Entropies, Capacities, and Colorings of Graphs

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Contents

Bevezetés és áttekintés (Summary in Hungarian)

Az egyik legfontosabb és legtöbbet vizsgált gráfelméleti paraméter a kromatikus szám, azon színek minimális száma, melyekkel a gráf csúcsai kiszínezhetők úgy, hogy szomszédos csúcsok színe különböző legyen.

A G gráf $\chi(G)$ kromatikus számának egyik legtermészetesebb alsó becslése a gráf $\omega(G)$ klikkszáma, a csúcshalmaz legnagyobb olyan részhalmazának elemszáma, amelyben minden csúcspár össze van kötve. Ez a becslés sokszor nagyon gyenge, egy háromszögmentes gráf kromatikus száma is lehet tetszőlegesen nagy, ezt bizonyítja pl. Mycielski [123] konstrukciója.

Az értekezésben fontos szerepet játszik gráfok konormálisnak nevezett szorzata, illetve hatványa, melynek bevezetését egyebek mellett számos információelméleti kérdés motiválja. E művelet során a klikkszám és a kromatikus szám ellentétesen viselkedik: előbbi szupermultiplikatív, utóbbi szubmultiplikatív erre a szorzásra nézve, melynek definíciója a következő.

Definíció. *Az F és G gráfok* konormális szorzata *az az $F \cdot G$ gráf, melyre*

$$\begin{aligned} V(F \cdot G) &= V(F) \times V(G) \\ E(F \cdot G) &= \{\{(f,g),(f',g')\} : \{f,f'\} \in E(F) \text{ vagy } \{g,g'\} \in E(G)\}. \end{aligned}$$

G^t a G gráf önmagával vett t-szeres konormális szorzata, amit G t-edik konormális hatványának hívunk.

A most definiált hatványozásra úgy érdemes gondolni, hogy amennyiben G élei a csúcsok valamiféle megkülönböztethetőségét jelentik, akkor a konormális hatványozás ezt a relációt terjeszti ki a csúcsok t hosszú sorozataira: két ilyen sorozat pontosan akkor megkülönböztethető, ha legalább egy koordinátában az.

Nem nehéz belátni, hogy, mint fent említettük, tetszőleges F és G gráfra fennáll az

$$\omega(F \cdot G) \geq \omega(F)\omega(G) \quad \text{valamint a} \quad \chi(F \cdot G) \leq \chi(F)\chi(G)$$

egyenlőtlenség.

A fentiből következik, hogy a

$$\chi^*(G) := \lim_{t\to\infty} \sqrt[t]{\chi(G^t)}$$

és a

$$c(G) := \lim_{t\to\infty} \sqrt[t]{\omega(G^t)}$$

határértékek egyaránt léteznek, és rájuk $\omega(G) \le c(G) \le \chi^*(G) \le \chi(G)$ teljesül.

Az elsőként felírt $\chi^*(G)$ határérték jól ismert mennyiség. A kromatikus szám felírható egy egészértékű programozási feladat megoldásaként. Ennek valós relaxációját megoldva jutunk a frakcionális kromatikus szám fogalmához, mely McEliece és Posner [120] egy tételéből adódóan (ld. Berge és Simonovits [19] dolgozatát is) megegyezik a fenti $\chi^*(G)$ határértékkel.

A klikkszám is felírható egészértékű programozási feladat megoldásaként, ennek valós relaxációja a frakcionális klikkszám fogalmához vezet, amelynek értéke a lineáris programozás dualitástétele révén mindig azonos a frakcionális kromatikus szám, tehát $\chi^*(G)$ értékével. Várható volna mindezek alapján, hogy $c(G)$ is ezzel a közös értékkel legyen egyenlő. Ez azonban nincs így.

A $c(G)$ mennyiség, pontosabban annak logaritmusa, Shannon [139] információelméleti vizsgálataiban bukkant fel először.

Definíció. (Shannon [139]) *Egy G gráf (logaritmikus) Shannon kapacitása*[1] *a*

$$C(G) = \lim \frac{1}{t} \log_2 \omega(G^t)$$

mennyiség.

A logaritmálás oka az információelméleti háttér, a fenti mennyiség ugyanis egy zajos csatorna bitekben mért ún. zéró-hiba kapacitását fejezi ki. Innen adódik az is, hogy a logaritmus alapja 2. A továbbiakban is minden logaritmus kettes alapú lesz, ezentúl ezt nem írjuk ki.

A Shannon kapacitás a modern kombinatorika egyik különösen érdekes fogalma. Vizsgálatát számos váratlan kapcsolat, valamint némely vele kapcsolatos probléma meglepő nehézsége egyaránt indokolja. Az a talán ártatlannak látszó kérdés például, hogy egy háromszögmentes gráfra a $C(G)$ érték lehet-e tetszőlegesen nagy, ekvivalens Erdősnek egy máig megoldatlan problémájával, mely azt kérdezi, hogy az $R(3:t) := R(3,3,\dots,3)$ Ramsey szám (az a legkisebb r szám, amire a K_r teljes gráf éleit t színnel színezve biztosan keletkezik egyszínű háromszög) gyorsabban nő-e, mint bármilyen rögzített c konstans t-edik hatványa (ld. Erdős, McEliece és Taylor [49], Alon és Orlitsky [6], valamint Rosenfeld és Nešetřil [125] cikkeit).

Shannon [139] meghatározta minden legfeljebb 4 csúcsú gráf Shannon kapacitását és az 5 csúcsúakét is egy kivétellel. Az 5 hosszúságú C_5 kör Shannon kapacitásáról csak 23 évvel később bizonyította be Lovász [109], hogy a Shannon által megadott alsó korláttal, $\log\sqrt{5}$-tel egyenlő. A Shannon kapacitás probléma nehézségét a fentieken túl az is jól mutatja,

[1] Megjegyezzük, hogy számos tárgyalás a $C(\bar{G})$ mennyiséget nevezi a G gráf Shannon kapacitásának, ahol $\bar{G}$ a G gráf komplementere. Maga Shannon is ezt a nyelvezetet használja, mi azért nem ezt követjük, mert az irányított gráfok kapacitásainak tárgyalása így természetesebb lesz. Erről bővebben ld. a dolgozat 1.5. Megjegyzését.

hogy az ötnél hosszabb páratlan körökre mindmáig ismeretlen a $C(G)$ érték, és még azt is csak 2003-ban igazolta Bohman és Holzman [24], hogy $C(C_{2k+1}) > \log 2$ minden k-ra, azaz minden háromnál hosszabb páratlan kör Shannon kapacitása meghaladja a klikkszámából adódó egyszerű alsó korlátot.

A Shannon kapacitás vizsgálata által inspirálva vezette be Berge [15, 16, 17] a perfekt gráfokat (ld. Ramírez-Alfonsín és Berge [18]).

Definíció. (Berge [16]) *Egy G gráf* perfekt*, ha minden G' feszített részgráfjára $\chi(G') = \omega(G')$ teljesül.*

A perfekt gráfok rendkívül fontos és sokat vizsgált gráfosztályt alkotnak. Ennek legfőbb oka az, hogy kapcsolatot teremtenek olyan látszólag távolabbi területek között, mint a gráfelmélet, a poliéderes kombinatorika és az információelmélet. Számos ilyen kapcsolatot részletesen tárgyal a Ramírez-Alfonsín és Reed által szerkesztett [128] könyv, valamint Schrijver [136] monumentális monográfiájának három perfekt gráfokról szóló fejezete. Igen sok érdekes gráf perfekt. Ilyenek például a páros gráfok és élgráfjaik, az intervallumgráfok, vagy a részben rendezett halmazokhoz rendelhető ún. összehasonlítási gráfok.

A perfekt gráfok számos szép struktúrális tulajdonsága önmagában figyelemre méltó. Kiemelkednek ezek közül a Berge [15, 16] híres sejtéseiből mára tétellé vált állítások, a megfogalmazása után körülbelül egy évtizeddel Lovász [106] által bebizonyított Perfekt Gráf Tétel és a kimondása után negyven évvel igazolt Erős Perfekt Gráf Tétel, melyről 2002-ben jelentette be Chudnovsky, Robertson, Seymour és Thomas [30, 31], hogy bebizonyították. A Perfekt Gráf Tétel szerint egy gráf akkor és csak akkor perfekt, ha a komplementere az. Az ezt általánosító Erős Perfekt Gráf Tétel azt állítja, hogy egy gráf pontosan akkor perfekt, ha feszített részgráfként nem tartalmaz páratlan kört vagy ilyennek komplementerét. Utóbbi, a megoldásáig Erős Perfekt Gráf Sejtésként ismert állítás, a gráfelmélet kiemelkedő problémája volt az elmúlt évtizedekben.

Közvetve tehát mindez Shannon [139] zéró-hiba kapacitás vizsgálataiból eredt.

Cohennel és Körnerrel a [33] cikkben a Shannon kapacitás fogalmát gráfcsaládokra terjesztettük ki, majd Körnerrel a [98] dolgozatban egy olyan extremális halmazelméleti kérdést vizsgáltunk, ami lefordítható volt irányított gráfcsaládok egy kapacitás típusú paraméterének vizsgálatára. Ezt általánosítva Gargano, Körner és Vaccaro [60] bevezette az irányított gráfokra értelmezett Sperner kapacitás fogalmat és ennek gráfcsaládokra való kiterjesztését. Ezzel megteremtették számos érdekes extremális halmazelméleti probléma közös tárgyalásának lehetőségét és [61, 62] cikkeikben bebizonyítottak egy mély tételt, mely számos ilyen problémát egyszerre megold. Ezek között a legnevezetesebb Rényinek az ún. kvalitatív 2-függetlenségre vonatkozó problémája, mely így felvetése után több, mint húsz évvel szintén megoldást nyert. Megjegyezzük, hogy a [33]-beli problémafelvetés eredetileg infomációelméleti indíttatású volt, a gráfcsaládok Shannon kapacitásaként értelmezett fogalom az ún. összetett csatorna zéró-hiba kapacitásának felel meg. Nayak és Rose [124] nemrégiben észrevette, hogy gráfcsaládok Sperner kapacitására is adható ehhez hasonló információelméleti interpretáció.

A Sperner kapacitás a Shannon kapacitás formális általánosításának tekinthető amennyiben az irányítatlan gráfokat olyan irányított gráfoknak tekintjük, melyek minden élüket mindkét lehetséges irányításukkal tartalmazzák. Látva, hogy a Shannon kapacitás értékét konkrét kis gráfokra sem mindig könnyű meghatározni, nem meglepő, hogy a helyzet hasonló a Sperner kapacitás esetében is. (Már egy ciklikusan irányított háromszög Sperner kapacitásának megállapítása sem triviális, ld. Calderbank, Frankl, Graham, Li, Shepp [28] és Blokhuis [23] dolgozatait.) Ugyanakkor nem egyszerűen egy megoldatlan probléma még nehezebbé tételéről van szó, hiszen a Sperner kapacitás meghatározása sokszor olyan gráfokra is érdekes, melyek irányítatlan verziójára ismerjük a Shannon kapacitás értékét. Különösen érdekes továbbá az irányítás hatását figyelni, vagyis összehasonlítani egy irányítatlan gráf Shannon kapacitását irányított változatai Sperner kapacitásával. Utóbbi sohasem lehet nagyobb az előbbinél, az egyenlőség pontos feltételei nem ismertek.

A Shannon kapacitáshoz hasonlóan az információelméletből származó gráfelméleti fogalom a gráfentrópia. Körner [88] vezette be 1973-ban megjelent cikkében és a frakcionális kromatikus szám egyfajta valószínűségi finomításaként is felfogható. A gráfentrópia teljesít egy szintén Körner [89] által észrevett szubadditivitási egyenlőtlenséget (ld. (2)), mely alkalmassá tette különféle kombinatorikus becslésekre, ld. pl. Körner [89], Newman, Ragde, Wigderson [126], Radhakrishnan [127] dolgozatait. Körner és Marton [93] a gráfentrópia és a rá vonatkozó alapvető egyenlőtlenség közvetlen általánosításával uniform hipergráfok entrópiájának segítségével adtak jobb becslést a Körner által a [89] dolgozatban Fredman és Komlós [54] nyomán vizsgált ún. "perfect hashing" problémára. A gráfentrópia talán legnagyobb sikere Kahn és Kim [83] áttörést jelentő eredménye, melyben először adtak meg konstans szorzó erejéig optimális számú összehasonlítást használó, és ezeket determinisztikusan és polinomidőben megválasztó algoritmust arra a sokat vizsgált rendezési problémára, melyben egy ismert részben rendezést kell minél kevesebb elempár összehasonlításával kiterjeszteni teljes rendezéssé. Ebben már az a Körner és Marton [92] által sejtett és a Csiszár, Körner, Lovász és Marton társszerzőkkel írt [38] cikkben bizonyított eredmény is szerepet játszott, mely szoros összefüggést állapított meg a gráfentrópia és a perfekt gráfok között.

Az értekezés [140] dolgozaton alapuló 1.1. Alfejezete és a [142] dolgozat felhasználásával készült 1.2. Alfejezet a perfekt gráfok és a gráfentrópia kapcsolatát kimondó [38]-beli tétel egy-egy kiterjesztését tárgyalja. Az első fejezet [143] cikken alapuló harmadik alfejezete a Witsenhausen ráta[2] nevű rokon fogalom gráfcsaládos változatát vezeti be és erre bizonyít egy információelméleti tartalmát tekintve talán meglepő tételt.

A második fejezetben gráfkapacitásokkal foglalkozunk. A GallucciovaI, Garganoval és Körnerrel közös [59] cikken alapuló első alfejezetben irányított gráfoknak a Sperner kapacitáshoz hasonlóan extremális halmazelméleti kérdésekre is lefordítható kapacitás jellegű paraméterét vizsgáljuk, majd ennek egy irányítatlan gráfokra vonatkozó rokonát.

[2] Az angol *rate* szót az információelméletben gyakran *sebesség*nek fordítják, ha annak maximalizálása a cél. Itt azonban minimalizálni szeretnénk, ezért választottuk inkább az idegenebbül hangzó, de talán kevésbé félrevezető szót.

Közben fogalkozunk a Sperner kapacitással is, és megmutatjuk, hogy az öt hosszú körnek van olyan irányítása, aminek Sperner kapacitása eléri a $C(C_5) = \log \sqrt{5}$ értéket. A Salival közös [132] cikken alapuló második alfejezetben ezt az észrevételt általánosítjuk tetszőleges csúcstranzitív önkomplementer gráfra. A Körnerrel és Pilotoval közös [97] cikken alapuló harmadik alfejezetben Alon [2] korábbi eredményét általánosító új felső korlátot adunk a Sperner kapacitásra. Ebben főszerepet játszik az Erdős, Füredi, Hajnal, Komjáth, Rödl és Seress [47] által bevezetett lokális kromatikus szám nevű paraméter, illetve annak irányított gráfokra való általánosítása. Azt is megmutatjuk, hogy a lokális kromatikus szám sohasem kisebb a frakcionális kromatikus számnál. Ez utóbbi eredmény a harmadik fejezet vizsgálatainak kiindulópontja. A lokális kromatikus számról nyilvánvaló, hogy a kromatikus számnál sohasem nagyobb, így az előbbi eredmény szerint mindig a frakcionális kromatikus szám és a kromatikus szám közé esik. Ez motiválja, hogy olyan gráfokra próbáljuk meghatározni az értékét, amire ez utóbbi két paraméter távol esik egymástól.

Viszonylag kevés olyan gráfcsalád ismert, amelynél e két paraméter messze van egymástól, és az ilyenekbe tartozó gráfoknál gyakran magának a kromatikus számnak a meghatározása is nehézségekbe ütközik. E nehézséget sok esetben azzal a váratlan, Lovász [108] Kneser gráfokkal kapcsolatos úttörő munkájából származó technikával lehet legyőzni, amely megfelelő előkészületek után az algebrai topológia híres tételét, a Borsuk-Ulam tételt hívja segítségül. A harmadik fejezet Tardos Gáborral közös [144] cikken alapuló első alfejezetében látni fogjuk, hogy ez a technika, az ún. topologikus módszer, a lokális kromatikus szám, sőt egy másik színezési paraméter, a cirkuláris kromatikus szám vizsgálatára is alkalmas. A lokális kromatikus számra sok esetben éles alsó becslést adunk. Az élességet kombinatorikus úton látjuk be, majd bizonyos topológiai következményeit is megfogalmazzuk. A cirkuláris kromatikus számra kapott eredményünk részlegesen (páros kromatikus gráfok esetén) igazolja Johnson, Holroyd és Stahl [81], valamint Chang, Huang és Zhu [29] egy-egy sejtését. Az előbbi sejtéssel kapcsolatos eredményt tőlünk függetlenül Meunier [121] is elérte. A szintén Tardos Gáborral közös [145] cikken alapuló 3.2. Alfejezetben szintén a topologikus módszert használva a fejezet első felében is vizsgált G gráfok optimális ($\chi(G)$ színt használó) színezéseiről látjuk be, hogy bennük minden elépzelhető $\chi(G)$ csúcsú teljesen tarka teljes páros gráf megjelenik részgráfként. Ez egyfajta ellenpontja a lokális kromatikus számra bizonyított eredményeinknek, melyek interpretálhatók úgy, hogy ha a kromatikus számnál csak eggyel több színt is használhatunk, akkor ezen tarka teljes páros gráfok közül egy kivételével mindegyik elkerülhető.

