.1.1) which is satisfied by the conditions (4.1.2), (4.1.3)

(4.2.14) $$u(z,r) = c_o r^4 + r^2 \sum_{k=1}^{\infty} c_k \cos\frac{\pi k z}{Z}\, I_2\left(\frac{\pi k r}{Z}\right).$$

The function $\bar{f}(z)$, which is supposed to satisfy the boundary condition (4.1.3) $\left(\frac{d\bar{f}}{dz}(0) = \frac{d\bar{f}}{dz}(Z) = 0\right)$ can be expressed by the Fourier series as follows

(4.2.15) $$\bar{f}(z) = a_o + \sum_{k=1}^{\infty} a_k \cos\frac{\pi k z}{Z},$$

where the Fourier coefficients of $\bar{f}(z)$ have the following form

$$(4.2.16) \qquad a_o = \frac{1}{Z}\int_0^Z \bar{f}(y)dy, \quad a_k = \frac{2}{Z}\int_0^Z \bar{f}(y)\cos\frac{\pi k y}{Z}dy,$$

$k = 1,2,\ldots.$

Let us consider the Fourier coefficients estimation. From (4.2.15) we get that

$$(4.2.17) \qquad \bar{f}''(z) = -\frac{\pi^2}{Z^2}\sum_{k=1}^{\infty} a_k k^2 \cos\frac{\pi k z}{Z}.$$

Since $u^p \in W_2^2$ we have that $\bar{f}''(z) \in L_2$. Hence the estimation of (4.2.17) is [8]

$$(4.2.18) \qquad |a_k k^2| \xrightarrow[k \to \infty]{} 0 .$$

To obtain the values of the coefficients c_o and c_k in (4.2.16) let us use the condition (4.1.4) and so we get

$$(4.2.19) \qquad c_o = \frac{a_o}{R^4}, \quad c_k = \frac{a_k}{R^2}\left[I_2\left(\frac{\pi k R}{Z}\right)\right]^{-1}, \quad k = 1,2,\ldots.$$

By substituting (4.2.19) into (4.2.14) we get that

$$(4.2.20) \qquad u(z,r) = a_o\frac{r^4}{R^4} + \frac{r^2}{R^2}\sum_{k=1}^{\infty} a_k \cos\frac{\pi k z}{Z}\,\frac{I_2\left(\frac{\pi k r}{Z}\right)}{I_2\left(\frac{\pi k R}{Z}\right)}.$$

4.3. An interation process construction to obtain the free boundary. Let $\tilde{R}$ be the approximated free boundary on Ω_e such that

$$(4.3.1) \qquad \tilde{R} = R + \delta,$$

where R represents the unknown exact boundary of Ω_e, and δ is supposed to be small number. We assumed before that u^p is a given function on Ω. Therefore on $\tilde{R}$ = const the function

$$(4.3.2) \qquad \tilde{f}(z) = u^p(z,\tilde{R})$$

is known. Similarly as in (4.2.15) we can write that

$$(4.3.3) \qquad \tilde{f}(z) = \tilde{a}_o + \sum_{k=1}^{\infty} \tilde{a}_k \cos\frac{\pi k z}{Z}$$

and for the coefficients of $\tilde{f}(z)$ it is true the relations in (4.2.16).

Now for the domain $\tilde{\Omega}_e \subset \Omega$ (with the boundary $\tilde{R}$) the solution of (4.1.1) (let us denoted by $\tilde{u}(z,r)$), which satisfies the boundary function (4.3.2), can be written as follows

$$(4.3.4) \qquad \tilde{u}(z,r) = \tilde{a}_o \frac{r^4}{\tilde{R}^4} + \frac{r^2}{\tilde{R}^2} \sum_{k=1}^{\infty} \tilde{a}_k \cos \frac{\pi k z}{Z} \frac{I_2(\frac{\pi k r}{Z})}{I_2(\frac{\pi k \tilde{R}}{Z})} .$$

Let us look for the relations between a_k and $\tilde{a}_k$. From the Taylor's expansion and (4.3.1) the boundary function $\tilde{f}(z)$ (4.3.2) can be written as follows

$$(4.3.5) \qquad \tilde{f}(z) = u^P(z,R) + \delta \frac{\partial u^P}{\partial r} (z,R) + O(\delta^2).$$

Hence from (4.2.16) and (4.3.5) we get that

$$(4.3.6) \qquad \tilde{a}_o = a_o + \delta \frac{1}{Z} \int_o^Z \frac{\partial u^P}{\partial r}\Big|_R dy + O(\delta^2),$$

and similary for $\tilde{a}_k$ $(k = 1,2,\ldots)$ we have

$$(4.3.7) \qquad \tilde{a}_k = a_k + \delta \frac{2}{Z} \int_o^Z \frac{\partial u^P}{\partial r}\Big|_R \cos \frac{\pi k y}{Z} dy + O(\delta^2).$$

From the formulation of the problem one can see that

$$(4.3.8) \qquad \tilde{u}(z,\tilde{R}) = u^P(z,\tilde{R}) = \tilde{f}(z),$$

$$(4.3.9) \qquad \frac{\partial \tilde{u}}{\partial z} (z,\tilde{R}) \equiv \frac{\partial u^P}{\partial z} (z,\tilde{R}) = \frac{d\tilde{f}(z)}{dz} .$$

But for $\tilde{R} \neq R$ it is obvious that

$$(4.3.10) \qquad \frac{\partial \tilde{u}}{\partial r} (z,\tilde{R}) \neq \frac{\partial u^P}{\partial r} (z,\tilde{R}) .$$

This means that the second condition of (4.1.5) is not satisfied. We shall use (4.3.10) to determine, from the difference between $\partial\tilde{u}/\partial r$ and $\partial u^P/\partial r$ in $r = \tilde{R}$, the approximate value of δ.

From (4.3.4) let us obtain $\partial\tilde{u}/\partial r$ on the line $r = \tilde{R}$, and so

$$(4.3.11) \qquad \left.\frac{\partial \tilde{u}}{\partial r}\right|_{r=\tilde{R}} = \frac{4\tilde{a}_o}{\tilde{R}} + \frac{2}{\tilde{R}} \sum_{k=1}^{\infty} \tilde{a}_k \cos \frac{\pi k z}{Z} +$$

$$+ \sum_{k=1}^{\infty} \tilde{a}_k \cos \frac{\pi k z}{Z} \frac{I_2'(\frac{\pi k \tilde{R}}{Z})}{I_2(\frac{\pi k \tilde{R}}{Z})} \frac{\pi k}{Z} .$$

where I_2' denotes the first derivative over its argument. By substituting (4.3.1), (4.3.6), (4.3.7) into (4.3.11) and using that

$$\frac{I_2'(\frac{\pi k (R+\delta)}{Z})}{I_2(\frac{\pi k (R+\delta)}{Z})} = \frac{I_2'(\frac{\pi k R}{Z})}{I_2(\frac{\pi k R}{Z})} + \frac{\pi k}{Z} \delta \left[\frac{I_2'(\frac{\pi k R}{Z})}{I_2(\frac{\pi k R}{Z})}\right]_R' + O(\delta^2)$$

we get with an arrangment to the terms the following

$$(4.3.12) \qquad \left.\frac{\partial \tilde{u}}{\partial r}\right|_{r=\tilde{R}} = \frac{4a_o}{R} + \frac{2}{R} \sum_{k=1}^{\infty} a_k \cos \frac{\pi k z}{Z}$$

$$+ \sum_{k=1}^{\infty} a_k \frac{\pi k}{Z} \cos \frac{\pi k z}{Z} I_2' \left(\frac{\pi k R}{Z}\right) \left[I_2\left(\frac{\pi k R}{Z}\right)\right]^{-1} +$$

$$+ \delta \left[- \frac{2a_o}{R^2} + \frac{2}{R^2} \left(a_o + \sum_{k=1}^{\infty} a_k \cos \frac{\pi k z}{Z}\right) + \right.$$

$$+ \frac{2}{RZ} \int_0^Z \left.\frac{\partial u^p}{\partial r}\right|_R dy \; \frac{2}{R} \left(\frac{1}{Z} \int_0^Z \left.\frac{\partial u^p}{\partial r}\right|_R dy + \right.$$

$$+ \sum_{k=1}^{\infty} \left(\frac{2}{Z} \int_0^Z \frac{\partial u^P}{\partial r}\Big|_R \cos \frac{\pi k y}{Z} dy \right) \frac{\pi k}{Z} *$$

$$* \cos \frac{\pi k z}{Z} I_2'\left(\frac{\pi k R}{Z}\right) \left[I_2\left(\frac{\pi k R}{Z}\right) \right]^{-1} + \sum_{k=1}^{\infty} a_k *$$

$$* \left(\frac{\pi k}{Z}\right)^2 \cos \frac{\pi k z}{Z} \left[\frac{I_2'\left(\frac{\pi k R}{Z}\right)}{I_2\left(\frac{\pi k R}{Z}\right)} \right]_R' + O(\delta^2).$$

To simplify the material it can be seen that the first three terms in (4.3.12) denote $\partial u/\partial r(z,R)$ and by Taylor's expansion one can write

$$(4.3.13) \qquad \frac{\partial u}{\partial r}\Big|_R = \frac{\partial u^P}{\partial r}\Big|_{\tilde{R}} - \delta \frac{\partial^2 u^P}{\partial r^2}\Big|_R - O(\delta^3).$$

Let us assume that

$$(4.3.14) \qquad \frac{\partial u^P}{\partial r}\Big|_R = g_0 + \sum_{k=1}^{\infty} g_k \cos \frac{\pi k z}{Z},$$

where

$$(4.3.15) \qquad g_0 = \frac{1}{Z} \int_0^Z \frac{\partial u^P}{\partial r}\Big|_R dy,$$

$$g_k = \frac{2}{Z} \int_0^Z \frac{\partial u^P}{\partial r}\Big|_R \cos \frac{\pi k y}{Z} dy, \quad k = 1,2,\dots .$$

By substituting (4.3.13), (4.3.14) into (4.3.12) and using the Taylor's expansions to express all the values of (4.3.12) on the boundary $r = \tilde{R}$ one can obtain the difference between $\frac{\partial \tilde{u}}{\partial r}\Big|_{\tilde{R}}$ and $\frac{\partial u^P}{\partial r}\Big|_{\tilde{R}}$ in form of a power series of the small parameter δ as follows

$$(4.3.16) \qquad \left.\frac{\partial \tilde{u}}{\partial r}\right|_{\tilde{R}} - \left.\frac{\partial u^P}{\partial r}\right|_{\tilde{R}} = \delta \left[\left.\frac{\partial^2 u^P}{\partial r^2}\right|_{\tilde{R}} - \frac{2\tilde{a}_o}{\tilde{R}^2} - \frac{2}{\tilde{R}^2} u^P(z,\tilde{R}) + \right.$$

$$+ \frac{2}{\tilde{R}^2}\tilde{g}_o + \frac{2}{\tilde{R}^2}\left.\frac{\partial u^P}{\partial r}\right|_{\tilde{R}} + \sum_{k=1}^{\infty} \tilde{g}_k \frac{\pi k}{Z} \cos\frac{\pi k z}{Z} \frac{I_2'(\frac{\pi k \tilde{R}}{Z})}{I_2(\frac{\pi k \tilde{R}}{Z})} +$$

$$\left. + \sum_{k=1}^{\infty} \tilde{a}_k \left(\frac{\pi k}{Z}\right)^2 \cos\frac{\pi k z}{Z} \left[\frac{I_2'(\frac{\pi k \tilde{R}}{Z})}{I_2(\frac{\pi k \tilde{R}}{Z})}\right]' \right] + O(\delta^2),$$

where $\tilde{g}_k$ are the Fourier coefficients of $\partial u^P/\partial r$ on $\tilde{R}$.