A topologikus módszer, ezen belül is a Borsuk-Ulam tételt használó technika jelentőségét nehéz túlbecsülni, itt most csak Matoušek [116] remek könyvére hivatkozunk, további előzményeket pedig a harmadik fejezet eredményeinek bővebb bemutatásakor tárgyalunk.

Az alábbiakban az egyes fejezetek néhány főbb eredményét ismertetem kicsit részletesebben. Ezen áttekintés végén megtalálható az egyes alfejezetek elkészítéséhez felhasznált dolgozatok felsorolása.

Gráfok entrópiái

Az értekezés azonos című fejezete három alfejezetének alapjául a [140], [142] egy része és a [143] dolgozat szolgált.

Gráfentrópia

A gráfentrópia nevű információelméleti függvényt Körner [88] definiálta. A kiindulópont itt is egy információelméleti probléma volt, ez vezetett a következő mennyiség bevezetéséhez, melyet Körner a G gráf P eloszláshoz tartozó entrópiájának nevezett el:

$$H_\varepsilon(G,P) := \lim_{t\to\infty} \frac{1}{t} \log \min_{P^t(U)>1-\varepsilon} \chi(G^t[U]),$$

ahol $G^t[U]$ a G gráf fentebb bevezetett t-edik konormális hatványának az $U \subseteq [V(G)]^t$ csúcshalmazon feszített részgráfja, $\varepsilon \in (0,1)$, P pedig egy $V(G)$-n adott valószínűségeloszlás, mely $P^t(\boldsymbol{x}) = \prod_{i=1}^t P(x_i)$ módon adja az $\boldsymbol{x} = x_1 \dots x_t$ sorozat valószínűségét és $P^t(U) = \sum_{\boldsymbol{x}\in U} P^t(\boldsymbol{x})$. Körner [88] megadott $H(G,P)$-re egy másik formulát is, a kettő egyenlőségének bizonyításával belátta, hogy a fenti határérték létezik és független ε-tól. Ennek a második formulának a további alakítása a [38] cikkben elvezetett ahhoz a harmadikhoz, amit az értekezésben használunk. Egy F (hiper)gráfban független halmaznak nevezzük a csúcsok minden olyan részhalmazát, amely nem tartalmaz élet, a $VP(F)$ csúcspakolási politóp pedig a független halmazok karakterisztikus vektorainak konvex burka.

Definíció. *Legyen F (hiper)gráf a $V(F) = \{1,...,n\}$ csúcshalmazon, $P = (p_1,...,p_n)$ pedig valószínűségeloszlás $V(F)$-en. Ekkor az F (hiper)gráf P eloszlásra vonatkozó entrópiája a*

$$H(F,P) = \min_{\boldsymbol{a}\in VP(F)} \sum_{i=1}^n p_i \log \frac{1}{a_i} \tag{1}$$

mennyiség.

Körner [89] a nyolcvanas években észrevette, hogy a gráfentrópia teljesíti a következő szubadditivitási tulajdonságot. Ha F és G két gráf ugyanazon a V csúcshalmazon, P tetszőleges V-n vett eloszlás, $F \cup G$ pedig a $V(F\cup G) = V, E(F\cup G) = E(F) \cup E(G)$ módon megadható gráf, akkor

$$H(F \cup G, P) \le H(F,P) + H(G,P). \tag{2}$$

Ez az egyenlőtlenség lényegében az egyszerűen belátható $\chi(F \cup G) \le \chi(F)\chi(G)$ összefüggés következménye. A bevezetőben már említettük, hogy a fenti egyenlőtlenségre alapozva nemtriviális becslések nyerhetők egyes kombinatorikai problémákban, ilyen alkalmazások találhatók például Körner [89], Newman, Ragde és Wigderson [126], vagy Radhakrishnan [127] cikkeiben. Mindez felvetette a szubadditivitási egyenlőtlenség élességének kérdését, melynek már információelméleti megfontolások szerint is kitüntetett speciális esete volt az, amikor a két gráf egymás komplementere (ld. Körner és Longo [91]). Körner és Marton [92] sejtette, a [38] cikkben pedig bizonyítást nyert, hogy a $H(G,P)+H(\bar{G},P)=H(K_{|V|},P)=H(P)$ egyenlőség pontosan akkor áll fenn minden P eloszlás esetén, ha G perfekt gráf. Itt $H(P)$ a P eloszlás entrópiája, ami megegyezik a $|V(G)|$ csúcsú teljes gráf P-hez tartozó entrópiájával.

Ennek az eredménynek tárgyaljuk két különböző kiterjesztését az első fejezet első két alfejezetében.

Az 1.1. Alfejezetben karakterizáljuk azon 3-uniform hipergráfokat, amelyek a fentivel analóg azonosságot teljesítenek. A problémát az általános k-uniform esetre is megoldjuk, de $k>3$-ra az derül ki, hogy a kívánt egyenlőség csak a triviális esetekben teljesül.

Hipergráfok entrópiáját Körner és Marton [93] definiálták a gráfentrópia általánosításaként (ld. a fenti definíciót), a szubadditivitási egyenlőtlenség itt is érvényben marad és a korábbiakhoz hasonlóan alkalmazható, ld. [93]. Ha F k-uniform hipergráf a V csúcshalmazon, akkor F-nek $\bar{F}$ komplementerén azt a V-n megadható hipergráfot értjük, melynek élei az F-nek $E(F)$ élhalmazában nem szereplő V-beli k-asok.

Egy 3-uniform F hipergráfot nevezzünk *levélmintá*nak, ha reprezentálható a következőképpen. Legyen T fa, melyben a legalább 2 fokú csúcsok mindegyike meg van jelölve 0-val vagy 1-gyel. Az így megjelölt T fához tartozó levélminta az a 3-uniform hipergráf, melynek csúcsai T levelei (1 fokú csúcsai), élei pedig azon $\{x,y,z\}$ hármasok, melyekre a fabeli xy, yz és xz utak egyetlen közös pontja 1-gyel van megjelölve. Az n csúcsú teljes 3-uniform hipergráfot jelölje $K_n^{(3)}$.

1.1.1. Tétel. *Az F 3-uniform hipergráfra akkor és csak akkor teljesül minden P eloszlás mellett a*

$$H(F,P)+H(\bar{F},P)=H(K_{|V|}^{(3)},P),$$

egyenlőség, ha F levélminta.

A tétel kiterjeszthető arra az esetre is, amikor $K_{|V|}^{(3)}$-at kettőnél több hipergráf uniójára bontjuk.

Az 1.2. Alfejezetben a gráfentrópia és az ún. imperfektségi hányados kapcsolatát tárgyaljuk. Az imperfektségi hányados fogalmát Gerke és McDiarmid vezették be [65] dolgozatukban. Frekvenciakiosztási problémákat vizsgálva minden G gráfhoz hozzárendeltek egy $\mathrm{imp}(G)\geq 1$ mennyiséget, mely pontosan akkor egyenlő 1-gyel, ha G perfekt. Az új fogalom további szép tulajdonsága, hogy $\mathrm{imp}(G)=\mathrm{imp}(\bar{\mathrm{G}})$ teljesül minden G gráfra. A [142] dolgozatban[3] a következő összefüggést sikerült igazolni.

1.2.6. Tétel. *Tetszőleges G gráfra teljesül, hogy*

$$\log \mathrm{imp}(G)=\max_P\{H(G,P)+H(\bar{G},P)-H(P)\}.$$

[3]Ennek az összefoglaló dolgozatnak a megírására nagyrészt a most tárgyalt eredmény révén került sor, részben emiatt kért fel a [128] könyv egyik szerkesztője, Bruce Reed, korábbi összefoglaló cikkem [141] ezt is tartalmazó átdolgozására.

A tétel tehát azt mutatja, hogy a Gerke és McDiarmid által definiált, imperfektséget mérő mennyiség és a [38]-beli eredményből adódó imperfektségi mérőszám lényegében ugyanaz.

A gráfentrópia definíciójának alábbi általánosítása szintén [38]-ból való.

Egy $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$ halmazt *konvex sarok*nak nevezünk, ha zárt, konvex, belseje nemüres, és teljesül rá, hogy amennyiben $\boldsymbol{a} \in \mathcal{A}$ és $0 \leq a'_i \leq a_i$ minden i-re, akkor $\boldsymbol{a}' \in \mathcal{A}$.

A gráfentrópia általánosításaként értelmezhető egy konvex sarok entrópiája is az alábbi formulával:

$$H_{\mathcal{A}}(P) := \min_{\boldsymbol{a} \in \mathcal{A}} \sum_{i=1}^{n} p_i \log \frac{1}{a_i}.$$

McDiarmid [119] bevezeti két konvex sarok, $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n_{+,0}$ *dilatációs hányados*át az alábbi módon:

$$\mathrm{dil}(\mathcal{A}, \mathcal{B}) := \min\{t : \mathcal{B} \subseteq t\mathcal{A}\}.$$

Gerke és McDiarmid egyik eredménye szerint ez a fogalom általánosítása az imperfektségi hányadosnak, utóbbi ugyanis kifejezhető két, a szóbanforgó gráfhoz rendelt speciális konvex sarok dilatációs hányadosaként.

Az 1.2. Alfejezetben megmutatjuk, hogy az 1.2.6. Tételhez hasonlóan bizonyítható annak következő általánosítása is.

1.2.8. Tétel.

$$\log \mathrm{dil}(\mathcal{A}, \mathcal{B}) = \max_P \{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\}.$$

Witsenhausen ráta

A gráfentrópia eredeti definíciójában másik (az ún. normális) gráfhatványozást alkalmazva valamivel kisebb értékű mennyiséghez jutunk, melyet Körner és Longo [91] vezetett be szintén információelméleti megfontolásból. Ez a jelen dolgozatban $\bar{H}(G, P)$-vel jelölt függvény tekinthető úgy, mint a (csak valamivel később bevezetett) Witsenhausen rátaként ismert mennyiség valószínűségi finomítása. A Witsenhausen [158] dolgozatában definiált Witsenhausen ráta azt fejezi ki, hogy 0 hibavalószínűségű dekódolást elvárva átlagosan mekkora hányadára lehet összetömöríteni egy üzenetet, ha a vevő rendelkezik valamilyen, az adó által nem ismert, de az üzenet tartalmával korreláló mellékinformációval. A Shannon kapacitás esetéhez hasonlóan itt is egy gráffal jellemezhető az információelméleti szituáció (a csúcsok a lehetséges üzenetek, és kettő össze van kötve éllel, ha van olyan mellékinformáció, mely nem különbözteti meg őket, tehát a hozzájuk tartozó üzeneteknek különbözniük kell). Jelölje $G^{\wedge t}$ a G gráf már említett normális hatványát, mely legegyszerűbben a $G^{\wedge t} = \overline{(\bar{G})^t}$ egyenlőséggel definiálható, ahol mindkét felülvonás komplementálást jelent. (Két különböző csúcsot alkotó sorozat pontosan akkor van összekötve, ha minden olyan koordinátában, ahol nem egyenlők, G-nek élét alkotják.)

Definíció. (Witsenhausen [158]) *A G gráf Witsenhausen ráta nevű paramétere az*

$$R(G) = \lim_{t\to\infty} \frac{1}{t} \log \chi(G^{\wedge t})$$

mindig létező határérték.

Az 1.3. Alfejezetben azt vizsgáljuk, hogy ha egyetlen adó sok különböző vevőnek küldi ugyanazt az üzenetet, s ezen adók mind más-más mellékinformációval rendelkeznek, akkor minden egyes vevőnél 0 hibavalószínűségű dekódolást elvárva, átlagosan mekkora hányadára tömöríthető össze az üzenet. A meglepő válasz az, hogy ugyanakkorára, mint amekkorára akkor lenne, ha csak azzal az egy vevővel kellene kommunikálnia az adónak, amelyik a leggyengébb tömörítést teszi lehetővé. Formálisan, ha k vevő van és G_i írja le az i-edik vevővel való kommunikációhoz tartozó gráfot ($i = 1, \ldots, k$), $\mathcal{G} = \{G_1, \ldots, G_k\}$ és a keresett mennyiséget $R(\mathcal{G})$ jelöli (meggondolható, hogy $R(\mathcal{G}) = \lim_{t\to\infty} \frac{1}{t} \log(\chi(\cup_i G_i^{\wedge t})))$, akkor a következő igaz.

1.3.1. Tétel.

$$R(\mathcal{G}) = \max_{G_i\in\mathcal{G}} R(G_i).$$

A bizonyítás Gargano, Körner és Vaccaro [62] mély tételén alapul, mely gráfok kapacitásainak valószínűségi finomítására mond ki erős eredményt. A Witsenhausen rátára ez azért alkalmazható, mert a Witsenhausen ráta már említett valószínűségi finomítása és a Shannon kapacitás Csiszár és Körner [37] által bevezetett valószínűségi finomítása között egy Marton [115] által igazolt szoros öszefüggés áll fönn.

Gráfok kapacitásai

Az értekezés ezen fejezetének egyes alfejezetei rendre a Gallucioval, Garganoval és Körnerrel közös [59], a Salival közös [132], valamint a Körnerrel és Pilottoval közös [97] cikkek felhasználásával készültek, eredményeik a megfelelő cikk társszerzőivel közös eredmények.

Variációk kapacitásfogalmakra

Legyen $\mathcal{F}$ tetszőleges, a konormális szorzásra zárt gráfcsalád. A G gráfban feszített részgráfként megjelenő legnagyobb (legtöbb csúcsú) $\mathcal{F}$-beli gráf csúcsainak számát $c_{\mathcal{F}}(G)$-vel jelölve $c_{\mathcal{F}}(G^t) \geq [c_{\mathcal{F}}(G)]^t$ nyilvánvalóan teljesül. Ilyenkor létezik a $C_{\mathcal{F}}(G) := \lim_{t\to\infty} \frac{1}{t} \log c_{\mathcal{F}}(G^t)$ határérték, ami $\mathcal{F}$-nek a teljes gráfok családját választva éppen a Shannon kapacitás. Ha az $\mathcal{F}$ gráfcsaládra még azt a további természetes feltételt is szabjuk, hogy a feszített részgráf képzésre is zárt legyen, akkor egy egyszerű észrevétel (Proposition 2.1.11) mutatja, hogy $\mathcal{F}$ megválasztására mindössze néhány lehetőségünk marad. Triviális eseteket leszámítva $\mathcal{F}$ nem lehet más, mint az összes üres (vagyis éleket nem tartalmazó), az összes teljes, vagy az összes teljes sokrészes gráf családja. (Utóbbiba azon gráfok tartoznak, melyek csúcshalmaza partícionálható néhány élet nem tartalmazó osztályra úgy, hogy bármely két különböző osztályba eső csúcs össze legyen kötve. Ez a gráfcsalád tartalmazza mindkét előzőt.) A megfelelő $C_{\mathcal{F}}(G)$ értékek közül az első mindig a függetlenségi szám logaritmusával lesz egyenlő, mert a legnagyobb független halmaz mérete pontos multiplikativitást mutat a konormális szorzás esetén. A második családhoz tartozó érték a Shannon kapacitás. Egyedül a harmadik család ad új, nemtriviális mennyiséget. Egyebek mellett ezt a mennyiséget vizsgáljuk *kaszkád kapacitás* néven a második fejezet első alfejezetében, mely Anna Galluccioval, Luisa Garganoval és Körner Jánossal közös eredményeket ismertet. A G-beli legtöbb csúcsú feszített teljes sokrészes részgráf csúcsainak számát $W(G)$-vel jelölve belátjuk, hogy egy alkalmasan definiált G^* segédgráf kromatikus számának logaritmusa felső korlátja G kaszkád kapacitásának. Ez közvetlen következménye az alábbi tételnek, melynek kimondásához definiáljuk a G^* gráfot.

A $G = (V, E)$ irányítatlan gráf *függetlenségi gráf*ja az a G^* irányítatlan gráf, melynek csúcsai és élei az alábbi módon adhatók meg:

$$\begin{aligned} V(G^*) &= \{(x, A) : A \subseteq V \text{ független halmaz } G\text{-ben és } x \in A\}, \\ E(G^*) &= \{\{(x, A), (y, B)\} : A = B \text{ és } x \neq y, \text{ vagy } \forall a \in A, \forall b \in B, \{a, b\} \in E\}. \end{aligned}$$

2.1.10. Tétel. *Tetszőleges G irányítatlan gráfra fennáll a*

$$W(G^t) \leq [\chi(G^*)]^t$$

egyenlőtlenség.

A tétel alkalmazásaként mutatunk olyan gráfosztályokat, amelyekre a $W(G)$ mennyiség multiplikatívan viselkedik.