Now it can be seen that the first five terms in the right hand side of (4.3.16) are bounded because they have known vales. Therefore let us look for the estimations of the last two series in (4.3.16). Since $u^P \in W_2^2$ and in addition we suppose that $\frac{\partial^3 u^P}{\partial z^2 \partial r} \in L_2$ then for $\tilde{g}_k$ the following estimation is true

$$(4.3.17) \qquad |g_k k^2| \underset{k\to\infty}{\longrightarrow} 0 ,$$

which is similar to the estimation obtained in (4.2.18) for a_k. It is obvious that $\tilde{g}_k$ k tends to zero more fastly than k^{-1}.

It can be shown that for I_2 of a big argument y the following asymptotic expressions are true

$$(4.3.18) \qquad \frac{I_2'(y)}{I_2(y)} = 1 - \frac{1}{2y} + O(y^{-2}) , \quad \left[\frac{I_2'(y)}{I_2(y)}\right]' =$$

$$= 1 - \frac{1}{2y^2} + O(y^{-3}) .$$

Therefore from the relations (4.2.18), (4.3.17) and (4.3.18) the convergence of the last two series in (4.3.16) are followed. We want to remark that the numerical computations have shown that the first expression in (4.3.18) is less than 1 starting with y = 5, and the absolute value for the product of the second expression in (4.3.18) with y^2 is less than 1 for every y starting with y = 4.

Notice that it is possible the both sides of (4.3.16) are functions of z while δ is supposed to be constant. To consider this case let us integrate the two sides of (4.3.16) on the interval (0,Z) and so we get

$$(4.3.19) \qquad \frac{1}{Z}\int_0^Z \frac{\partial\tilde{u}}{\partial r}\Big|_{\tilde{R}} dy - \tilde{g}_o = \delta\left[-\frac{1}{Z}\int_0^Z \frac{\partial^2 u^p}{\partial r^2}\Big|_{\tilde{R}} dy - \frac{4\tilde{a}_o}{\tilde{R}^2} - \frac{4}{\tilde{R}}\tilde{g}_o\right] + O(\delta^2).$$

Clearly it can be seen that the right-hand side of (4.3.19) have constant terms. Therefore the approximate value of δ which is independent from z can be obtained too. Thus formulas (4.3.16) and (4.3.19) can be applied as bases to the iteration processes for solving the formulated free boundary problem.

CHAPTER V

Two-Dimensional Elastic-Plastic Problem Solution By Iteration Method

5.0. For the one-dimensional discretized Newton iteration methods:

$$x^{k+1} = x^k - \left[\frac{f(x^k + h^k) - f(x^k)}{h^k}\right]^{-1} f(x^k), \quad k = 0,1,\ldots$$

an important special case is obtained if $h_k = x^{k-1} - x^k$:

$$x^{k+1} = x^k - \left[\frac{f(x^{k-1}) - f(x^k)}{x^{k-1} - x^k}\right]^{-1} f(x^k), \quad k = 0,1,\ldots$$

This special case is known as secant method. The method can be extended to the general one in n dimensions. For the secant method and its convergence we refer to the books of [21], [30], [31], [34]. In [34] it is shown that the secant method asymptotically have 1.44 iteration procesess more than the Newton's method.

5.1. Introduction. In this chapter we propose a method for the solution of an elastic-plastic axisymmetric cylinder problem. The cylinder is subjected to a torque T applied at both ends (see FIG.1.1. in Ch.I)

The problem can be formulated in a two-dimensional domain $\Omega = \Omega^e \cup \Omega^p$ (Ω^e and Ω^p are the elastic and plastic domains respectively) in the rz-plane because of the axial symmetry (FIG.5.1).

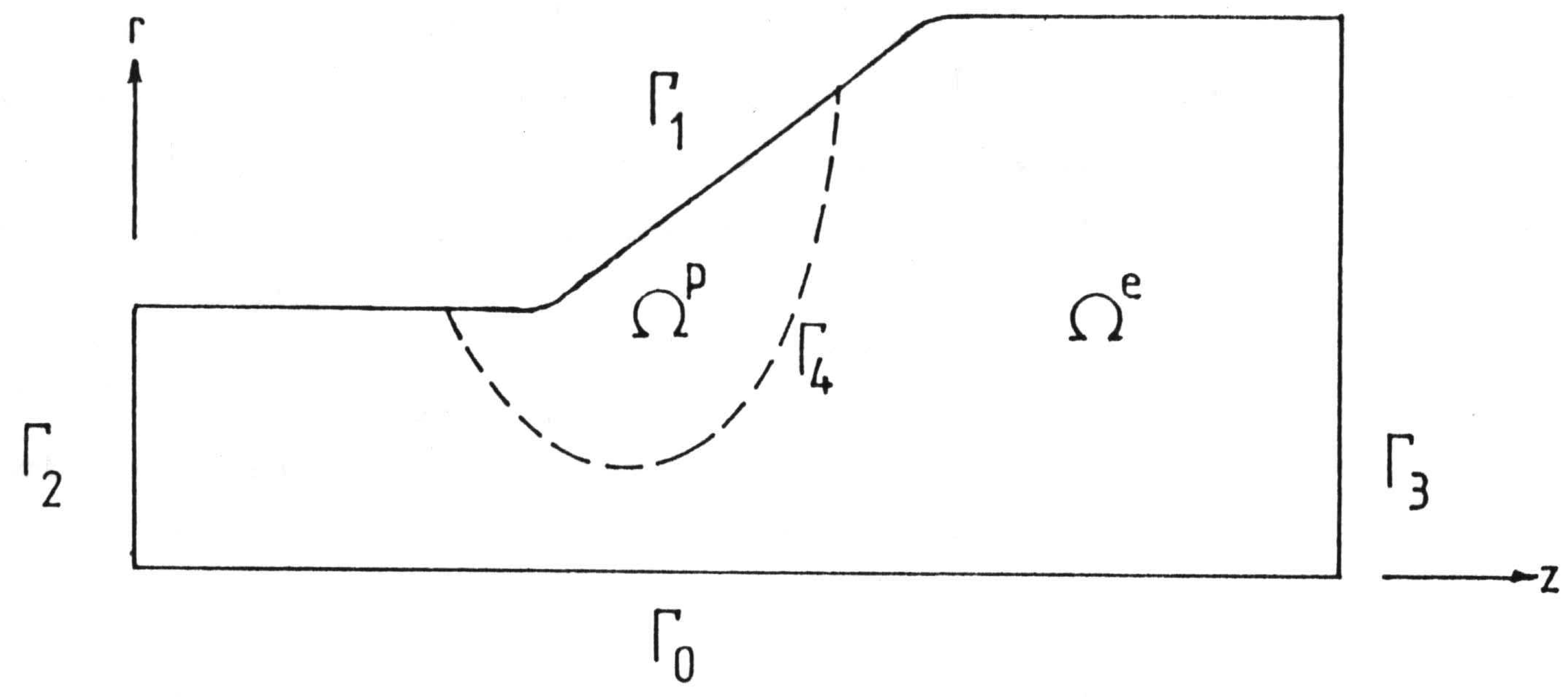

FIGURE 5.1. The elastic-plastic domain Ω

Let us summarize the problem as follows (for more description about the problem formulation see [1]): Find the function u such that

(5.1.1) $$Au := -\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial u}{\partial z}\right) = 0, \text{ in } \Omega^e,$$

(5.1.2) $$q := \frac{1}{r^2}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{1/2} = k, \text{ in } \Omega^p,$$

(5.1.3) $$q \le k, \text{ in } \Omega,$$

(5.1.4) $$u = 0, \text{ on } \Gamma_0,$$

(5.1.5) $$u = F(z), \text{ on } \Gamma_1,$$

(5.1.6) $$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial z} = \phi_1(r), \text{ on } \Gamma_2,$$

(5.1.7) $\frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = \phi_2(r)$, on Γ_3,

(5.1.8) u, $\frac{\partial u}{\partial r}$, $\frac{\partial u}{\partial z}$ are continuous across Γ_4,

where Γ_i ($i = \overline{0,4}$) are as shown in FIG.5.2., k is a given constant and Γ_4 is the free boundary between Ω^e and Ω^p. Let $R = R(z)$ be the function of the free boundary Γ_4. For $R(z)$ we suppose the following

(5.1.9) $\left|\frac{d\ R(z)}{dz}\right| < \infty$.

Notice, physicaly the condition (5.1.3) means $q < k$ inside Ω^e and $q = k$ only strating from Γ_4 and inside Ω^p.

5.2.1. Mode of solving the problem. The first approximation of the free boundary.

(i) Iteration process is proposed to obtain the unknown free boundary.

(ii) The plastic problem can be solved once through out the iteration process.

(iii) The elastic problem need to be solved in every step of the iteration process in corrigated elastic domain with the new approximate boundary, using the value of the plastic function on this approximate boundary line.

(iv) For starting the iteration process it need to put a method to find out the first approximation of the free boundary separating the elastic and plastic domains.

(v) After obtaining the first approximation the elastic problem can be solved in the new elastic domain. Then the next approximation of the free boundary can be obtained from the comparsion between the gradient of the solutions of the elastic and plastic problems. Hence iteration process can be constructed.

Let us illustrate below method to obtain the first approximation of the free boundary. In the begining suppose that the whole domain Ω (FIG.5.1) is occupied by the elastic state material only. The elastic problem (5.1.1), (5.1.5)-(5.1.7) in Ω

can be solved by spline function method introduced in ch.I. Let us denote the problem solution by $\tilde{u}^e := S_{ij}$ in Ω_{ij}.

After that let us go to obtain the first approximation of the free boundary. This can be done by examining the derivatives of $\tilde{u}^e$ with use of (5.1.3) in every subdomain Ω_{ij} starting at the neighborhood of Γ_1. For the spline $S_{ij} \in W_2^1$ we proved in ch.I that this spline and its first derivatives approximated the elastic problem function and its first derivatives too. Now to construct an algorithm we can write the following

(5.2.1) $$\nabla\tilde{u}^e = \nabla S_{ij} = \left[\left(\frac{\partial S_{ij}}{\partial r}\right)^2 + \left(\frac{\partial S_{ij}}{\partial z}\right)^2\right]^{1/2}.$$

Let us choose the mid-points $\left(z_i + \frac{h_z}{2},\ r_j + \frac{h_r}{2}\right)$ of the rectangular subdomains and the same mid-points on the curved sides of the triangular subdomains (see ch.I) in which ∇S_{ij} need to be evaluated. Now from using the inequalities $\nabla S_{ij} \lessgtr kr^2$ we can separate the elastic and plastic domains as follows

(5.2.2) $$\text{if } \nabla S_{ij} < k\left(r_j + \frac{h_r}{2}\right)^2 \text{ then } \Omega_{ij} \subset \Omega_o^e ,$$

and

(5.2.3) $$\text{if } \Delta S_{ij} > k\left(r_j + \frac{h_r}{2}\right)^2 \text{ then } \Omega_{ij} \subset \Omega_o^e ,$$

where Ω_o^e and Ω_o^p are the elastic and plastic domains, respectively, in the first approximation of the problem.

5.2.2. Iteration process based on qualititive strategy.

The process of finding the values of ∇S_{ij} and the using of (5.2.2), (5.2.3) start in the Ω_{ij} at the neighborhood of Γ_1. For the next step of the examinition we go to the below subdomains and so on step by step from Ω_{ij} to $\Omega_{i,j-1}$. The process will be ended if the following is satisfied

(5.2.4) $$|\nabla S_{ij}| \geq k\left(r_j + \frac{h_r}{2}\right)^2 \text{ in } \Omega_{ij},\ |\nabla S_{ij}| < k\left(r_j + \frac{h_r}{2}\right)^2 \text{ in } \Omega_{ij,-1}.$$

It is possible there are not exist subdomains for which $|\nabla S_{ij}| \geq k\left(r_j + \frac{h_r}{2}\right)^2$ in Ω_{ij}. This means that the whole domain Ω is an elastic one, therefore, it is not need to solve the elastic-plastic problem.