A fenti eredményhez irányított gráfok analóg paramétereit vizsgálva jutottunk el. A konormális szorzás egyszerűen kiterjeszthető irányított gráfokra: az F és G irányított gráfok $F \cdot G$ konormális szorzata az a $V(F) \times V(G)$ csúcshalmazú irányított gráf, melyben az (f_1, g_1) csúcsból pontosan akkor megy (irányított) él (f_2, g_2)-be, ha $(f_1, f_2) \in E(F)$ vagy $(g_1, g_2) \in E(G)$ teljesül. Az irányított esetben is G^t jelöli a G gráf önmagával vett t-szeres konormális szorzatát. Érdemes megjegyezni, hogy G^t-ben két csúcs között futhat mindkét irányban él olyankor is, ha G-ben ez nem fordul elő.

Irányított gráfok konormális hatványozása segítségével definiálta Gargano, Körner és Vaccaro [60] a Sperner kapacitás fogalmát, amiről említettük a bevezetőben, hogy különböző extremális halmazelméleti problémák közös tárgyalását tette lehetővé. Hasonló motivációk alapján a 2.1. Alfejezetben bevezetjük az *antilánc kapacitás* fogalmát, mely olyan G^t-beli részgráfok maximális csúcsszámának aszimptotikus exponensét méri, melyekben bármely két csúcs vagy összekötetlen, vagy mindkét irányban összekötött.

Jelöléssel: ha $M(G,t)$ a legnagyobb ilyen tulajdonságú részgráf csúcsszáma G^t-ben, akkor G antilánc kapacitása az

$$A(G) := \lim_{t\to\infty} \frac{1}{t} \log M(G,t)$$

mennyiség. A teljes sokrészes gráfokhoz mindez annak révén kapcsolódik, hogy $A(G)$ legtermészetesebb alsó becslését bizonyos speciálisan irányított G-beli teljes sokrészes gráfok maximális csúcsszámának logaritmusa adja.

Ha a G alapgráfban tetszőleges két csúcs legfeljebb az egyik irányban lehet összekötve, akkor az antilánc kapacitásra is felső becslést ad egy segédgráf kromatikus számának logaritmusa. Ennek a szintén G^*-gal jelölt gráfnak a csúcshalmaza ugyanúgy adható meg, mint az irányítatlan esetben, élhalmaza pedig az alábbi:

$$E(G^*) = \{\{(x,A),(y,B)\} : A = B \text{ és } x \neq y, \text{ vagy } \forall a \in A, \forall b \in B, (a,b) \in E\}.$$

Az irányítatlan gráfok függetlenségi gráfjának definíciójához képest tehát annyi az eltérés, hogy ha A és B olyan független halmazok, melyekre minden $a \in A$ és $b \in B$ között van él, akkor az is lényeges, hogy az A és B között valamelyik kitüntetett irányba futó valamennyi él jelen van-e. Ha az irányítatlan gráfokat olyan irányított gráfokkal azonosítjuk, melyekben minden él mindkét irányban szerepel, akkor az újabb definíció magában foglalja a korábbit, s ez indokolja az azonos jelölést.

Az antilánc kapacitást becslő tétel tehát a következő.

2.1.3. Tétel. *Ha a G irányított gráfban bármely két pont között legfeljebb az egyik irányban van él, akkor*

$$A(G) \leq \log \chi(G^*).$$

Pontosabban, fennáll az

$$M(G,t) \leq [\chi(G^*)]^{t-1}\alpha(G)$$

egyenlőtlenség, ahol $\alpha(G)$ a G gráf függetlenségi számát jelöli.

A 2.1.3. Tétel és az előbb már kimondott 2.1.10. Tétel bizonyítása sokban hasonlít ugyan, ennek ellenére a 2.1.10. Tétel bizonyítása nem teljesen automatikus a 2.1.3. Tétel bizonyításának ismeretében sem. Ennek az az oka, hogy míg az irányított probléma esetén a kielégítendő feltétel csúcspárok között áll fönn, az irányítatlan esetben csúcsok hármasait kell vizsgálni annak megállapításához, hogy szerepelhetnek-e együtt egy számunkra megfelelő halmazban, vagyis egy teljes sokrészes gráfban.

Sperner kapacitás becslései

Gargano, Körner és Vaccaro [60] a következő módon általánosította a Shannon kapacitás fogalmát irányított gráfokra. Jelentse $\omega_s(G)$ a G irányított gráf legnagyobb olyan $U \subseteq V(G)$ csúcshalmazának elemszámát, amire $x,y \in U$-ból $(x,y) \in E(G)$ és $(y,x) \in E(G)$ is következik. (Az ilyen U által feszített részgráfot *szimmetrikus klikk*nek nevezzük.)

Definíció (Gargano, Körner, Vaccaro [60]) *A G irányított gráf (logaritmikus) Sperner kapacitása a*

$$\Sigma(G) = \lim_{t\to\infty} \frac{1}{t} \log \omega_s(G^t)$$

mennyiség.

A definícióból adódik, hogy ha egy irányítatlan gráfot ismét azzal az irányított gráffal azonosítunk, mely minden éle helyén annak mindkét lehetséges irányított változatát tartalmazza, akkor az így kapott irányított gráf Sperner kapacitása az eredeti irányítatlan gráf Shannon kapacitásával lesz azonos. Ebből az is azonnal látható, hogy egy irányítatlan G gráf összes lehetséges $\hat{G}$ irányított változatára $\Sigma(\hat{G}) \le C(G)$ igaz. Felmerül a kérdés, hogy ha $\hat{G}$ csak olyan irányított gráfot jelenthet, ami G minden élének egyik, de csak egyik irányítását tartalmazza, akkor van-e az így kapható irányított változatok között mindig olyan, amire $\Sigma(\hat{G}) = C(G)$. Általánosságban a kérdés nyitott, alább néhány speciális esetéről szólunk.

Nem nehéz belátni, hogy minden irányított G gráfra fennáll $\Sigma(G) \ge \log \mathrm{tr}(G)$, ahol $\mathrm{tr}(G)$ a gráf *tranzitív klikkszáma*, vagyis a benne lévő legnagyobb olyan klikk mérete, melynek csúcsai megcímkézhetők különböző egész számokkal úgy, hogy kisebb címkéjű csúcsból nagyobb címkéjűbe mindig menjen él[4]. Ebből könnyen adódik, hogy amennyiben egy G irányítatlan gráfra $\chi(G) = \omega(G)$ teljesül, akkor egy legnagyobb klikkjét tranzitívan (többi élét pedig tetszőlegesen) irányítva a keletkező $\hat{G}$ irányított gráfra fenn fog állni a $\Sigma(\hat{G}) = C(G)$ egyenlőség. A 2.1. Alfejezet vizsgálatainak egyik mellékterméke az az észrevétel (Proposition 2.1.17), hogy C_5-nek van olyan irányítása, melynek négyzetében megjelenik egy öt csúcsú tranzitív klikk. Ebből $C(C_5) = \log \sqrt{5}$ alapján azonnal adódik, hogy C_5 is rendelkezik a fenti tulajdonsággal (noha $\chi(C_5) > \omega(C_5)$). A 2.2. Alfejezetben ezt az észrevételt általánosítjuk. Az itt ismertetett, Sali Attilával közös eredmény szerint a vizsgált tulajdonsággal minden csúcstranzitív önkomplementer gráf rendelkezik. A bizonyításhoz az alábbi tételt igazoljuk. (A tétel kimondásában egy halmaz ρ-val jelölt lineáris rendezése úgy értendő, hogy $\rho(k)$ a halmaz azon elemét jelöli, ami ρ szerint a k-adik helyre kerül.)

2.2.1. Tétel. *Legyen $G = (V, E)$ önkomplementer gráf a $V = \{1, 2, \ldots, n\}$ csúcshalmazon és legyen $\tau : V \to V$ a V elemeinek az önkomplementerséget tanúsító permutációja, vagyis olyan egy-egy értelmű leképezés, amire $\{i, j\}$ akkor és csak akkor nem éle G-nek, ha $\{\tau^{-1}(i), \tau^{-1}(j)\} \in E$. Ekkor létezik V elemeinek olyan σ lineáris rendezése, amire fennáll, hogy ha $\{i, j\} \notin E$ és $\sigma^{-1}(i) < \sigma^{-1}(j)$, akkor $\sigma^{-1}(\tau^{-1}(i)) < \sigma^{-1}(\tau^{-1}(j))$.*

Kevésbé formálisan ez azt jelenti, hogy minden önkomplementer gráf irányítható úgy a komplementerével együtt, hogy irányított gráfként is izomorfak legyenek (ráadásul egy előre megadott, irányítatlan változataik között érvényes izomorfizmus szerint), és

[4]Röviden azt mondhatnánk, hogy egy klikk tranzitív, ha nincs benne irányított kör, de mivel a hatványokban oda-vissza élek is megjelenhetnek, biztosabban kerüli el a félreértést a kicsit bonyolultabb fogalmazás.

az uniójukként előálló irányított teljes gráf irányítása tranzitív legyen. (Ilyen irányítást kapunk, ha a tételbeli lineáris rendezéssel konzisztensen irányítjuk a gráf éleit.)

Ebből Lovász [109] egy tételét is felhasználva adódik a következő.

2.2.3. Tétel. *Ha G csúcstranzitív és önkomplementer gráf, akkor az összes irányításán vett Sperner kapacitások maximuma egyenlő a Shannon kapacitásával.*

Calderbank, Frankl, Graham, Li és Shepp [28] bizonyították először, hogy egy irányított gráf Sperner kapacitása lehet ténylegesen kisebb, mint irányítatlan megfelelőjének Shannon kapacitása. Azt mutatták meg, hogy egy ciklikusan irányított háromszög Sperner kapacitása $\log 2$, míg $C(C_3) = \log 3$ nyilvánvaló. A bizonyítás lineáris algebrai módszert használ. Blokhuis [23] rövidesen másik elegáns lineáris algebrai bizonyítást közölt ugyanerre az állításra. Az ő bizonyítását általánosította valamivel később Alon [2], aki belátta, hogy

$$\Sigma(G) \leq \log(\min\{\Delta_+(G), \Delta_-(G)\} + 1),$$

ahol $\Delta_+(G)$ és $\Delta_-(G)$ a G gráf egy-egy csúcsából kiinduló, illetve oda befutó élek maximális számát jelöli. Ezt az eredményt sikerült tovább általánosítani Körner Jánossal és Concetta Pilottoval közösen, erről szól a 2.3. Alfejezet. Az eredmény kimondásához definiálnunk kell az irányított lokális kromatikus szám fogalmát, mely egy Erdős, Füredi, Hajnal, Komjáth, Rödl és Seress [47] által bevezetett, irányítatlan gráfokon definiált fogalom általánosítása. Először ez utóbbit definiáljuk.

Definíció. ([47]) *Egy G irányítatlan gráf lokális kromatikus száma a*

$$\psi(G) := \min_{c:\ V(G)\to\mathbb{N}} \max_{v\in V(G)} |\{c(u) : u \in \Gamma_G(v)\}|$$

mennyiség, ahol a minimalizálást az összes c jó színezésre végezzük, $\mathbb{N}$ a természetes számok halmaza, $\Gamma_G(v)$ pedig a v csúcs "zárt szomszédsága", vagyis $\Gamma_G(v) = \{u \in V(G) : \{u, v\} \in E(G) \text{ vagy } u = v\}$.

A lokális kromatikus szám tehát az a minimális szám, amire igaz, hogy ennyi színnek minden jó színezésben elő kell fordulnia valamely zárt szomszédságban.

Definíció. *A G irányított gráf irányított lokális kromatikus száma a*

$$\psi_d(G) := \min_{c:\ V(G)\to\mathbb{N}} \max_{v\in V(G)} |\{c(w) : w \in \Gamma_G^+(v)\}$$

mennyiség, ahol a minimalizálást az összes c jó színezésre végezzük, vagyis olyanokra, amelyekben összekötött csúcsok színe nem lehet azonos, $\mathbb{N}$ a természetes számok halmaza, $\Gamma_G^+(v)$ pedig a v csúcs "zárt kiszomszédsága", vagyis $\Gamma_G^+(v) = \{u \in V(G) : (v, u) \in E(G) \text{ vagy } u = v\}$.

Nyilvánvaló, hogy egy irányítatlan gráf minden élét két ugyanazon csúcsok között vezető ellentétes irányítású élre cserélve az utóbbi fogalom az előbbit adja vissza.

2.3.1. Tétel.

$$\Sigma(G) \leq \log \psi_d(G).$$

(A dolgozatban e tétel kimondásakor a logaritmálás nélküli $\sigma(G) = 2^{\Sigma(G)}$ jelölést használjuk.)

A 2.3.1. Tétel azonnali következménye például, hogy egy irányított páratlan kör Sperner kapacitása csak úgy lehet nagyobb a triviális log 2 alsó korlátnál, ha részgráfként tartalmaz egy "alternáló" irányítású páratlan kört, vagyis egy olyat, melyben egy kivételével minden csúcsnak 0 vagy 2 a kifoka. Bohman és Holzman [24] 2003-ban megjelent, a bevezetőben már említett eredménye, hogy minden páratlan kör Shannon kapacitása nagyobb a triviális log 2-nél. Ha tehát a Shannon kapacitás értékét C_{2k+1} valamely irányított változatának Sperner kapacitása eléri, akkor ez az irányított változat csakis az alternáló módon irányított lehet. (Öt hosszú páratlan kör esetén szükségképpen ezt az irányítást szolgáltatta a 2.2.1. Tétel bizonyítása.)

A 2.3.1. Tételt irányítatlan (azaz ezzel egyenértékűen, szimmetrikusan irányított) gráfokra alkalmazva azt kapjuk, hogy $C(G) \leq \log \psi(G)$ igaz. Szemben az irányított esettel, ez itt nem jelent új korlátot. Jelölje ugyanis $\chi^*(G)$ ismét a G gráf frakcionális kromatikus számát. Jól ismert (ld. Shannon [139], Lovász [109]) és a bevezetőben (logaritmálás nélkül) láttuk is, hogy $C(G) \leq \log \chi^*(G)$, ugyanakkor azt is belátjuk, hogy a lokális kromatikus számra fennáll a következő.

2.3.7. Tétel.

$$\psi(G) \geq \chi^*(G).$$

A 2.3. Alfejezetben bevezetjük még $\psi_d(G)$ egy frakcionális változatát, ami a 2.3.1. Tétel erősítéséhez vezet, s elemezzük ennek néhány következményét.

Gráfok színezései

Az értekezés harmadik fejezete a Tardos Gáborral közös [144] és [145] cikkeken alapul, az alábbiakban a dolgozatból idézett valamennyi eredmény Tardos Gáborral közös.

Lokális színezés

Az előző fejezetben már szereplő lokális kromatikus számnak triviális felső korlátja a kromatikus szám. Erdős, Füredi, Hajnal, Komjáth, Rödl és Seress [47] belátták, hogy az eltérés tetszőlegesen nagy lehet: minden $k \geq 3$-hoz megadható olyan G gráf, amire $\psi(G) = 3$ és $\chi(G) \geq k$. Az előző fejezet végén láttuk, hogy a lokális kromatikus szám ugyanakkor nem lehet kisebb a gráf frakcionális kromatikus számánál. Ahogy a bevezetőben már említettük, ez indokolja, hogy a lokális kromatikus szám viselkedését olyan gráfokra vizsgáljuk, amelyekre e két korlát, a kromatikus szám és a frakcionális kromatikus szám távol esik egymástól.

Ilyen tulajdonságú gráfokra alapvető példák a Kneser gráfok és a Mycielski gráfok (ld. Scheinerman és Ullman [133] könyvét), valamint ezek különféle variánsai, például az ún. Schrijver gráfok és általánosított Mycielski gráfok. Ha $\chi(G)$ távol esik $\chi^*(G)$-től, akkor a kromatikus számnak nem lehet éles becslése az egyébként triviális $|V(G)|/\alpha(G)$ alsó korlát (ahol $\alpha(G)$ ismét a G gráf függetlenségi száma), mivel ez a mennyiség a frakcionális kromatikus számot is alulról becsli. Részben ez a magyarázata, hogy számos ilyen gráf kromatikus számának meghatározásához a megszokott kombinatorikus módszerek nem elegendőek.

A Kneser gráfok kromatikus számát Kneser 1955-ben leírt sejtését bizonyítva Lovász [108] határozta meg több, mint két évtizeddel később. A bizonyítás az algebrai topológia híres tételét, a Borsuk-Ulam tételt használta, s a topológiai kombinatorika elnevezésű terület egyik kiindulópontjává vált, (ld. de Longueville [43] jubileumi cikkét, Björner [21] összefoglalóját és Matoušek [116] már említett könyvét). Szintén a topologikus módszerrel igazolta Schrijver [134], hogy a Kneser gráfok róla elnevezett (csúcs-)színkritikus feszített részgráfjainak kromatikus száma megegyezik a megfelelő Kneser gráfok kromatikus számával.

Mycielski [123] konstrukciója tetszőleges (legalább egy élet tartalmazó) gráfból olyan másikat állít elő melynek klikkszáma változatlan, kromatikus száma pedig 1-gyel nagyobb az eredeti gráfénál. A kromatikus szám növekedése itt kombinatorikus érveléssel (is) igazolható. (Mycielski gráfoknak az egyetlen élből kiindulva a konstrukció iterált alkalmazásával kapható gráfokat szokás hívni.) Az általánosított Mycielski konstrukció ennek a konstrukciónak olyan módosítása, mely (egy triviális eset kivételével) szintén változatlanul hagyja a klikkszámot, a kromatikus számot pedig minden olyan esetben növeli 1-gyel, amikor a gráf eleget tesz egy bizonyos topologikus feltételnek. Ennek a ténynek szintén topológiai bizonyítása Stiebitz [149] eredménye (ld. még Gyárfás, Jensen, Stiebitz [72], Matoušek [116]).

A 3.1. Alfejezet alapjául szolgáló, Tardos Gáborral közös [144] cikkben azt kezdtük vizsgálni, hogy a topológiai módszer segítségével tudunk-e valamit mondani a fenti típusú gráfok lokális kromatikus számáról.

A Lovász-Kneser tétel bizonyításának mára számos (szintén topológiát használó, vagy legalábbis azon alapuló) variánsa ismert (ld. pl. Bárány [11], Dolnyikov [46], Greene [69]). Matoušek és Ziegler [118] cikke, valamint Matoušek [116] könyve Alon, Frankl, Lovász [5] és Kříž [101] munkái nyomán ún. *box komplexusok* bevezetésével hoz közös nevezőre sok rokon, de számos fontos részletben mégis különböző bizonyítást. E box komplexusok segítségével topologikus terek rendelhetők gráfokhoz, melyeknek egyes topológiai paraméterei alsó becslést szolgáltatnak a gráf kromatikus számára. Ennek részletes leírását valamint a box komplexusok definícióját ebben a rövid összefoglalóban terjedelmi okokból mellőzzük, mindez megtalálható a 3.1. Alfejezetben. Az eredmények kimondásához alkalmazzuk a 3.1. Alfejezet elején is szereplő konvenciót, miszerint *topologikusan t-kromatikus*nak mondunk egy G gráfot, ha egy bizonyos box komplexusának egy bizonyos paramétere (a $B_0(G)$-vel jelölt komplexus $\mathbb{Z}_2$-coindexe) olyan értéket vesz föl, hogy abból $\chi(G) \geq t$ következik. (E konvenció pontos jelentéséhez ld.

a 34. Definíciót a dolgozat 79. oldalán.) Megjegyezzük, hogy a t-kromatikus Kneser és Schrijver gráfok toplogikusan t-kromatikus gráfok. Szintén ilyenek azok az általánosított Mycielski konstrukcióval nyerhető gráfok, melyeknél az eredeti gráf, amire a konstrukciót alkalmazzuk, topologikusan $(t-1)$-kromatikus.

A Borsuk-Ulam tétel egy Ky Fan-tól származó általánosítását [52] használva a következő alsó becslést adjuk a lokális kromatikus számra.

3.1.1. Tétel. *Ha G topologikusan t-kromatikus gráf, akkor*

$$\psi(G) \geq \left\lceil \frac{t}{2} \right\rceil + 1.$$

Ez a tétel a következő, a dolgozatban Cikk-cakk tételnek nevezett, Ky Fan tételéből adódó általánosabb tétel közvetlen következménye, melynek Kneser gráfokra vonatkozó speciális esetét Ky Fan maga is igazolta [53].

Cikk-cakk tétel. *Legyen G topologikusan t-kromatikus gráf és c ennek tetszőleges jó színezése tetszőleges számú színnel, melyekről feltesszük, hogy lineárisan rendezettek. Ekkor G tartalmaz egy olyan $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ teljes páros részgráfot, melynek a c színezés szerint mind a t csúcsa különböző színű és ezek a színek természetes sorrendjükben felsorolva felváltva helyezkednek el a teljes páros gráf két oldalán.*

A 3.1.1. Tétel alsó becslése sok esetben pontos. Ilyen eredményeket megfelelő paraméterű Schrijver gráfokra, általánosított Mycielski gráfokra és ún. Borsuk gráfokra bizonyítunk. Itt példaként a Schrijver gráfok esetét részletezzük, ehhez először megadjuk pontos definíciójukat. Használni fogjuk az $[n] = \{1, \ldots, n\}$ jelölést.

Definíció. (Schrijver [134]) *Tetszőleges k és $n \geq 2k$ pozitív egészekhez az* $\mathrm{SG}(n,k)$ Schrijver gráf *csúcsainak és éleinek halmaza így adható meg:*

$$\begin{aligned} V(\mathrm{SG}(n,k)) &= \{A \subseteq [n] : |A| = k, \forall i\ \{i, i+1\} \nsubseteq A \ \text{ és } \ \{1, n\} \nsubseteq A\}, \\ E(\mathrm{SG}(n,k)) &= \{\{A, B\} : A \cap B = \emptyset\}. \end{aligned}$$

Megemlítjük, hogy a $\mathrm{KG}(n,k)$ Kneser gráf definíciója ettől annyiban tér el, hogy ott a csúcshalmaz $[n]$-nek minden k elemű részhalmazát tartalmazza.

Schrijver [134] tétele szerint $\chi(\mathrm{SG}(n,k)) = n - 2k + 2$ és bármely csúcs elhagyása esetén a kromatikus szám csökken.

A Schrijver gráfok lokális kromatikus számára vonatkozik a következő eredmény.

3.1.3. Tétel. *Ha $t = n - 2k + 2 > 2$ páratlan és $n \geq 4t^2 - 7t$, akkor*

$$\psi(\mathrm{SG}(n,k)) = \left\lceil \frac{t}{2} \right\rceil + 1.$$

A tételbeli egyenlőséghez az alsó becslést a 3.1.1. Tétel és az a tény szolgáltatja, hogy a Schrijver gráfok toplogikusan t-kromatikusak a kromatikus számukkal egyenlő t-re. A felső becslés bizonyítása kombinatorikus módszerrel történik. Ennek fő ötlete, hogy a gráfot úgy színezzük kromatikus számának megfelelő számú színnel, hogy mindazon csúcsok, amelyek túl sok (az összes felénél több) színt látnak, együttesen független szomszédsággal rendelkezzenek. Ekkor az összes ilyen szomszédság alkotta független halmaz kiszínezhető egyetlen új színnel, és ezzel az egy csúcs által látható színek maximális száma körülbelül a felére csökken. (Akkor mondjuk, hogy egy csúcs "lát" egy színt, ha az szerepel a szomszédainak színei között.)

Bizonyos, az előbbi feltételt teljesítő (az új szín bevezetése előtti) színezéseket *széles színezés*eknek hívunk. Nem minden G gráfnak van széles színezése $\chi(G)$ színnel. A Kneser gráfoknak például nincs, a Schrijver gráfoknak viszont a fenti tételben előírt paraméterek esetén van, és ez adja a felső korlátot.

Mivel $\mathrm{SG}(n,k)$ feszített részgráfja $\mathrm{SG}(n+1,k)$-nak, a 3.1.3. Tételből az is azonnal következik, hogy ha $t=n-2k+2$ rögzített páros szám és n,k kellően nagy, akkor

$$\psi(\mathrm{SG}(n,k)) \in \left\{\frac{t}{2}+1, \frac{t}{2}+2\right\}.$$

A megfelelő paraméterű általánosított Mycielski gráfoknak szintén megadható kromatikus számuknak megfelelő számú színt használó széles színezésük, így rájuk is a fentihez hasonló eredmény kapható (3.1.5. Tétel). Ez azt jelenti, hogy megfelelő gráfból kiindulva és megfelelő paraméterekkel iterálva az általánosított Mycielski konstrukciót, a lokális kromatikus szám iterációnként átlagosan 1/2-del nő. Ezzel kapcsolatban megemlítjük még, hogy a hagyományos Mycielski konstrukció viszont ugyanúgy 1-gyel növeli a lokális kromatikus számot, mint a kromatikus számot (Proposition 3.1.13).

A $B(d,\alpha)$ Borsuk gráf (ld. Erdős és Hajnal [48], Lovász [111]) az $\mathbb{S}^{d-1}$ egységgömb pontjain mint csúcsokon adott végtelen gráf, melyben két pont akkor van összekötve, ha távolságuk legalább α valamilyen rögzített $0<\alpha<2$ valós számra. A Borsuk-Ulam tételből következik, hogy $\chi(B(d,\alpha)) \geq d+1$, és egyenlőség áll, ha $\alpha<2$ elér egy bizonyos korlátot. A korábbiak alapján megmutatható, hogy a Borsuk gráfok lokális kromatikus száma is a már látottakhoz hasonlóan viselkedik.

3.1.20. Következmény. *Páros d esetén létezik olyan $\alpha_d<2$, hogy $\alpha_d<\alpha<2$ esetén*

$$\psi(B(d,\alpha)) = \frac{d}{2}+2.$$

Topológiai következmények

Az előbbi eredményeknek topológiai következményei is megfogalmazhatóak, melyek kapcsolatosak Micha Perles alábbi kérdésével. (A kérdést Matatyahu Rubin egy rokon kérdése motiválta, s azért, hogy mindez eljutott hozzánk, Bárány Imrét és Gil Kalai-t illeti köszönet.)

Definíció. *Legyen h nemnegatív egész, és jelölje $Q(h)$ azt a legkisebb ℓ számot, amire az $\mathbb{S}^h$ egységgömb lefedhető (tetszőleges számú) nyílt halmazzal úgy, hogy e halmazok egyike sem tartalmazza a gömbnek átellenes pontjait, és $\mathbb{S}^h$ egyetlen pontja sincs benne ℓ-nél több halmazban.*

Ky Fan tételéből $\frac{h}{2}+1 \le Q(h)$ következik. A lokális kromatikus számra adott felső becsléseinkből pedig adódik, hogy ez páratlan h esetén pontos, páros h esetén pedig majdnem pontos.

3.1.23. Következmény.

$$\frac{h}{2}+1 \le Q(h) \le \frac{h}{2}+2.$$

A fenti következményt erősebb formában is kimondhatjuk. Ehhez először kimondjuk Ky Fan tételét (egyik lehetséges formájában).

Ky Fan tétele. ([52]) *Legyen $\mathcal{A}$ az $\mathbb{S}^h$ egységgömb nyílt halmazainak (vagy zárt halmazainak véges) családja, melyre teljesül, hogy $\cup_{A\in\mathcal{A}}(A\cup(-A))=\mathbb{S}^h$. Tegyük fel, hogy $\mathcal{A}$ elemein adott egy "$<$" rendezés, továbbá, hogy $A\cap(-A)=\emptyset$ áll minden $A\in\mathcal{A}$ halmazra. Ekkor létezik $\boldsymbol{x}\in\mathbb{S}^h$ pont és $A_1<A_2<\ldots<A_{h+1}$ halmazok $\mathcal{A}$-ban, amikre $(-1)^i\boldsymbol{x}\in A_i$ teljesül minden $i=1,\ldots,h+1$ esetén.*

A lokális kromatikus számra vonatkozó felső korlátaink bizonyításából az alábbi állítás adódik, ami a Ky Fan tételében szereplő paraméterek optimalitásaként értelmezhető.

3.1.21. Következmény. *Megadható az $\mathbb{S}^h$ egységgömb nyílt (zárt) halmazainak olyan $h+2$ halmazból álló $\mathcal{A}$ családja, melyekre $\cup_{A\in\mathcal{A}}(A\cup(-A))=\mathbb{S}^h$, $A\cap(-A)=\emptyset$ teljesül minden $A\in\mathcal{A}$ halmazra, továbbá igaz, hogy egyetlen $\boldsymbol{x}\in\mathbb{S}^h$ pont sincsen benne $\lceil\frac{h+1}{2}\rceil$-nél több $A\in\mathcal{A}$ halmazban. Emellett még az is teljesül minden $\boldsymbol{x}\in\mathbb{S}^h$ pontra, hogy az $\boldsymbol{x}$-et vagy $-\boldsymbol{x}$-et tartalmazó $\mathcal{A}$-beli halmazok együttes száma legfeljebb $h+1$.*

Cirkuláris színezés

Egy G gráf Vince [157] által bevezetett $\chi_c(G)$ cirkuláris kromatikus száma a következő módon definiálható.

Valamely p,q pozitív egészekre egy gráf (p,q)-színezésén a csúcsok olyan $c: V(G)\to[p]$ színezését értjük, amire igaz, hogy ha u és v szomszédos csúcsok, akkor $q\le|c(u)-c(v)|\le p-q$. G cirkuláris kromatikus száma a következő mennyiség:

$$\chi_c(G)=\inf\left\{\frac{p}{q}: \text{létezik } G\text{−nek } (p,q)\text{-színezése}\right\}.$$

A cirkuláris kromatikus számra mindig fennáll, hogy $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, ezért szokás a kromatikus szám egyfajta finomításának tekinteni. A cirkuláris kromatikus számot az utóbbi időben nagyon sokat vizsgálták, ld. pl. Zhu [159] összefoglaló cikkét, mely idézi az alábbi két sejtést.

Sejtés. (Johnson, Holroyd, Stahl [81]) *A* $\mathrm{KG}(n,k)$ *Kneser gráfra minden* $n \geq 2k$ *esetén fennáll, hogy*

$$\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k)).$$

Johnson, Holroyd és Stahl [81] belátták a sejtést a $k = 2$, valamint az $n = 2k + 1$, $n = 2k + 2$ esetekre. Schrijver gráfok cirkuláris kromatikus számát is vizsgálta Lih és Liu [105] valamint Hajiabolhassan és Zhu [73]. Utóbbi szerzők [73]-ban megmutatták, hogy minden k-hoz létezik olyan $n_0(k)$ küszöb, hogy $n \geq n_0(k)$ esetén $\chi_c(\mathrm{SG}(n,k)) = \chi(\mathrm{SG}(n,k))$, amiből az ilyen esetekben $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$ is következik.

Sejtés. (Chang, Huang, Zhu [29]) *A* K_n *teljes gráfból a Mycielski konstrukció* d*-szeres alkalmazásával kapható* $M^d(K_n)$*-nel jelölt* $(n + d)$*-kromatikus gráfra* $n \geq d + 2$ *esetén fennáll*

$$\chi_c(M^d(K_n)) = \chi(M^d(K_n)).$$

Chang, Huang és Zhu [29] a $d = 1, 2$ esetre belátták a sejtést, valamint megmutatták, hogy ha $\chi(G) = d + 1$, akkor a G gráfból a Mycielski konstrukció d-szeres alkalmazásával kapható $M^d(G)$ gráfra $\chi_c(M^d(G)) \leq \chi(M^d(G)) - 1/2$ igaz. Szintén a Mycielski gráfok cirkuláris kromatikus számát vizsgálta Fan [51] és Hajiabolhassan és Zhu [74]. Az utóbbi cikkben azt mutatták meg, hogy $n \geq d + 2$ helyett $n \geq (2^d + 2)$-t írva már igaz a sejtés.

A Cikk-cakk tétel segítségével egyszerűen belátható a következő állítás, amely mindkét fenti sejtést igazolja azokban az esetekben, amikor a bennük szereplő gráf kromatikus száma páros. A Kneser és Schrijver gráfokra vonatkozó speciális esetet tőlünk függetlenül Frédéric Meunier [121] is bebizonyította.

3.1.6. Tétel. *Ha* G *topologikusan* t*-kromatikus gráf és* t *páros, akkor* $\chi_c(G) \geq t$.

3.1.24. Következmény. (ld. Meunier [121] is) *A Johnson-Holroyd-Stahl sejtés igaz minden páros* n*-re. Páros* n*-re az erősebb*

$$\chi_c(\mathrm{SG}(n,k)) = \chi(\mathrm{SG}(n,k))$$

egyenlőség is fönnáll.

3.1.25. Következmény. *Ha* $n + d$ *páros, akkor* $\chi_c(M^d(K_n)) = \chi(M^d(K_n))$.

Megjegyezzük, hogy a 3.1.25. Következményt a dolgozatban erősebb formában mondjuk ki: K_n helyén számos más gráf is állhat és a Mycielski konstrukció helyett az általánosított Mycielski konstrukció is alkalmazható.

Lam, Lin, Gu és Song [102] megadott egy pontos formulát olyan gráfok cirkuláris kromatikus számára, melyek egy teljes gráfból az általánosított Mycielski konstrukció egyszeri alkalmazásával állnak elő. Eredményüket felhasználva megmutattuk, hogy a 3.1.24. Következményben Schrijver gráfok esetén nem hagyható el a párossági feltétel.

3.1.27. Tétel. *Minden $\varepsilon > 0$ és $t \geq 3$ páratlan szám esetén, ha $t = n-2k+2$ és $n \geq t^3/\varepsilon$, akkor fennáll*

$$1 - \varepsilon < \chi(\mathrm{SG}(n,k)) - \chi_c(\mathrm{SG}(n,k)) < 1.$$

Kneser jellegű gráfok tarka részgráfjai

A lokális kromatikus számmal kapcsolatos eredményeink megmutatták, hogy egy topologikusan t-kromatikus gráf sok esetben kiszínezhető úgy $t+1$ színnel, hogy benne a Cikk-cakk tétel által előírt $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ részgráfok mellett ne forduljon elő más t csúcsú teljes páros részgráf, aminek minden csúcsa különböző színű. (Páratlan t esetén láttuk, hogy egy ilyen páros gráf mindkét oldalán legfeljebb $\lceil \frac{t}{2} \rceil$ csúcs lehet, különben a $\psi(G)$-re adott alsó becslés nem volna pontos. A felső becslést adó színezést kicsit közelebbről szemügyre véve adódik, hogy nem lehet mindkét oldalon ennyi csúcs.)