Clearly the process of the examinition with the satisfication of (5.2.4) lead us to the first approximate free boundary Γ_4^0 which is between Ω_0^e and Ω_0^p. Now the solution u^p of the plastic problem in Ω_0^p can be obtained by finite diference method introduced in ch.II. We remark that the solution u^p need to be obtained once because it is not depend upon the position of the free boundary. After that the new solution u_0^e of the elastic problem in Ω_0^e can be obtained by spline function method; therefore

(5.2.5) $$u_0^e := S_{ij}^0 \ , \ \text{in } \Omega_{ij}.$$

For the solution u_0^e the following condition need to be satisfied

(5.2.6) $$u_0^e\Big|_{\Gamma_4^0} = u^p\Big|_{\Gamma_4^0} .$$

Now let us examine the ∇S_{ij}^0 in Ω_{ij} at the neighborhood of Γ_4^0. If in these Ω_{ij}

(5.2.7) $$|\nabla S_{ij}^0| \cong k\left(r_j + \frac{h_r}{2}\right)^2 ,$$

then Γ_4^0 is the good approximation for Γ_4. But if (5.2.7) is not satisfied at these Ω_{ij} it need to construct the new approximation Γ_4^1 in the neighborhood of Γ_4^0. To find the position of the exact free boundary Γ_4 compared with the position of Γ_4^0 we consider the following

(5.2.8) $$\text{if } |\nabla S_{ij}^0| > k\left(r_j + \frac{h_r}{2}\right)^2 \text{ then } \Gamma_4 \text{ is below } \Gamma_4^0 ,$$

and

(5.2.9) $$\text{if } |\nabla S_{ij}^0| < k\left(r_j + \frac{h_r}{2}\right)^2 \text{ then } \Gamma_4 \text{ is above } \Gamma_4^0 .$$

From these relations we can find the direction of the algorithm for only one step for the next approximation, i.e. to go up or down Γ_4^0 for one step for the next approximation Γ_4^1 towards Γ_4. This is simple iteration based on qualititive strategy. In the every step of this iteration it is need to solve the elastic problem in the new corrigated domain. Notice that the solution of the plastic problem don't depend upon the new elastic domains.

5.3. Iteration process based on quantitive strategy. For this iteration it is need to consider the plastic problem in the new domain Ω_1^p

(5.3.1) $$\Omega_1^p := \Omega_0^p + \text{neighborhood of } \Gamma_4^0 .$$

Let the neighborhood of Γ_4^0 be bounded from below by Γ_4^1. The thickness of this neighborhood could be small if Γ_4^0 is good first approximation otherwise it is need to be big one. After that we solve the elastic problem in the new corrigated elastic domain Ω_1^e where

(5.3.2) $$u_1^e := S_{ij}^1 \text{ , in } \Omega_{ij} \in \Omega_1^e$$

denotes the problem solution for which the following condition need to be satisfied

(5.3.3) $$u_1^e\Big|_{\Gamma_4^1} = u^p\Big|_{\Gamma_4^1} .$$

It is known that the tangentinal derivatives of u_j^e and u^p on Γ_4^j $(j = 0,1\ldots)$ are equal but their normal derivatives are not. Therefore as in ch.III and ch.IV we shall use the function

(5.3.4) $$\zeta(z,r) := \frac{\partial u_j^e}{\partial n} - \frac{\partial u^p}{\partial n}$$

for every $z,r \in \Gamma_4^j$ to be as basis for the iteration process.

To build up an algorithm for the iteration process let for a given $z = z^*$ the points $r_j \in \Gamma_4^j$ $(j = 0,1\ldots)$ and the unknown point $\bar{r}(z^*) \in \Gamma_4$. For a given $z = z^*$ let us expand the function $\zeta(z^*,r)$ about r_j in a Taylor's series. Hence for $r = \bar{r}$ we get

(5.3.5) $$\zeta(z^*,\bar{r}) \cong \zeta(z^*,r_j) + (\bar{r} - r_j)\frac{d\zeta}{dr}(z^*,r_j).$$

But $\zeta(z^*,\bar{r}) = 0$, therefore

(5.3.6) $$\bar{r} \cong r_j - \frac{\zeta(z^*,r_j)}{\frac{d\zeta}{dr}(z^*,r_j)}.$$

In this approximate relation the derivative of the function ζ is unknown. But this derivative can be obtained approximatly as follows

(5.3.7) $$\frac{d\zeta}{dr}(z^*,r_j) \cong \frac{\zeta(z^*,r_j) - \zeta(z^*,r_{j-1})}{r_j - r_{j-1}}.$$

Now from (5.3.6) and (5.3.7) we get following iteration

(5.3.8) $$r_{j+1} = r_j - \frac{\zeta(z^*,r_j)(r_j - r_{j-1})}{\zeta(z^*,r_j)-\zeta(z^*,r_{j-1})}, \qquad j = 1,2,\ldots$$

In [34] it is shown that the secant method asymptotically have 1.44 iteration processes more than the Newton's method.

Remark 1. Because we have not the derivatives of the function ζ we can not use the Newton's method for the solution of the problem.

Remark 2. For a given z the iteration (5.3.8) is known as secant method, (see 5.0) but the only differenc that the function ζ in our problem has two variables z and r.

Remark 3. In the quantitive strategy it is possible to go up (or down) not only along one subdomain as the case in the qualititive strategy. Therefore using the quantitive strategy the process can be accelerated.

Remark 4. By using the iteration method presented in this chapter for the one-dimensional elastic-plastic problem introduced in ch.III we obtained for the function ζ the following explicit formula

(5.3.9) $$\zeta(\tilde{r}) = \frac{k}{3}\left(\tilde{r}^2 - \frac{\bar{r}^3}{\tilde{r}}\right),$$

where $\tilde{r}$ is the approximate free boundary and $\bar{r}$ is the exact one. Now for $\tilde{r} > \bar{r}$ in (5.3.9) one can see that $\zeta(\tilde{r}) > 0$, $\zeta'(\tilde{r}) > 0$, $\zeta''(\tilde{r}) > 0$, and for $\tilde{r} = \bar{r}$ one have $\zeta(\tilde{r}) < 0$, $\zeta'(\tilde{r}) > 0$, $\zeta''(\tilde{r}) <$ < 0, and for $\tilde{r} = \bar{r}$; $\zeta(\tilde{r}) = 0$. These results assure that the iteration process is convergence. For the two-dimensional case we have not the explicit formula for the function ζ. But we hope that the results obtained in the one-dimensional problem is also true for the two-dimensional problem.

Remark 5. It is possible to use another iteration methods (see [31]) to the solution of our problem for instance Steffensen's method, etc.

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MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

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THESIS FOR CANDIDATE /PH.D/ DISSERTATION

SPLINE APPROXIMATION FOR THE SOLUTION OF TWO-DIMENSIONAL ELASTIC-PLASTIC PROBLEM

written by

ALI SHAUKET MOHAMED

supervised by

Associate Prof. RUSZLAN FARZAN PH.D

Computer Center of

Lorand Eötvös University

Budapest

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MAGYAR TUDOMÁNYOS AKADÉMIA KÖNYVTÁRA

1. The motivation of elasticity theory and the elastic-plastic problem investigated

The mathematical theory of elasticity /which has practicaly important applications in architecture, engineering, physics and all other useful arts in which the material of construction is solid/ owes its development to demand for more realistic methods of determing the safty factors of structures or machine parts. The rapid and intensive development of computer engineering has aroused the considerable interest of researchers for the development of effective numerical methods for the solution of elasticity problems. Together with the methods in continum mechaincs and engineering calculations.

One of the important and complicated cases in the elasticity problems are those in which the body has been strained by a load or subjected to very great pressure,etc. surpassing the limite of perfect elasticity, and is set free, the set gradually diminishes. The body never returns to its primitive condition, and the ultimate deformation is the "permanent set", the part of strain that gradually disappears is called elastic after-strain. In other elastic problems some plasticity of the material appears as soon as the limit of linear elasticity is exceeded. This leads to the elastic-plastic problems which are still under considerable investigation.

The particular problem investigated in the dissertation is an elastic cylinder of rotational symmetry. The cylinder is subjected to a torque applied at both ends. The torque of sufficient magnitude is to cause portions of the material of the cylinder to yield. The material is assummed to be isotropic and yields according to the condition of von

Mises. This condition means that below the yield point the behavior is perfectly elastic and after yield the material exhibits perfect plasticity. The yield conditions requires that the maximum shearing stress has the constant value (k) equal to the yield stress in pure shear. Because of the axial symmetry the formulation of the problem reduced to a two-dimensional domain Ω in the rz-plane (FIG.1).

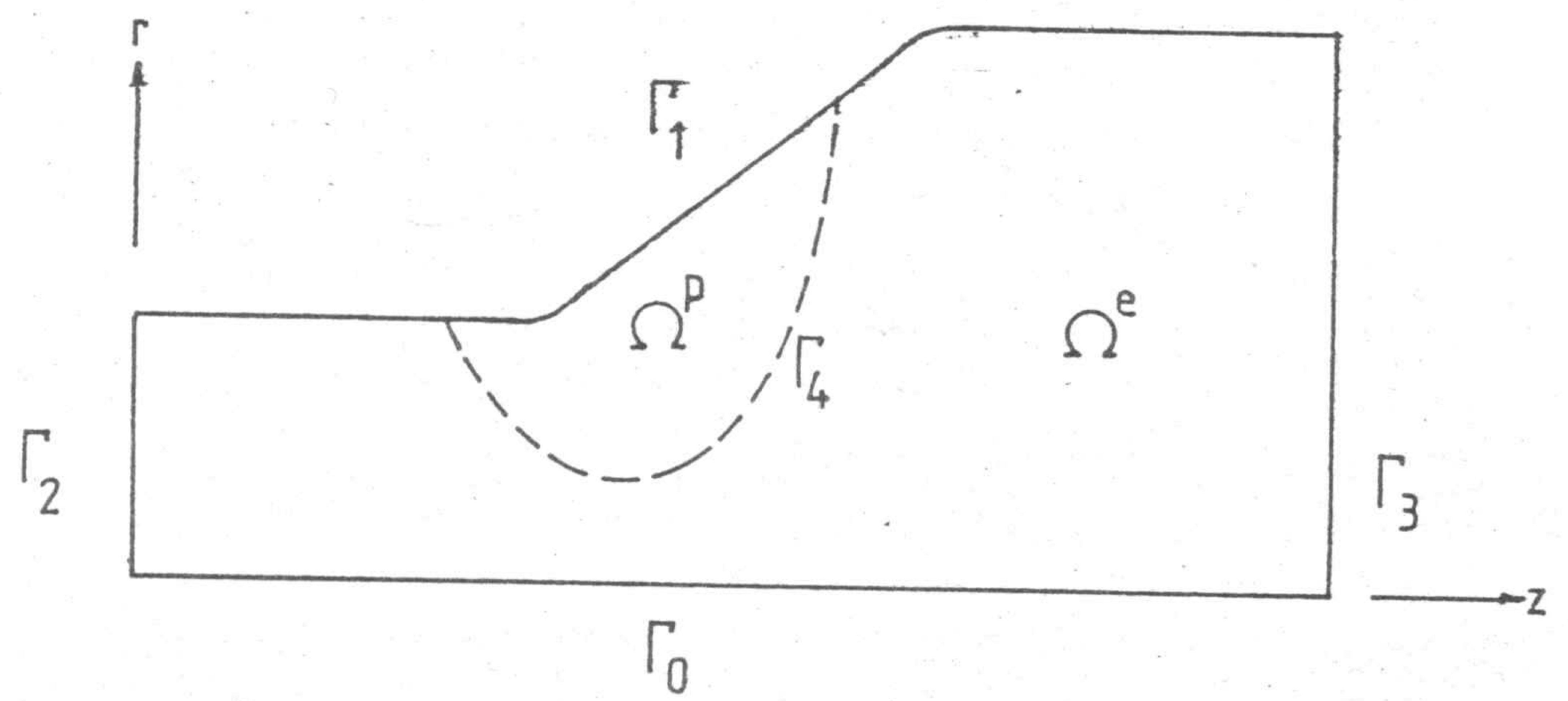

FIGURE 1. The domain $\Omega = \Omega^e \cup \Omega^p$

In obtaining the elastic and plastic equations it is assummed that the only two non-zero componens of the stress tensor are $\tau_{r\vartheta}, \tau_{\vartheta z}$ [1] and these are given in terms of the stress function u by