Ha nem használhatunk a kromatikus számnál több színt, akkor a helyzet drasztikusan megváltozik. Könnyen belátható, hogy tetszőleges t-kromatikus gráf t színnel való színezésében lesz (minden színosztályban) olyan csúcs, ami az összes sajátjától eltérő színt látja a szomszédságában. A Tardos Gáborral közös [145] dolgozaton alapuló 3.2. Alfejezetben megmutatjuk, hogy topologikusan t-kromatikus gráfokra jóval több is igaz.

Az eredmény kimondása előtt megemlítjük még Csorba, Lange, Schurr és Waßmer [42] tételét, ami azt mondja ki, hogy egy a topologikus t-kromatikusságnál valamivel enyhébb feltételt teljesítő gráf részgráfként tartalmaz minden olyan $K_{\ell,m}$ teljes páros gráfot, amire $\ell + m = t$. Olyan topologikusan t-kromatikus gráfok esetén, melyek kromatikus száma pontosan t, az alábbi tétel általánosítja ezt az eredményt.

3.2.2. Tétel. *Legyen G topologikusan t-kromatikus gráf, $\chi(G) = t$ és $c : V(G) \to [t]$ G-nek jó színezése. Legyen továbbá $A, B \subseteq [t]$ a színhalmaznak tetszőleges bipartíciója, vagyis $A \cup B = [t]$ és $A \cap B = \emptyset$.*

Ekkor van G-nek olyan $K_{\ell,m}$ teljes páros részgráfja, aminek minden csúcsa különböző színű, $\ell = |A|, m = |B|$, és az ℓ méretű oldalon az A-beli, az m méretű oldalon a B-beli színek szerepelnek.

A tétel bizonyításához a Borsuk-Ulam tételnek egy Tucker [153] és Bacon [10] nevéhez köthető általánosítását használjuk.

A 3.2.2. Tétel alkalmazható minden olyan gráfra, melyre a topologikus t-kromatikusságot definiáló topologikus paraméter éles becslést ad a kromatikus számra. Ilyenek a Kneser, a Schrijver, a(z általánosított) Mycielski, valamint a Borsuk gráfok. A fejezet eredményei révén ez a lista kibővíthető néhány olyan gráffal, melyekről a fenti eredmények éppen azt mutatták meg, hogy a most felsorolt gráfok valamelyike éltartóan (azaz homomorf módon) beleképezhető. A 3.2.4. Következményben felsoroljuk ezeket a gráfokat.

A 3.2. Alfejezetben egy további tételt is bizonyítunk, mely Greene [69] és Matoušek [116] ötleteit Ky Fan tételével kombinálva a Lovász-Kneser tételt általánosító Dolnyikov tételnek [46] adja további általánosítását.

Köszönetnyilvánítás

Sokaknak tartozom köszönettel, mert tanítottak, segítettek, figyelemmel követték a munkámat. Kandidátusi dolgozatom témavezetőjeként Körner János alapvetően alakította az érdeklődésemet. Számos tőle hallott szép probléma a mai napig meghatározó a munkámban. Sok-sok figyelmet és bátorítást köszönök Lovász Lászlónak, Simonovits Miklósnak és T. Sós Verának. Mindig bizalommal fordulhattam mások mellett Csiszár Imréhez, Győri Ervinhez, Katona Gyulához és Recski Andráshoz. Köszönöm Bárány Imre, Füredi Zoltán, Gyárfás András és Marton Katalin inspiráló érdeklődését egy-egy dolgozatom iránt. A közös munka élményét köszönöm minden társszerzőmnek, a még nem említettek közül külön is Sali Attilának és Tardos Gábornak. Végül köszönöm még számos név szerint nem említett kollégámnak azt a légkört, amiben mindig örömmel dolgozhattam.

Az értekezés egyes alfejezetei az alábbi dolgozatok felhasználásával készültek:

1.1. Alfejezet:
G. Simonyi, Entropy splitting hypergraphs, *J. Combin. Theory Ser. B*, **66** (1996), 310–323.

1.2. Alfejezet:
G. Simonyi, Perfect Graphs and Graph Entropy. An Updated Survey, Chapter 13 in: *Perfect Graphs* (J. L. Ramírez-Alfonsín, B. A. Reed eds.), John Wiley and Sons, 2001, 293–328.

(A felhasznált rész a 13.4.2 jelű, melynek alcíme "Imperfection ratio", valamint az ennek tárgyalásához szükséges bevezető részek.)

1.3. Alfejezet:
G. Simonyi, On Witsenhausen's zero-error rate for multiple sources, *IEEE Trans. Inform. Theory,* **49** (2003), 3258–3261.

2.1. Alfejezet:
A. Galluccio, L. Gargano, J. Körner, G. Simonyi, Different capacities of a digraph, *Graphs Combin.*, **10** (1994), 105–121.

2.2. Alfejezet:
A. Sali, G. Simonyi, Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities, *European J. Combin.*, **20** (1999), 93–99.

2.3. Alfejezet:
J. Körner, C. Pilotto, G. Simonyi, Local chromatic number and Sperner capacity, *J. Combin. Theory Ser. B*, **95** (2005), 101-117.

3.1. Alfejezet:
G. Simonyi, G. Tardos, Local chromatic number, Ky Fan's theorem, and circular colorings, *Combinatorica*, közlésre elfogadva, arXiv:math.CO/0407075.

3.2. Alfejezet:
G. Simonyi, G. Tardos, Colorful subgraphs in Kneser-like graphs, *European J. Combin.*, közlésre elfogadva, arxiv:math.CO/0512019.

Chapter 1

Graph Entropies

1.1 Entropy splitting hypergraphs

This section is based on the paper [140].

1.1.1 Introduction

Graph entropy $H(G, P)$ is an information theoretic functional on a graph G with a probability distribution P on its vertex set. It was introduced by Körner in [88]. A basic property of graph entropy, proved also by Körner [89], is its subadditivity under graph union. Let F and G be two graphs on the same vertex set V with edge sets $E(F)$ and $E(G)$, respectively, and $F \cup G$ is the graph on V with edge set $E(F) \cup E(G)$. Then for any fixed probability distribution P on V we have

$$H(F \cup G, P) \leq H(F, P) + H(G, P). \tag{1.1}$$

This inequality has become a useful tool for obtaining lower bounds in graph covering and complexity problems, for various applications, see e.g. Körner [89], Körner and Marton [94], Boppana [26], Newman, Ragde, and Wigderson [126], Radhakrishnan [127], and Kahn and Kim [83]. In [93] Körner and Marton introduced hypergraph entropy to improve upon the Fredman-Komlós bound of [54] generalizing its proof that relied on the subadditivity of graph entropy in [89]. This generalization was based on a similar inequality for hypergraphs. (For another application of hypergraph entropy, see Körner and Marton [95].)

Realizing the central role of inequality (1.1) the natural question of its sharpness arose. Conditions of equality were already asked for in a special case during the information theory investigations of Körner and Longo [91]. Similar questions were considered in [92], [38], and [99]. The results of these investigations showed that there are close connections between graph entropy and some classical concepts of combinatorics, e.g., perfect graphs.

One of the main questions in [91] was to characterize those graphs G that satisfy equality in (1.1) with $F = \bar{G}$ (where $\bar{G}$ stands for the complementary graph of G) and every P. It was conjectured in [92] and proved in [38] that these graphs are exactly the perfect graphs. (For perfect graphs cf. Lovász [106], [112].) In this paper we investigate conditions for the similar equality in case of complementary uniform hypergraphs.

1.1.2 Basic definitions

The usual notation, $V(G), E(G)$, for the vertex and edge set of a (hyper)graph G will be used throughout. Logarithm's are always meant to be binary. (This holds for this entire work, i.e., also for the later sections and chapters.)

Definition 1 *The* vertex packing polytope $VP(F)$ *of a hypergraph F is the convex hull of the characteristic vectors of the independent sets of F.*

We remark that an independent set of a hypergraph F is a subset of its vertex set $V(F)$ that contains no edge.

Definition 2 *Let F be a hypergraph on the vertex set $V(F) = \{1, ..., n\}$ and let $P = (p_1, ..., p_n)$ be a probability distribution on $V(F)$ (i.e., $p_1 + ... + p_n = 1$ and $p_i \geq 0$ for all i). The entropy of F with respect to P is then defined as*

$$H(F, P) = \min_{\boldsymbol{a} \in VP(F)} \sum_{i=1}^{n} p_i \log \frac{1}{a_i}. \tag{1.2}$$

Remark 1.1. The results in [88] provide two equivalent definitions for graph entropy. A third equivalent definition was given in [38]. This is the one we have adopted. (Körner and Marton [93] generalized one of the earlier definitions when they introduced hypergraph entropy. The proof of equivalence in [38], however, literally applies to the hypergraph case, too.) $\diamondsuit$

The union of two hypergraphs on the same vertex set V is a third hypergraph on V having as its edge set the union of the edge sets of the two original hypergraphs.

A hypergraph is k-uniform if all of its edges have size k. We denote the complete k-uniform hypergraph on n vertices by $K_n^{(k)}$. (Instead of $K_n^{(2)}$, however, we usually write simply K_n.) The complement of a k-uniform hypergraph F on n vertices is the k-uniform hypergraph $\bar{F}$ on the same vertex set that has a disjoint edge set from that of F and satisfies $F \cup \bar{F} = K_n^{(k)}$.

Considering graphs as 2-uniform hypergraphs, Definition 2 gives graph entropy as a special case. We remark that it is not difficult to see (cf. Lemma I.3.1 in [36]) from this definition that the entropy of the complete graph, K_n, equals the Shannon-entropy of the probability distribution involved:

$$H(K_n, P) = H(P) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i}.$$

For a somewhat more complicated formula to compute $H(K_n^{(k)}, P)$ for $k > 2$ see [44]. (The same formula was found independently by Gerards and Hochstättler [63], the statement of this result is quoted also in [141].)

In [93] Körner and Marton proved that hypergraph entropy is subadditive in general, i.e., (1.1) holds not only for graphs but also for hypergraphs F and G.

The following definition is from [91] generalized to hypergraphs.

Definition 3 *A k-uniform hypergraph F is* strongly splitting *if for every probability distribution P on $V(F) = V$, we have*

$$H(F, P) + H(\bar{F}, P) = H(K_{|V|}^{(k)}, P). \tag{1.3}$$

As we have already mentioned, it was conjectured in [92] and proved in [38] that a graph is strongly splitting if and only if it is perfect.

Our aim here is to characterize strongly splitting k-uniform hypergraphs for $k \geq 3$. The main results are Theorems 1.1.1 and 1.1.2 of the next subsection that give this characterization for $k = 3$ and its generalization involving more than two 3-uniform hypergraphs. It turns out that for $k > 3$ no strongly splitting hypergraph exists except the trivial ones, $K_n^{(k)}$ and its complement. This is shown in Subsection 1.1.4. Subsection 1.1.5 contains some further remarks.

1.1.3 Splitting 3-uniform hypergraphs

All hypergraphs in this subsection will be 3-uniform, so we will often omit the full description and write simply hypergraph. (Graphs, however, still mean 2-uniform hypergraphs.) To state our results on 3-uniform hypergraphs we need the following definition.

Definition 4 *Let T be a tree and let us be given a two-coloring of its internal vertices with two colors that we call 0 and 1. The* leaf-pattern *of the two-colored tree T is the following* 3*-uniform hypergraph F. The vertices of F are the leaves of T and three leaves x, y, z form an edge if and only if the unique common point of the paths joining pairs of x, y and z is colored by* 1.

A 3*-uniform hypergraph F is said to be a leaf-pattern if there exists a two-colored tree T such that F is the leaf-pattern of T.*

It is obvious that the degree two vertices of a tree will have no effect on its leaf-pattern, so when concerned about the leaf-pattern, we can always think about trees with no degree two vertices. In fact, if a 3-uniform hypergraph F is the leaf-pattern of some tree then there is a unique two-colored tree not containing degree two vertices and having a proper coloring (i.e., a coloring in which neighbouring nodes have different colors), for which F is its leaf-pattern. For example, $K_n^{(3)}$ is the leaf-pattern of a star on $n+1$ points having 1 as the color of the central point.

Strongly splitting 3-uniform hypergraphs are characterized by the following theorem.

Theorem 1.1.1 *A 3-uniform hypergraph is strongly splitting if and only if it is a leaf-pattern.*

For the proof we need some preparation. Motivated by game theoretic questions leaf-patterns were already investigated by Gurvich [70]. We will make use of his theorem characterizing leaf-patterns by forbidden subhypergraphs. To state this theorem we give a name to a particular configuration. Consider the five points $0,1,2,3,4$ and the five hyperedges of the form $\{i, i+1, i+2\}$ where $i = 0,1,2,3,4$ and addition is intended modulo 5. Notice that the hypergraph defined this way is isomorphic to its complement and let us call it *flower*. Gurvich's theorem is the following.

Theorem G ([70]) *A 3-uniform hypergraph is a leaf-pattern if and only if it has an even number of hyperedges on every four vertices and it does not contain an induced flower.*

Duplicating a vertex x of a hypergraph F means that a new vertex x' is added to $V(F)$ thereby creating a new hypergraph F' as follows. For any set of vertices $S \subseteq V(F)$ not containing x, the set $S \cup \{x'\}$ is an edge of F' if and only if $S \cup \{x\}$ is an edge of F. A set $S \subseteq V(F)$ itself forms an edge of F' if and only if it is an edge in F. Notice that no edge of the new hypergraph contains both x and x'.

Definition 5 *A uniform hypergraph is called* reducible *if it can be obtained from a single edge by successive use of the following two operations in an arbitrary order:*
(i) duplication of a vertex,
(ii) taking the complementary uniform hypergraph.

It is a more or less trivial observation that 3-uniform reducible hypergraphs are equivalent to leaf-patterns. This also implies that every subhypergraph of such a hypergraph is reducible.

Let us call two vertices siblings in a hypergraph F if they are duplicates of each other either in F or in $\bar{F}$. Observe that a leaf-pattern on at least 4 points always has two disjoint pairs of siblings. (Considering a longest path in the underlying tree T containing no degree two vertex the two ends of this path each have a sibling and these two pairs are disjoint or can be chosen so in case T is a star. This simple argument is due to one of the anonymous referees.)

Now we recall some consequences of already known results about graph entropy. As an immediate consequence of the definition of hypergraph entropy, notice that if $p_i > 0$

for all i, then the minimizing vector $\boldsymbol{a}$ in (1.2) is always a maximal vector of $VP(F)$. (We call a vector $\boldsymbol{b}$ maximal in some set of vectors, if this set does not contain a $\boldsymbol{b}' \neq \boldsymbol{b}$ with $b'_i \geq b_i$ for every i.) This also implies that all the independent sets of F that appear with positive coefficients in some convex combination representation of this minimizing $\boldsymbol{a}$ must be maximal. (Thus for an arbitrary distribution this is always so within the induced subgraph of vertices with positive probability.)

For a hypergraph F let us denote by $VP'(F)$ the set of all those vectors $\boldsymbol{a} \in VP(F)$ for which $(1+\varepsilon)\boldsymbol{a} \notin VP(F)$ for any $\varepsilon > 0$. In other words, $VP'(F)$ is the closure of that part of the boundary of $VP(F)$ that is not contained in any of the coordinate hyperplanes $x_i = 0$.

The following lemma is an immediate consequence of Corollary 7 in [38].

Lemma A *For every $\boldsymbol{a} \in VP'(F)$ there exists a probability distribution P such that $H(F,P) = -\sum_{i=1}^n p_i \log a_i$. Furthermore, if $\boldsymbol{a}$ is a maximal positive vector in $VP(F)$ then there is an everywhere positive such P and conversely, if no $p_i = 0$ for our P then $\boldsymbol{a}$ is the unique minimizing vector in the definition of $H(F,P)$.*

The following lemma is also not new (cf. [91], [92], [38] Corollary 10, [99] Lemma 3), but since it is easy, we give a short proof for the sake of clarity.