$$\tau_{r\vartheta} := -\frac{1}{r^2}\frac{\partial u}{\partial z}, \qquad \tau_{\vartheta z} := +\frac{1}{r^2}\frac{\partial u}{\partial r}$$

The complete statement of the elastic-plastic problem reduced to the following:

(a) In the elastic domain Ω^e the function u must satisfy the equation

(1) $$Au := -\frac{\partial}{\partial r}\left(\frac{1}{r^3}\frac{\partial u}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{1}{r^3}\frac{\partial u}{\partial z}\right) = 0$$

together with the overriding condition

(2) $$q := \frac{1}{r^2}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{1/2} \le k, \qquad k = \text{const.},$$

and the equality can be held only on the free boundary Γ_4.

(b) In the plastic domain Ω^p the function u satisfies the equation

(3) $$q := \frac{1}{r^2}\left[\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2\right]^{1/2} = k.$$

(c) We assummed that the cylinder is subjected to different external tractions. Therefore we used the following boundary conditions on the three sides of the cylinder and on its central line:

(i) $u(z,0) = 0$, on Γ_0,

(ii) $u(z,R) = F(z)$, on Γ_1,

(4) (iii) $$\frac{\partial u}{\partial n} = -\frac{\partial u}{\partial z} = \phi_1(r), \text{on } \Gamma_2,$$

(iv) $$\frac{\partial u}{\partial n} = \frac{\partial u}{\partial z} = \phi_2(r), \text{on } \Gamma_3.$$

(d) Across Γ_4 the unknown free boundary, the functions u, $\frac{\partial u}{\partial n}$, $\frac{\partial u}{\partial z}$ are continuous.

The problem formulated so as to find the stress functions u in the elastic and plastic domains and Γ_4 the unknown free boundary separating these two domains.

2. Scientific background and aims of this research.

Some mathematicians have investigated the above problem. Eddy and Show [2] obtained the approximate numerical solution of the problem by using relaxtion method. Gryer [1] reformulated the problem to the complementary and varitional inequality and used finite element method for their solutions.

The aims of this research are to use spline approximation method in the elastic domains, finite difference method in the plastic domain and iteration method for the elastic-plastic problem to find out approximate solutions to the following particular problems:

(i) The elastic problem in the cylinder of constans or variable diameter.

(ii) The plastic problem in the cylinder of variable diameter.

(iii) The one-dimensional elastic-plastic problem.

(iv) The elastic-plastic problem in the cylinder of constant diameter.

(v) The elastic-plastic problem in the cylinder of variable diameter.

3. New scientific results

In dealing with the five mentioned problems by means of the methods used, we have developed a number of new results. Some of the important new results are briefly given below:

In chapter I, we constructed special spline function S_Δ for approximating the solution of the elastic problem in two different cylinder domains: of constant and variable diameters, where the problem has singularity on the centeral

line of the cylinder in which the coefficients tend to infinity. In solving these problems we used the more general boundary conditions (4) than those used in [1] and [2]. In theorems (1.2.1, 1.3.1) we proved that our spline of fourth degree with only four coefficients (see 1.2.9), which exists, satisfies the second-order elliptic differential equation (1) in the subdomains with the main boundary condition (the first two boundaries in (4)), and is continuous over the whole domains considered. For the one-dimensional and two-dimensional elastic problems in domains of constant diameter with certain boundary conditions this spline give the exact solution. This lead us to suggest that our spline gives better approximate solution to the problems investigated than the piecewise linear spline. We proved that the generalized solutions of the problems considered are exist in the energetical Hilbert space H_A (see 1.2.34). The coefficients of the spline are obtained by Ritz method with using of the Lagrange multiplyers (see 1.2.39). We proved that our spline in a limit sense is dense in H_A. For the one dimensional problem this result was formulated in theorems (1.2.2,1.2.3). For the two-dimensional problem in the domain of constant diameter, in theorem 1.2.4, we obtained result which is come from the generalization of the one-dimensional case: If for every $f(z,r) \in W_2^1$ there exists spline function $\bar{S}_\Delta$ then the following estimations are true

$$[f(z,r) - \bar{S}_\Delta(z,r)] \to 0 \text{ if } h_z, h_r \to 0, (n,m \to \infty),$$

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{L_2} \to 0 \text{ if } h_z, h_r \to 0,$$

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{W_2^1} \to 0 \text{ if } h_z, h_r \to 0.$$

For the above problem but if $f(z,r) \in W_2^2$ we obtained in theorem 1.2.5 the following better estimations:

$$[f(z,r) - \bar{S}_\Delta(z,r)] = O(h_z + h_r),$$

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{L_2} = O(h_z^2 + h_r^2),$$

$$||f(z,r) - \bar{S}_\Delta(z,r)||_{W_2^1} = O(h_z + h_r).$$

In theorems 1.3.2 and 1.3.3 for the two-dimensional problem in a cylinder with variable diameter similar results were obtained.

In the lemmas 1.2.1, 1.2.2 and 1.3.1 for all the elastic problems considered we proved that there is a connection between the norms of the generalized solution u in the W_2^1-Sobolev space and in H_A-the energetical Hilbert space as follows

$$||u||_{W_2^1} \le c_1[u]$$

where c_1 is a positive constant.