Lemma B *Let F and G be two hypergraphs, P an everywhere positive probability distribution and let $\boldsymbol{a} \in VP(F)$, $\boldsymbol{b} \in VP(G)$, $\boldsymbol{c} \in VP(F \cup G)$ be the vectors achieving $H(F,P)$, $H(G,P)$, and $H(F \cup G, P)$, respectively. Now, if $H(F,P) + H(G,P) = H(F \cup G, P)$ then necessarily $a_i b_i = c_i$ for every i. Furthermore, then any two independent sets appearing with positive coefficients in some convex combination representations of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, must intersect in a maximal independent set of $F \cup G$.*

Proof. Observe that the intersection of an independent set of F and an independent set of G is always an independent set of $F \cup G$. (In fact, subadditivity is a consequence of this observation, cf. [93].) This implies that the vector $(a_1b_1, ..., a_nb_n) \in VP(F \cup G)$. So, if $H(F,P) + H(G,P) = -\sum_{i=1}^n p_i \log(a_ib_i) = H(F \cup G, P)$, then this vector should be the minimizing vector defining $H(F \cup G, P)$. The statement about the intersection is then obvious by the remark that only maximal independent sets can appear with positive coefficients in the representation of a vector achieving entropy with respect to an everywhere positive P. $\square$

Now we are ready to prove Theorem 1.1.1.
For vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$ we will use the notation $\boldsymbol{a} \circ \boldsymbol{b} = (a_1b_1, a_2b_2, ..., a_nb_n)$.

Proof of Theorem 1.1.1.

First we prove that a strongly splitting hypergraph can contain neither four vertices inducing an odd number of edges nor an induced flower. To this end it is enough to show that the flower and also the 3-uniform hypergraphs with an odd number of edges on four vertices are not strongly splitting. This will imply that no strongly splitting hypergraph can contain these configurations. Indeed, otherwise we could concentrate a probability distribution violating (1.3) on this particular subhypergraph, all the entropy values would be the same as if the zero-probability vertices did not exist, and so (1.3) would be violated by the entire hypergraph. But if no strongly splitting hypergraph contains these subhypergraphs, then all strongly splitting hypergraphs are leaf-patterns by Theorem G, so this proves one direction of the theorem.

Consider the first pair of forbidden configurations, the 3-uniform hypergraph on four vertices with one edge and its complement that has three edges. Let us denote them by F and $\bar{F}$, respectively, and their four vertices by x, y, z, t, in such a way that the only edge of F is $\{x, y, z\}$. We will show that for no $\boldsymbol{a} \in VP(F)$ and $\boldsymbol{b} \in VP(\bar{F})$ can $a_i b_i = \frac{1}{2}, (i = 1, 2, 3, 4)$ be satisfied. Since $\boldsymbol{c} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in VP'(K_4^{(3)})$ is a maximal vector in $VP(K_4^{(3)})$, this implies the statement by Lemmas A and B. (In fact, instead of the above $\boldsymbol{c}$ we could consider any $\boldsymbol{c} \in VP'(K_4^{(3)})$ satisfying $0 < c_i < 1$ for every i.)

First observe that every maximal independent set of $\bar{F}$ containing t has only two elements, and for having $b_t > 0$ it is necessary for at least one of these sets to get a positive coefficient in the convex combination representation of $\boldsymbol{b}$. We may assume that the set $\{x, t\}$ gets a positive coefficient. However, this set is a maximal independent set of $K_4^{(3)}$, too, therefore by Lemma B, all maximal independent sets of F that will get a positive coefficient in the representation of $\boldsymbol{a}$ must contain $\{x, t\}$ completely. There are only two such maximal independent sets in F: $\{x, y, t\}$ and $\{x, z, t\}$. Both of these two sets should get a positive coefficient in the representation of $\boldsymbol{a}$ in order to have $a_y > 0$ and $a_z > 0$. Now going back to $\bar{F}$, apart from $\{x, t\}$, it has only one maximal independent set that intersects both of the previous two independent sets of F in a maximal independent set of $K_4^{(3)}$, this is $\{x, y, z\}$. So, again by Lemma B, apart from $\{x, t\}$ only this set can get a positive coefficient in the representation of $\boldsymbol{a}$. Now observe that all the above mentioned sets contain x, so whatever convex combination of them is taken, we will have $a_x = b_x = 1$, therefore $a_x b_x = \frac{1}{2}$ will not be satisfied. By Lemma B, this proves that the hypergraphs in our first pair of forbidden configurations are not strongly splitting.

For the flower a similar proof can be carried out. The following argument, however, is shorter. It was suggested by one of the referees. Let M denote a flower and let the vector $\boldsymbol{c} \in VP'(K_5^{(3)})$ we want to have in the form $\boldsymbol{c} = \boldsymbol{a} \circ \boldsymbol{b}$ with $\boldsymbol{a} \in VP(M)$, $\boldsymbol{b} \in VP(\bar{M})$, be $c_i = \frac{2}{5}, i = 1, .., 5$. Assume we have such an $\boldsymbol{a}$ and $\boldsymbol{b}$. Since the independence number of M is 3, we have $a_1 + ... + a_5 \leq 3$ and similarly for the b_i's. By the convexity of the function $\frac{1}{x}$ we can write

$$3 \geq \sum_{i=1}^{5} b_i = \frac{2}{5} \sum_{i=1}^{5} \frac{1}{a_i} \geq (\frac{2}{5}) 5 (\frac{5}{3}) = \frac{10}{3},$$

a contradiction. With Theorem G this concludes the proof of the first part of the theorem.

Now we have to prove that all leaf-patterns are strongly splitting. To this end we use the observation that leaf-patterns are equivalent to reducible 3-uniform hypergraphs.

We use induction on $n = |V(F)|$. For $n = 3$ the statement is trivial. We assume it is true for $n = m$ and prove it for $n = m + 1$. Consider a reducible hypergraph F on $m + 1$ vertices. Let us be given an arbitrary everywhere positive probability distribution P on the vertices of F and let $\boldsymbol{c} \in VP(K_{m+1}^{(3)})$ be the vector achieving $H(K_{m+1}^{(3)}, P)$. (In case of a not everywhere positive probability distribution we are done by the induction hypothesis.) Observe that $VP(K_n^{(3)}) = \{\boldsymbol{h} : 0 \leq h_i \leq 1, \sum_{i=1}^{n} h_i \leq 2\}$ and since $\boldsymbol{c}$ must be a maximal vector in $VP(K_{m+1}^{(3)})$ we surely have $\sum_{i=1}^{m+1} c_i = 2$.

We know there exist two disjoint pairs of siblings in F, let them be x, y and z, t. By $\sum_{i=1}^{m+1} c_i = 2$ we have that at least one of the two inequalities, $c_x + c_y \leq 1$ and $c_z + c_t \leq 1$, holds. (Note that this need not be true for $k > 3$.) We may assume that the first one is valid and label the vertices so that $x = 1$ and $y = 2$. Then we have

$$\boldsymbol{c}' = (c_1 + c_2, c_3, c_4, ..., c_m, c_{m+1}) \in VP'(K_m^{(3)}),$$

and $\boldsymbol{c}'$ is a maximal vector in $VP(K_m^{(3)})$, thus by Lemma A there exists a nowhere vanishing probability distribution P' for which $H(K_m^{(3)}, P')$ is achieved by $\boldsymbol{c}'$. Now consider the hypergraph on m vertices that we obtain by identifying the vertices x and y (i.e., 1 and 2) of F in the obvious manner. The new vertex will be denoted by x', and the hypergraph obtained this way we denote by F'. By the induction hypothesis, F' is strongly splitting, in particular, we have

$$H(F', P') + H(\bar{F}', P') = H(K_m^{(3)}, P').$$

By Lemma B this means that the vectors $\boldsymbol{a}'$ and $\boldsymbol{b}'$ achieving $H(F', P')$ and $H(\bar{F}', P')$, respectively, satisfy $\boldsymbol{a}' \circ \boldsymbol{b}' = \boldsymbol{c}'$. Now we obtain an $\boldsymbol{a} \in VP(F)$ and a $\boldsymbol{b} \in VP(\bar{F})$ from $\boldsymbol{a}'$, $\boldsymbol{b}'$, respectively, that will satisfy $\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{c}$. To this end we assume that 1 and 2 (the former x and y) are duplicates in F, otherwise we could change notation and consider $\bar{F}$. Look at the maximal independent sets of F' and $\bar{F}'$ that appear with positive coefficients in some representations of $\boldsymbol{a}'$ and $\boldsymbol{b}'$, respectively. Let the coefficient of the independent set I of F' be $\alpha'(I)$ in the representation of $\boldsymbol{a}'$. For $x' \notin I$ let $\alpha(I) = \alpha'(I)$ and for $x' \in I$ let $\alpha((I \setminus \{x'\}) \cup \{x, y\}) = \alpha'(I)$. The coefficient of an independent set J of $\bar{F}'$ we denote by $\beta'(J)$. For $x' \notin J$ we let $\beta(J) = \beta'(J)$ while for $x' \in J$ we let $\beta((J \setminus \{x'\}) \cup \{x\}) = \beta'(J)\frac{c_1}{c_1+c_2}$ and $\beta((J \setminus \{x'\}) \cup \{y\}) = \beta'(J)\frac{c_2}{c_1+c_2}$. It is easy to check that this way we gave coefficients to independent sets of F and $\bar{F}$, and that the $\boldsymbol{a} \in VP(F)$ and $\boldsymbol{b} \in VP(\bar{F})$ they represent are:

$$\boldsymbol{a} = (a_1', a_1', a_3', a_4', ..., a_m', a_{m+1}')$$

and

$$\boldsymbol{b} = (b_1' \frac{c_1}{c_1 + c_2}, b_1' \frac{c_2}{c_1 + c_2}, b_3', b_4', ..., b_m', b_{m+1}').$$

Using $a_i' b_i' = c_i'$ this immediately gives $a_i b_i = c_i$ for every i and so

$$H(K_{m+1}^{(3)}, P) = -\sum_{i=1}^{m+1} p_i \log c_i = -\sum_{i=1}^{m+1} p_i \log a_i - \sum_{i=1}^{m+1} p_i \log b_i \geq H(F, P) + H(\bar{F}, P).$$

Together with the subadditivity of hypergraph entropy this implies equality above and so F is strongly splitting. $\square$

Remark 1.2. We note that the second part of the above proof does make use of the fact that we are in case $k = 3$, that is, though it may sound plausible, it is not proven, moreover, it is not true in general that vertex duplication keeps the splitting property of a uniform hypergraph. If this were true then all reducible uniform hypergraphs were strongly splitting contradicting Theorem 1.1.3 of the next section. (In case of $k = 2$ the analogous statement is true and follows from the easy fact that vertex duplication preserves perfectness of a graph. The more subtle fact that it also preserves perfectness of the complement is a key lemma in Lovász's paper [106].) $\lozenge$

In [70] Gurvich has proved his Theorem G in a somewhat more general setting. This we can use to obtain a generalization of Theorem 1.1.1. First a generalization of the concept of leaf-pattern is needed.

Definition 6 *Let T be a tree with its inner nodes colored by colors* 1, 2,..., r. *The* leaf-factorization *of the r-colored tree T is a collection $\{F_1, F_2, ..., F_r\}$ of 3-uniform hypergraphs with the following properties. The vertex set of F_i ($i = 1, .., r$) is the set of leaves of T and three leaves x, y, z form an edge in F_i if and only if the unique common point of the paths xy, yz, and zx is colored with color i in T.*

The collection of hypergraphs $\{F_1, ..., F_r\}$ is called a leaf-factorization if it is the leaf-factorization of some r-colored tree T.

The general result of Gurvich is the following.

Theorem GG *A collection $\{F_1, ..., F_r\}$ of 3-uniform hypergraphs is a leaf-factorization if and only if no F_i contains an induced flower or an odd number of vertices on any four points.*

Using this result we have

Theorem 1.1.2 *Let $F_1, ..., F_r$ be 3-uniform hypergraphs on a common vertex set V and their union be the complete 3-uniform hypergraph on V. Then having*

$$\sum_{i=1}^{r} H(F_i, P) = H(K^{(3)}_{|V|}, P)$$

for every distribution P on V is equivalent to $\{F_1, ..., F_r\}$ forming a leaf-factorization.

Proof. By subaditivity and Theorem 1.1.1 the equality in the statement implies that the F_i's are edge-disjoint and every F_i is a leaf-pattern, i.e., none of them contains the forbidden configurations. Then by Theorem GG they form a leaf-factorization. All we have to show is that leaf-factorizations satisfy the above equality. This goes by a similar induction as that in the second part of the proof of Theorem 1.1.1.

Let $\{F_1, ..., F_r\}$ be the leaf-factorization of the r-colored tree T. Since F_1 is a leaf-pattern it has two disjoint pairs of siblings. Let one such pair be x and y with the additional property that $c_x + c_y \leq 1$ where $(c_1, c_2, ..., c_{|V|})$ denotes the vector in $VP(K_{|V|}^{(3)})$ that gives $H(K_{|V|}^{(3)}, P)$ for some arbitrarily fixed P. Now observe that x and y are siblings in all F_i's, moreover, they are duplicates in each F_i except one, F_j, say. (This is because, if we exclude degree 2 vertices in T, that we can, then x and y must be two leaves of T with a common neighbour that is colored by j.) After this observation we can more or less literally repeat the corresponding part of the proof of Theorem 1.1.1 with F_j playing the role of $\bar{F}$ there. □

Remark 1.3. Theorem 1.1.2 is the analogue of Corollary 1 in [99] which states that if $\{G_1, ..., G_r\}$ is a collection of edge disjoint graphs with their union being the complete graph on their common vertex set, then

$$\sum_{i=1}^{r} H(G_i, P) = H(P)$$

for every P is equivalent to all G_i's being perfect and no triangle having its three edges in three different G_i's. It is interesting to note that while all G_i's being strongly splitting (i.e., perfect) is not enough for the above equality, all F_i's being strongly splitting is sufficient for the analogous equality in the 3-uniform case. ◊

1.1.4 The case $k \geq 4$

In this subsection we show that for $k > 3$ the only strongly splitting k-uniform hypergraphs are the two trivial ones.

Theorem 1.1.3 *If $k \geq 4$ and F is a strongly splitting k-uniform hypergraph on n vertices then $F = K_n^{(k)}$ or $F = \bar{K}_n^{(k)}$.*

Proof. It is enough to prove the above statement for $n = k + 1$. This is because being strongly splitting is a hereditary property and a k-uniform hypergraph which is complete or empty on every $k + 1$ vertices must be complete or empty itself. (The fact that being strongly splitting is hereditary follows from the argument that a probability distribution can be concentrated on any subset of the vertex set and then the entropy values are just the same as if the zero-probability vertices did not exist.) The proof for $n = k + 1$ will use similar arguments as the beginning of the proof of Theorem 1.1.1.

Consider a k-uniform hypergraph F with $k + 1$ vertices and m edges. Up to isomorphism, there is only one such hypergraph. Thus we may assume $E(F) = \{\{1, \ldots, k+1\} \setminus \{i\} : i = 1, \ldots, m\}$. Its complement $\bar{F}$ has $k + 1 - m$ edges. The maximal independent sets of F ($\bar{F}$) are the edges of $\bar{F}$ (F) and those $(k-1)$-element sets that are not contained in the former independent sets.

Like in the proof of Theorem 1.1.1 our setting is this. We consider an arbitrarily given everywhere positive probability distribution P. This singles out a vector $\boldsymbol{c} \in VP(K_{k+1}^{(k)})$ that achieves the entropy of $K_{k+1}^{(k)}$ with respect to P. Now we look for an $\boldsymbol{a} \in VP(F)$ and a $\boldsymbol{b} \in VP(\bar{F})$ giving $\boldsymbol{a} \circ \boldsymbol{b} = \boldsymbol{c}$, and thereby additivity of hypergraph entropy for the given P. We will investigate which independent sets of F and $\bar{F}$ may have positive coefficients in the convex combination representations of $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively. It will follow that not every maximal $\boldsymbol{c}$ in $VP(K_{k+1}^{(k)})$ can be represented this way if neither F nor $\bar{F}$ is complete, and then by Lemmas A and B the theorem follows.

So our next task is to choose a maximal positive $\boldsymbol{c}$ in $VP(K_{k+1}^{(k)})$ that we will not be able to obtain in the required form. By Lemma A this is enough, since then a corresponding P exists for which $\boldsymbol{c}$ is the unique minimizing vector achieving $H(K_{k+1}^{(k)}, P)$. Let this $\boldsymbol{c}$ be such that $0 < c_i < 1$ for every i, and furthermore, none of $\sum_{i=1}^{m}(1-c_i) = 1$ and $\sum_{i=m+1}^{k+1}(1-c_i) = 1$ holds. (In fact, the latter two are equivalent, since $\sum_{i=1}^{k+1} c_i = k-1$ for every maximal $\boldsymbol{c}$ in $VP(K_{k+1}^{(k)})$.) It is easy to check that such a maximal $\boldsymbol{c}$ in $VP(K_{k+1}^{(k)})$ always exists. We show it cannot be represented as $\boldsymbol{a} \circ \boldsymbol{b}$ with $\boldsymbol{a} \in VP(F)$, $\boldsymbol{b} \in VP(\bar{F})$.

Assume the contrary. First observe that it cannot happen that in the representations of both, $\boldsymbol{a}$ and $\boldsymbol{b}$, some $(k-1)$-element independent set occurs with positive coefficient, because (since these sets could not be identical) the intersection of such two sets would not be a maximal independent set of $K_{k+1}^{(k)}$, thereby contradicting Lemma B. We distinguish between two cases: either there is at least one $(k-1)$-element set with positive coefficient in the representation of, say, $\boldsymbol{a}$, or no $(k-1)$-element set appears with positive coefficient at all.

In the second case, for every vertex i there is at most one independent set with positive coefficient not containing i. This implies that for every i this unique independent set must get coefficient $(1-c_i)$. We get convex combinations this way only if $\sum_{i=1}^{m}(1-c_i) = 1$ and $\sum_{i=m+1}^{k+1}(1-c_i) = 1$. But this is not satisfied by the $\boldsymbol{c}$ we have chosen.

In the first case, only those two maximal independent sets may have positive coefficients in the representation of $\boldsymbol{b}$ that contain the $(k-1)$-element set appearing in the representation of $\boldsymbol{a}$. (This is again by Lemma B.) Since we must have $b_i > 0$ for every i, these two independent sets must really get positive coefficients there. This implies that only one $(k-1)$-element set can get positive coefficient in the representation of $\boldsymbol{a}$ (again, by Lemma B). Now observe that this way there are $m-2$ points that will be contained in all the independent sets that may appear in the representations of $\boldsymbol{a}$ or $\boldsymbol{b}$ with positive coefficient. For all such points i we will have $a_i b_i = 1$, a contradiction, unless we have $m \le 2$.

If $m = 2$, then again, the coefficients of the k-element sets appearing in the representation of $\boldsymbol{a}$ are determined. Since the set missing element i is the only set that does not contain i, its coefficient must be $1 - c_i$. Labelling the vertices in such a way that 1 and 2 are the two vertices missed by our unique $(k-1)$-element set in the representation of $\boldsymbol{a}$, the previous observation implies $\sum_{i=3}^{k+1}(1-c_i) \le 1$. We may assume, however, that c_1 and c_2 are just the two largest coordinates of $\boldsymbol{c}$, implying $c_1 + c_2 \ge \frac{2(k-1)}{k+1}$, i.e., $\sum_{i=3}^{k+1}(1-c_i) = ((k-1) - (k-1-(c_1+c_2)) \ge \frac{2(k-1)}{k+1}$. But $\frac{2(k-1)}{k+1} \le 1$ implies $k \le 3$.

It is already implicit in the above argument that $m \neq 1$. Indeed, if $m = 1$, then there is a vertex which is not contained in any independent set of $\bar{F}$ that is larger than $k-1$. Since some independent set of $\bar{F}$ containing this vertex must get positive coefficient, there must be a $k-1$-element independent set with positive coefficient in the representation of $\boldsymbol{b}$. But we assumed we have a $k-1$-element independent set with positive coefficient in the representation of $\boldsymbol{a}$. Since the latter two have too small intersection, we have arrived to a contradiction.

The proof is complete now. □

Theorem 2 of [38] together with our Theorems 1.1.1 and 1.1.3 implies the following

Corollary 1.1.4 *A k-uniform hypergraph F is strongly splitting if and only if (at least) one of the following three statements holds:*
(i) $k = 2$ and F is a perfect graph
(ii) $k = 3$ and F is a leaf-pattern
(iii) F is $K_n^{(k)}$ or $\bar{K}_n^{(k)}$.

□

1.1.5 Connections with cographs

Cographs are defined as those graphs one can obtain starting from a single vertex by successively and iteratively using two operations: taking the complement and taking vertex disjoint union. (For their algorithmic importance, history, and other details, cf. [34].) By a theorem of Corneil, Lerchs, and Stewart Burlingham [34] cographs are identical to reducible graphs (i.e., reducible 2-uniform hypergraphs) in the sense of Definition 5. In fact, Corneil, Lerchs and Stewart Burlingham [34] show the equivalence of eight different characterizations of cographs, relying also on earlier results by Jung [82], Lerchs [104], Seinsche [138], and Sumner [150]. (Related results can also be found in [70], cf. also [85]). Among others, this theorem shows that cographs also admit a characterization by excluded configurations. In fact, they are equivalent to P_4-free graphs, i.e., graphs that have no induced subgraph isomorphic to a chordless path on 4 vertices.

The definition of reducible hypergraphs gives a natural (although not necessarily unique) way to describe the evolution of such a hypergraph. We obtain this description by simply ordering the vertices, telling for each vertex which preceding vertex it was originally a duplicate of and saying at which steps we should complement the hypergraph we have at hand. Since this means that after having fixed the first three vertices, the same description can describe a cograph and also a 3-uniform reducible hypergraph, it is natural that some correspondence can be found between them more directly. This is really easy to find.

Proposition 1.1.5 *A 3-uniform hypergraph F is reducible if and only if there exists a cograph G on $V(F)$ such that in each edge of F the number of edges of G has the same parity.*

□

The proof is straightforward and left to the reader.

Quoting results of Seidel [137], Hayward [76] defines the IP_3-structure of a graph G. This is the 3-uniform hypergraph on $V(G)$ the edges of which are exactly those triples of vertices that induce an even number of edges in G. (Thus every set of 3 vertices induces an independent set or a path P_3 explaining the term IP_3. It is shown (cf. [137], [76]) that the IP_3-structures of graphs are exactly those 3-uniform hypergraphs that on every four vertices have an even number of edges.) Using this terminology and the fact that the complement of a cograph is also a cograph, the previous proposition says that leaf-patterns (reducible 3-uniform hypergraphs) are equivalent to the IP_3 structures that arise from cographs. For further details on the related topic of "Seidel's switching" cf. also [100].

Finally, it is interesting to note, that since all cographs are perfect (cf. Lovász [106], Seinsche [138]), Corollary 1.1.4 together with the above proposition shows a kind of "monotonicity" as we consider strongly splitting graphs, strongly splitting 3-uniform hypergraphs and then strongly splitting k-uniform hypergraphs with $k > 3$.

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1.2 Imperfection ratio and graph entropy

In this section we present a result that is published as a new result in [142]. It is an extension of the characterization of perfectness by graph entropy proven in [38] (for the exact statement cf. Corollary 1.1.4 (i)) and it was this result that triggered the writing of the updated survey article [142] and its publication in the book [128].

Apart from some of the concepts already introduced in Section 1.1 the presentation of this result needs some further notions and facts that are discussed in earlier parts of [142]. Below we give a short summary of these preliminaries.

1.2.1 Some preliminaries

Recall that the entropy of a graph G with respect to a probability distribution P on its vertex set can be given by the formula

$$H(G,P) = \min_{\boldsymbol{a}\in VP(G)} \sum_{i\in V(G)} p_i \log \frac{1}{a_i} \tag{1.4}$$

which is just Definition 2 specified to the case when our hypergraph happens to be a graph.

It is immediate that Definition 2 can be generalized for more general polytopes, or in what follows, to *convex corners*.

Definition 7 *A set $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$ is called a* convex corner *if it is closed, convex, has a non-empty interior, and satisfies the property that if $0 \le a'_i \le a_i$ for $i = 1,..,n$ then $\boldsymbol{a} \in \mathcal{A}$ implies $\boldsymbol{a}' \in \mathcal{A}$.*

Definition 8 ([38]) *For a convex corner $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$ and probability distribution $P = (p_1, \ldots, p_n)$ the entropy of $\mathcal{A}$ with respect to P is defined as*

$$H_{\mathcal{A}}(P) = \min_{\boldsymbol{a}\in\mathcal{A}} \sum_{i=1}^{n} p_i \log \frac{1}{a_i}.$$

The proof of the result in [38] stating that a graph is strongly splitting if and only if it is perfect is based on results related to the notion of antiblocker of a convex corner that we also need later.

Definition 9 (Fulkerson [55]) *Let $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$ be a convex corner. The* antiblocker $\mathcal{A}^*$ *of $\mathcal{A}$ is defined as*

$$\mathcal{A}^* = \{\boldsymbol{b} \in \mathbb{R}^n_{+,0} : \boldsymbol{b}^T \cdot \boldsymbol{a} \le 1 \ \forall \boldsymbol{a} \in \mathcal{A}\}.$$

Remark 1.4. It is a well-known fact (cf. Fulkerson [55]) that $(\mathcal{A}^*)^* = \mathcal{A}$. If $B = \mathcal{A}^*$ then $(\mathcal{A}, \mathcal{B})$ is called an *antiblocking pair.* $\Diamond$

In order to characterize strongly splitting graphs the following more general additivity result was proven in [38].

Theorem 1.2.1 *For a convex corner $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$*

$$H_{\mathcal{A}}(P) + H_{\mathcal{A}^*}(P) = H(P)$$

holds for every probability distribution P.

The mentioned characterization immediately follows from Theorem 1.2.1 using a well-known result of Fulkerson [56] and Chvátal [32]. To state the latter we have to define another polytope associated with a graph. (In what follows we often write *stable set* in place of *independent set.* The two terms are equivalent.)

Definition 10 *The* fractional vertex packing polytope $FVP(G)$ *of a graph G on n vertices is the antiblocker of $VP(\bar{G})$, i.e.,*

$$FVP(G) = \{\boldsymbol{b} \in \mathbb{R}^n_{+,0} : \sum_{i \in B \in S(\bar{G})} b_i \leq 1 \quad \forall B \in S(\bar{G})\},$$

where $S(F)$ denotes the set of stable sets of graph F.

Since a clique and a stable set can have at most one vertex in common $VP(G) \subseteq FVP(G)$ holds for every graph.

Theorem 1.2.2 (Fulkerson [56], Chvátal [32]) $VP(G) = FVP(G)$ *if and only if G is a perfect graph.*

We will also need another additivity result proved in [99].

The notion of substitution is defined in [106] by Lovász in relation with the proof of the perfect graph theorem. The definition is as follows. Let F and G be two vertex disjoint graphs and v be a vertex of G. By substituting F for v we mean deleting v and joining every vertex of F to those vertices of G which have been adjacent with v. We will denote the resulting graph by $G_{v \leftarrow F}$.

We extend the above concept also to distributions. If we are given a probability distribution P on $V(G)$ and a probability distribution Q on $V(F)$ then by $P_{v \leftarrow Q}$ we denote the distribution on $V(G_{v \leftarrow F})$ given by $P_{v \leftarrow Q}(x) = P(x)$ if $x \in V(G) - \{v\}$ and $P_{v \leftarrow Q}(x) = P(v)Q(x)$ if $x \in V(F)$.

Now we are ready to state

Lemma 1.2.3 *(Substitution Lemma) Let F and G be two vertex disjoint graphs, v a vertex of G, while P and Q are probability distributions on $V(G)$ and $V(F)$, respectively. Then we have*

$$H(G_{v\leftarrow F}, P_{v\leftarrow Q}) = H(G, P) + P(v)H(F, Q).$$

(We remark that the lemma called "Substitution Lemma" in [99] is formulated in a somewhat different way. Still, its proof together with the trivial "Contraction Lemma" of the same paper immediately gives the proof of our lemma above. The above formulation appears in [141] and [142].)

To conclude this preliminary subsection we mention one more property of graph entropy we will use. (We write minimum instead of infimum in the following definition as the minimum is known to exist, cf. e.g. [133].)

Definition 11 *The* fractional chromatic number $\chi^*(G)$ *of graph G is defined as the minimum sum of non-negative weights on the stable sets of G satisfying that for any vertex the sum of weights of those stable sets that contain this vertex is at least* 1.

The following statement follows easily from the original definition of graph entropy given in [88] and a theorem of McEliece and Posner [120] (see also as Problem 13.51 in [110].) It is also easy to deduce it from Definition 2 that we do below for the sake of completeness. Recall Lemma A from Subsection 1.1.3 that was proven in [38] and states that for any vector $\boldsymbol{a} \in VP(G)$ that belongs to $VP'(G)$ which is the closure of that part of the boundary of $VP(G)$ that is not contained in any of the coordinate hyperplanes $x_i = 0$, there is some probability distribution P on $V(G)$ such that the value of $H(G, P)$ is attained by $\boldsymbol{a}$.

Lemma 1.2.4

$$\max_P H(G, P) = \log \chi^*(G).$$

Proof. It is easy to verify that $\boldsymbol{h} := (\frac{1}{\chi^*(G)}, \frac{1}{\chi^*(G)}, \dots, \frac{1}{\chi^*(G)}) \in VP(G)$ holds for any graph G implying that the right hand side of (1.4) cannot be larger than $\log \chi^*(G)$ for any P.
It follows from the definition of the fractional chromatic number that $\boldsymbol{h} \in VP'(G)$ and thus by Lemma A there exists a probability distribution P for which $H(G, P) = \log \chi^*(G)$ proving the statement. $\square$

1.2.2 An entropy formula for the imperfection ratio

In a recent paper Gerke and McDiarmid [65] introduced a parameter called the imperfection ratio of graphs (cf. also [64], [66], [119]). Its definition is motivated by so-called radio channel assignment problems, the relation is explained in detail in [119]. After repeating the original definition and two possible characterizations of this new notion we will show that it can also be characterized in terms of graph entropy.

The definition of the imperfection ratio needs the notion of substitution in the sense of [106], cf. its definition before Lemma 1.2.3. For an integer vector $\boldsymbol{x} = (x_v)_{v \in V(G)}$ of non-negative entries let $G_{\boldsymbol{x}}$ denote the graph obtained from G by substituting a clique of size x_v into G at its vertex v for all $v \in V(G)$. (Vertices v with $x_v = 0$ are simply deleted.) The imperfection ratio is then defined as

$$\mathrm{imp}(G) = \max\{\frac{\chi^*(G_{\boldsymbol{x}})}{\omega(G_{\boldsymbol{x}})} : \ \boldsymbol{x} \in \{0, 1, \ldots\}^V, \ \boldsymbol{x} \neq \mathbf{0}\},$$

where $\omega(F)$ stands for the clique number of graph F. It is noted in [119] that the above maximum is always attained.

One of the main results in [119] contains the following statements we will use later.

Theorem 1.2.5 ([119]) *For every graph G the following holds:*

$$\mathrm{imp}(G) = \min\{t : FVP(G) \subseteq t \cdot VP(G)\} = \max\{\boldsymbol{x} \cdot \boldsymbol{y} : \boldsymbol{x} \in FVP(G), \boldsymbol{y} \in FVP(\bar{G})\}.$$

(The product $\boldsymbol{x} \cdot \boldsymbol{y}$ above simply means the scalar product of the two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. There is a slight abuse of the notation here since it does not show that one of the vectors should be transposed before the multiplication is carried out. We will use this notation similarly later on.)

It is worth recalling that the above theorem immediately implies the following results in [119].

1. A graph G is perfect if and only if $\mathrm{imp}(G) = 1$. (In view of Theorem 1.2.5 this is equivalent to Theorem 1.2.2. Observe that $\mathrm{imp}(G) \geq 1$ for any graph.)

2. Complementary graphs have the same imperfection ratio, i.e., $\mathrm{imp}(\bar{G}) = \mathrm{imp}(G)$.

The first statement above can be interpreted by saying that the larger $\mathrm{imp}(G)$ the more imperfect G is. The characterization of perfectness by graph entropy proven in [38] (cf. Corollary 1.1.4 (i)) suggests that the value

$$\max_P \{H(G, P) + H(\bar{G}, P) - H(P)\}$$

is also a possible "measure" of imperfectness of graph G. It turns out that this measure is essentially the same as $\mathrm{imp}(G)$. More precisely, the following is true.

Theorem 1.2.6

$$\max_P \{H(G, P) + H(\bar{G}, P) - H(P)\} = \log \mathrm{imp}(G)$$

for all graphs G.

In the proof we will refer to the following observation.

Lemma 1.2.7 *For any non-negative integer vector $\boldsymbol{x} \in \mathbb{R}^{|V(G)|}$ one has*

$$\max_P\{H(G,P) + H(\bar{G},P) - H(P)\} \geq \max_Q\{H(G_{\boldsymbol{x}},Q) + H(\overline{G_{\boldsymbol{x}}},Q) - H(Q)\}$$

with equality if $\boldsymbol{x}$ is non-zero at those points where the maximizing P of the left-hand side is non-vanishing.