In theorems 1.2.6 and 1.3.2 the convergence of the approximate solutions of the one-dimensional and the two-dimensional elastic problems in domains of constant or variable diameter are proved. In the case that the generalized solution $\tilde{u}_o \in W_2^1$ we obtained the following estimation

$$||\tilde{u}_o - \tilde{S}||_{W_2^1} \le c[\tilde{u}_o - \tilde{S}] \le c \inf_{S_\Delta}[\tilde{u}_o - S_\Delta] \to 0 \text{ if } h \to 0,$$

where $\tilde{S}$ minimized the functional defined by Ritz method on family of continuous splines S_Δ. If $f(z,r) \in W_2^2$ we get the following better estimation

$$||\tilde{u}_o - \tilde{S}||_{W_2^1} = O(h)$$

where $h = \max|h_r|$ in the one-dimensional problem and $h = \max(h_z, h_r)$ in the two-dimensional problems.

In proving the convergence of the approximate solutions the only property we used that the spline functions $\tilde{S} \in W_2^1$. Therefore we have proved that our spline with its derivatives are approximating the solution of the problem and its first derivatives too.

In chapter II the non-linear first-order hyperbolic equation (3), which expresses the plastic problem in a cylinder of variable diameter, with the given boundary conditions is approximated by finite difference method. We constructed explicit and implicit schemes (see 2.3.8, 2.4.1) for obtaining the approximate solution of the problem. In theorem 2.3.1 we proved that the explicit scheme has the following accuracy

$$\psi = O(h_z^2 + h_r).$$

The accuracy of the implicit scheme is more stabile and accurate than the implicit scheme, but the later is easer and can be better applied for the problems of varied boundaries such as the elastic-plastic problem.

In chapter III. new iteration process ideas applied to find out the unknown free boundary of the one-dimensional elastic-plastic problem. In theorems (3.3.1 and 3.4.1) we proved that the constructed iteration processes have the following quadratic convergence

$$|\tilde{r}_{j+1} - \bar{r}| = O(|\tilde{r}_j - \bar{r}|^2), \quad j = 0,1,...$$

where $\bar{r}$ is the exact value of the free boundary and $\tilde{r}_j$ its j-th approximation.

In chapter IV we considered the solution of free boundary problem in rectangular domain as a special case of the elastic-plastic problem formulated before. The Fourier method is used for the solution of the boundary value problem of the second-order elliptic differential equation (1). As consequence of solving equation (1) we get an eigenvalue problem for ordinary differential equation and the modified Bessel's equation. Therefore the solution of the elliptic boundary value problem is represented in form of infinity Fourier series (see 4.2.20). For obtaining the position of the free boundary we have used the difference between the values of the normal derivatives of the solutions of the elastic and plastic problems. This analysis leads to iteration process for approximating the free boundary. The convergence of the iteration process has been proved.

In chapter V we considered the solution of the elastic-plastic problem (1)-(3) in the domain Ω of variable diameter with using of the more general boundary conditions (4). Iteration process is proposed to obtain the unknown free boundary Γ_4. The elastic and plastic problems can be solved by the methods introduced in the first two chapters. During the iteration process the plastic problem need to be solved once. While the elastic problem need to be solved in every step of the iteration process in the corrigated elastic domain with the new approximate boundary, using the value of the plastic function on this boundary line. For separating the elastic and plastic domains in the first approximation we have examined the solution of the elastic problem, which can be obtained in the begining in whole Ω, with using of condition (2) starting at the neighborhood of Γ_1. After obtaining the first approximation the elastic problem can be solved in the new elastic domain. The next approximation of the free boundary can be found from the comparsion between the gradients of solutions of the elastic and plastic problems. Hence, iteration processes based on qualititive and quantitive strategies are constructed. The convergence of the iteration processes are discussed.

4. Publications

Most of the dissertation contents were presented in form of papers ([3]-[8]), and lectures before the following conferences and seminars:

(i) Approximate Solution for an Elastic Problem in a Cylinder with Circular Section by Spline Method. Lecture delivered in seminar held at Computer Centre of ELTE, 1987.

(ii) Some Results About Solving Elastic-Plastic Problem of Free boundary (in Russian) with R. Farzan. Lecture presented at "Taung über Probleme und Methoden der Mathematischen Physik" conference held in DDR, Karl-Marx-Stadt, 1988.

(iii) Spline Function Solution of Elastic Problem. Lecture before the "Fourth Conference of Program Designers" held in Budapest, 1988.

(iv) Spline Method for One Elastic Problem in the Body with Cylinderical Symmetry. Lecture delivered in seminar organized by Veszprém University in cooperation with "Society of Educational Science" Veszprém county organization, "Bolyai János Matematical Society" Veszprém county branch, and "Neumann János Society of Computer Science" Veszprém county organization, 1989.

(v) Iteration Method for an Elastic-Plastic Problem in a Cylinder with a Collar. Lecture befor the "Fifth Conference of Program Designers" held in Budapest, 1989.

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Appendix

By numerical experiments we proved that the spline method suggested in the dissertation is applicable to solve

the second-order elliptic differential equation be satisfied on the elastic domains (see ch.1).

For the numerical calculation we used the equation $Au = \tilde{F}$ together with homogenous boundary conditions. In the numerical examples the right-hand side functions $\tilde{F}$ are chosen so that the exact solutions are known. This give us the possiblity to do comparsion between the exact and the approximate solutions.

To compute the coefficients of the spline functions systems of linear algebraic equations are obtained by using Ritz method as well as the method of Lagrange multiplyers.

The above problem has been solved in three different domains, the first two are squres of different partitions, and the third one has variable diameter.

The results show that the numerical solutions are in accordance with the exact ones, and the difference between them are corresponded to the theoretical estimates obtained in the dissertation.

MAGYAR
TUDOMÁNYOS AKADÉMIA
KÖNYVTÁRA

Készült az ELTE Sokszorosítóüzemében
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Felelős kiadó: Dr. Kátai Imre
Felelős vezető: Arató Tamás
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