Proof. The Substitution Lemma (Lemma 1.2.3) implies that substituting any graph into another one at its vertex v in such a way that the sum of the probabilities of the new vertices is exactly $P(v)$ the entropy increase caused is independent of the original graph into which we substitute. Thus when cliques of some given size are substituted into G, then the entropy of G increases by the same amount as that of the complete graph if we substitute the same sized cliques into it. Since the entropy of the complete graph is $H(P)$, the difference $H(G,P) - H(P)$ equals $H(G_{\boldsymbol{x}},P_{\boldsymbol{x}}) - H(P_{\boldsymbol{x}})$ where $P_{\boldsymbol{x}}$ is any distribution on $V(G_{\boldsymbol{x}})$ satisfying that summing it on the vertices of a clique substituted into $v \in V(G)$ we get $P(v)$ as the sum. (The latter condition cannot be satisfied only if $\boldsymbol{x}$ is zero at some v with $P(v) \neq 0$. This explains why we have the technical condition for equality in the statement. If that is not satisfied it can only make the right-hand side smaller. In the following we assume the condition to be satisfied.) At the same time, $H(\bar{G},P) = H(\overline{G_{\boldsymbol{x}}},P_{\boldsymbol{x}})$ since the change of entropy caused by the substitution of stable sets is zero. Therefore the two maxima above must be the same. $\square$

Proof of Theorem 1.2.6. It is implied by Theorem 1.2.1 that $H(G,P) + H(\bar{G},P) - H(P) = H_{VP(G)}(P) - H_{FVP(G)}(P)$ for every G and P. Let $G_{\boldsymbol{x}}$ be the graph which attains $\mathrm{imp}(G)$. Observing that the vector $(\frac{1}{\omega(L)}, \frac{1}{\omega(L)}, \ldots, \frac{1}{\omega(L)})$ is in $FVP(L)$ for any graph L we have $H_{FVP(G_{\boldsymbol{x}})}(P) \leq \sum_{v \in V(G_{\boldsymbol{x}})} p_i \log \omega(G_{\boldsymbol{x}}) = \log \omega(G_{\boldsymbol{x}})$ for every P. Thus by Lemma 1.2.4 we have $\max_P\{H_{VP(G_{\boldsymbol{x}})}(P) - H_{FVP(G_{\boldsymbol{x}})}(P)\} \geq \log \chi^*(G_{\boldsymbol{x}}) - \log \omega(G_{\boldsymbol{x}}) = \log \mathrm{imp}(G)$. By Lemma 1.2.7 this proves one direction of our statement.

To prove the reverse inequality we use Theorem 1.2.5 from [119]. Applying Theorem 1.2.1 again, observe that $H(G,P) + H(\bar{G},P) - H(P)$ can also be expressed as $H(P) - H_{FVP(G)}(P) - H_{FVP(\bar{G})}(P)$. Let P_0 be the distribution maximizing this expression and let $\boldsymbol{a}$ and $\boldsymbol{b}$ be the vectors attaining $H_{FVP(G)}(P_0)$ and $H_{FVP(\bar{G})}(P_0)$, respectively. Using the inequality between the weighted arithmetic and geometric mean we can write by the foregoing

$$\max_P\{H(G,P) + H(\bar{G},P) - H(P)\} = H(P_0) - H_{FVP(G)}(P_0) - H_{FVP(\bar{G})}(P_0) =$$

$$\sum_{i \in V(G)} P_0(i) \log \frac{a_i b_i}{P_0(i)} = \log \prod_{i \in V(G)} (\frac{a_i b_i}{P_0(i)})^{P_0(i)} \leq \log \sum_{i \in V(G)} P_0(i) \frac{a_i b_i}{P_0(i)} = \log(\boldsymbol{a} \cdot \boldsymbol{b})$$

$$\leq \log \max\{\boldsymbol{x} \cdot \boldsymbol{y} : \boldsymbol{x} \in FVP(G), \boldsymbol{y} \in FVP(\bar{G})\} = \log \mathrm{imp}(G).$$

$\square$

We remark that McDiarmid [119] has somewhat simplified the above proof.

1.2.3 Dilation ratio and entropy of convex corners

The following generalization of $\mathrm{imp}(G)$ is also mentioned in [119]. Let $\mathcal{A}$ and $\mathcal{B}$ be convex corners in $\mathbb{R}^n_{+,0}$. Their *dilation ratio* is defined as

$$\mathrm{dil}(\mathcal{A},\mathcal{B}) = \min\{t : \mathcal{B} \subseteq t\mathcal{A}\}.$$

It is also given in [119] that

$$\mathrm{dil}(\mathcal{A},\mathcal{B}) = \max\{\boldsymbol{x}\cdot\boldsymbol{y} : \boldsymbol{x} \in \mathcal{A}^*, \boldsymbol{y} \in \mathcal{B}\}.$$

Along similar lines to those proving Theorem 1.2.6 one can show that the following more general statement also holds.

Theorem 1.2.8 *For any two convex corners $\mathcal{A},\mathcal{B} \subseteq \mathbb{R}^n_{+,0}$ one has*

$$\log \mathrm{dil}(\mathcal{A},\mathcal{B}) = \max_P\{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\}.$$

This was only noted in [142] but here we give more details. We also remark that McDiarmid's simplification applies again, see [119].

Showing

$$\max_P\{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\} \le \log \mathrm{dil}(\mathcal{A},\mathcal{B})$$

is essentially identical to that in the proof of Theorem 1.2.6. Namely, let P_0 be the disribution attaining the maximum in the left hand side and $\boldsymbol{a},\boldsymbol{b}$ be the vectors attaining the minima in the definition of $H^*_{\mathcal{A}}(P_0), H_{\mathcal{B}}(P_0)$, respectively. Then just as before, we can write

$$\max_P\{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\} = H(P_0) - H_{\mathcal{A}^*}(P_0) - H_{\mathcal{B}}(P_0) =$$
$$= \sum_{i=1}^n P_0(i)\log\frac{a_ib_i}{P_0(i)} = \log\prod_{i=1}^n(\frac{a_ib_i}{P_0(i)})^{P_0(i)} \le \log\sum_{i=1}^n P_0(i)\frac{a_ib_i}{P_0(i)} =$$
$$= \log(\boldsymbol{a}\cdot\boldsymbol{b}) \le \log\max\{\boldsymbol{x}\cdot\boldsymbol{y} : \boldsymbol{x}\in\mathcal{A}^*, \boldsymbol{y}\in\mathcal{B}\} = \log\mathrm{dil}(\mathcal{A},\mathcal{B}).$$

To prove the reverse inequality we use an analogue of the Substitution Lemma. (In fact, the main point in the proof simplification by McDiarmid is that he can avoid the use of this lemma. Still, I beleive that the proof using this lemma also has some advantages by the geometric intuition it may provide.) First we define what substitution (of a d-dimensional simplex, the analogue of a $(d+1)$-clique) means in this more general context.

Definition 12 *Let $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$ be a convex corner. Let $\mathcal{A}_{i\leftarrow\Delta_d}$ denote the convex corner in $\mathbb{R}^{n+d}_{+,0}$ defined as follows*

$$\mathcal{A}_{i\leftarrow\Delta_d} := \{\boldsymbol{a} = (a_1,\dots,a_{i-1},a_{i_1},\dots,a_{i_{d+1}},a_{i+1},\dots,a_n) \in \mathbb{R}^{n+d}_{+,0} :$$
$$(a_1,\dots,a_{i-1},\sum_{r=1}^{d+1} a_{i_r}, a_{i+1},\dots,a_n) \in \mathcal{A}\}.$$

The lemma that will play the role of the Substitution Lemma now is the following.

Lemma 1.2.9 *Let $P = (p_1, \dots, p_n)$ and $Q = (q_1, \dots, q_{d+1})$ be two arbitrary probability distributions of respective dimensions. Let $P_{i\leftarrow Q}$ denote the distribution with entries $(p_1, \dots, p_{i-1}, p_i q_1, \dots, p_i q_{d+1}, p_{i+1}, \dots, p_n)$. Then for any convex corner $\mathcal{A} \subseteq \mathbb{R}^n_{+,0}$ we have*

$$H_{\mathcal{A}_{i\leftarrow\Delta_d}}(P_{i\leftarrow Q}) = H_{\mathcal{A}}(P) + p_i H(Q).$$

Proof. Let $(a_1, \dots, a_{i-1}, a_{i_1}, \dots, a_{i_{d+1}}, a_{i+1} \dots, a_n) \in \mathcal{A}_{i\leftarrow\Delta_d}$ be the vector attaining $H_{\mathcal{A}_{i\leftarrow\Delta_d}}(P_{i\leftarrow Q})$. Let $a_i = \max\{x : (a_1, \dots, a_{i-1}, x, a_{i+1}, \dots, a_n) \in \mathcal{A}\}$. It is clear by the definition of $\mathcal{A}_{i\leftarrow\Delta_d}$ and that of the entropy of convex corners, that $\sum_{j=1}^{d+1} a_{i_j} = a_i$ should hold (at least whenever $p_i > 0$, but otherwise the statement is trivial). Thus there are nonnegative numbers $r_1, \dots, r_{d+1}$ for which $\sum_{j=1}^{d+1} r_j = 1$ and $a_{i_j} = r_j a_i$ holds for all $j \in \{1, \dots, d+1\}$.

Therefore we can write

$$H_{\mathcal{A}_{i\leftarrow\Delta_d}}(P_{i\leftarrow Q}) = \sum_{j\neq i} p_j \log\frac{1}{a_j} + p_i \sum_{j=1}^{d+1} q_j \log\frac{1}{a_{i_j}} = \sum_{j\neq i} p_j \log\frac{1}{a_j} + p_i \sum_{j=1}^{d+1} q_j \log\frac{1}{r_j a_i} =$$

$$\sum_{j=1}^{n} p_j \log\frac{1}{a_j} + p_i \sum_{j=1}^{d+1} q_j \log\frac{1}{r_j} \geq H_{\mathcal{A}}(P) + p_i H(Q),$$

where the last inequality is by the well-known fact expressed e.g. by Lemma I.3.2(c) in [36].

On the other hand, if we let $\boldsymbol{a}' = (a'_1, \dots, a'_n)$ be the vector attaining $H_{\mathcal{A}}(P)$ and $a'_{i_j} = q_j a'_i$ for all j, then essentially the same calculation (executed backwards) gives the reverse inequality. □

We remark that the above proof actually gives the proof of the special case of the Substitution Lemma used in the previous subsection, namely the case where the substituted graphs are always cliques.

Corollary 1.2.10 *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n_{+,0}$ be convex corners and $d_1, \dots, d_n$ be positive integers. Let $\hat{\mathcal{A}}$ and $\hat{\mathcal{B}}$ be obtained from $\mathcal{A}, \mathcal{B}$ by subsequently substituting the simplex Δ_{d_i} into i for $i = 1, \dots, n$ (simultaneously for both convex corners). Then*

$$\max_P\{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\} = \max_{\hat{P}}\{H_{\hat{\mathcal{A}}}(\hat{P}) - H_{\hat{\mathcal{B}}}(\hat{P})\}$$

Proof. It is enough to notice that the entropy increase caused by the substitution of simplices is the same for the two convex corners for any fixed distributions P and Q involved in the substitution. □

Lemma 1.2.11 *Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n_{+,0}$ be two convex corners. Then for every positive integer d*

$$\mathrm{dil}(\mathcal{A}, \mathcal{B}) = \mathrm{dil}(\mathcal{A}_{i\leftarrow\Delta_d}, \mathcal{B}_{i\leftarrow\Delta_d}).$$

Proof. It is clear from the definitions that $\mathcal{B} \subseteq t\mathcal{A}$ implies $\mathcal{B}_{i\leftarrow\Delta_d} \subseteq t\mathcal{A}_{i\leftarrow\Delta_d}$, therefore $\mathrm{dil}(\mathcal{A}_{i\leftarrow\Delta_d}, \mathcal{B}_{i\leftarrow\Delta_d}) \leq \mathrm{dil}(\mathcal{A}, \mathcal{B})$. On the other hand, if $\mathcal{B} \not\subseteq t\mathcal{A}$, then there exists a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{A}$ satisfying $(1+\varepsilon)\boldsymbol{a} \notin \mathcal{A}$ for any $\varepsilon > 0$ for which there is a $\boldsymbol{b} = (b_1, \ldots, b_n) \in \mathcal{B}$ such that $b_j > ta_j$ for all j with $a_j > 0$. Setting $a_{i,1} = a_i$, $b_{i,1} = b_i$ and $a_{i,j} = b_{i,j} = 0$ for all $j > 1$ we see that $\mathcal{B}_{i\leftarrow\Delta_d} \not\subseteq t\mathcal{A}_{i\leftarrow\Delta_d}$. This implies $\mathrm{dil}(\mathcal{A}_{i\leftarrow\Delta_d}, \mathcal{B}_{i\leftarrow\Delta_d}) \geq \mathrm{dil}(\mathcal{A}, \mathcal{B})$ completing the proof. □

Proof of Theorem 1.2.8.

We have already seen the proof of $\max_P\{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\} \leq \log \mathrm{dil}(\mathcal{A}, \mathcal{B})$ right after the statement. So we need to prove only the reverse inequality.

Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^n_{+,0}$ be convex corners and let $t = \mathrm{dil}(\mathcal{A}, \mathcal{B})$ be attained by the vectors $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathcal{A}$ and $\boldsymbol{b} = t\boldsymbol{a} \in \mathcal{B}$. Let $\beta_0 = \sum_{i=1}^n b_i$ and let $\varepsilon > 0$ be an arbitrarily small positive real. Let $d_1, \ldots, d_n$ be non-negative integers for which $(1+\varepsilon)\frac{b_j}{d_j+1} \geq \beta := \frac{\beta_0}{\sum_{i=1}^n (d_i+1)} \geq (1-\varepsilon)\frac{b_j}{d_j+1}$ for every j. It is clear that such a choice of the d_i's is possible. For every i set $b_{i,j} = \frac{b_i}{d_i+1}$ for $j = 1, \ldots, d_i + 1$. (Thus β is just the arithmetic mean of all the $b_{i,j}$'s and the d_i's are chosen so, that no $b_{i,j}$ differs too much from the arithmetic mean.)

Let $\hat{\mathcal{A}}^i$ be defined recursively by $\hat{\mathcal{A}}^0 = \mathcal{A}$ and $\hat{\mathcal{A}}^i = \mathcal{A}^{i-1}_{i\leftarrow\Delta_{d_i}}$ for $i = 1, \ldots, n$. Let $\hat{\mathcal{A}}$ denote $\hat{\mathcal{A}}^n$ and define $\hat{\mathcal{B}}$ similarly from $\mathcal{B}$. (Note that the vector $(\frac{\beta}{1+\varepsilon}, \ldots, \frac{\beta}{1+\varepsilon})$ of dimension $\sum_{i=1}^n (d_i+1)$ is in $\hat{\mathcal{B}}$.)

Let $a_{i,j} = b_{i,j}/t$ for all i, j. Corollary 7 of [38] implies the analogue of Lemma A (see on page 5 in Subsection 1.1.3) for any convex corner $\mathcal{D}$ in place of $VP(F)$. That is, if $\mathcal{D}'$ is the closure of that part of the boundary of $\mathcal{D}$ that is not contained in any of the coordinate hyperplanes $x_i = 0$, then for every $\boldsymbol{h} \in \mathcal{D}'$ there exists some probability distribution P such that $H_{\mathcal{D}}(P)$ is attained by $\boldsymbol{h}$. Applying this to $\hat{\mathcal{A}}$ and the vector $((a_{i,j})_{j=1}^{d_i+1})_{i=1}^n \in \hat{\mathcal{A}}'$ we have a distribution P_0 for which $H_{\hat{\mathcal{A}}}(P_0)$ is attained by $((a_{i,j})_{j=1}^{d_i+1})_{i=1}^n$. Now we can write the following. (In the calculation below we assume $a_{i,j} > 0$ for all i, j. If for some i, j this would not be the case then $P_0(i,j)$ would also vanish thus making the calculation correct.)

$$\max_P\{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\} = \max_P\{H_{\hat{\mathcal{A}}}(P) - H_{\hat{\mathcal{B}}}(P)\} \geq$$

$$H_{\hat{\mathcal{A}}}(P_0) - H_{\hat{\mathcal{B}}}(P_0) = \sum_{i=1}^n \sum_{j=1}^{d_i+1} P_0(i,j) \log \frac{1}{a_{i,j}} - H_{\hat{\mathcal{B}}}(P_0) \geq$$

$$\sum_{i,j} P_0(i,j) \log \frac{1}{a_{i,j}} - \sum_{i,j} P_0(i,j) \log \frac{1+\varepsilon}{\beta} \geq \sum_{i,j} P_0(i,j) \log \frac{1}{a_{i,j}} - \sum_{i,j} P_0(i,j) \log \frac{1+\varepsilon}{b_{i,j}(1-\varepsilon)} =$$

$$\sum_{i,j} P_0(i,j) \log \frac{b_{i,j}}{a_{i,j}} - \log \frac{1+\varepsilon}{1-\varepsilon} = \sum_{i,j} P_0(i,j) \log(\mathrm{dil}(\mathcal{A},\mathcal{B})) - \log \frac{1+\varepsilon}{1-\varepsilon} =$$

$$\log \mathrm{dil}(\mathcal{A},\mathcal{B}) - \log \frac{1+\varepsilon}{1-\varepsilon}.$$

Since the $\log \frac{1+\varepsilon}{1-\varepsilon}$ term in the right hand side can be made arbitrarily small by choosing ε small enough, this proves

$$\max_P \{H_{\mathcal{A}}(P) - H_{\mathcal{B}}(P)\} \geq \log \mathrm{dil}(\mathcal{A},\mathcal{B})$$

that we needed to complete the proof.

□

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1.3 Witsenhausen rate of multiple sources

Graph entropy was originally defined in terms of graph exponentiation and graph coloring in [88]. Using a different graph exponentiation the related concept of complementary graph entropy or co-entropy was introduced in [91]. The latter notion is in a sense a probabilistic refinement of the information theoretic concept called Witsenhausen's rate. In this section we use results of earlier research around these concepts to prove a perhaps surprising statement whose main interest lies in its information theoretic content.

This section is based on the paper [143]

1.3.1 Introduction

Let $X, Y^{(1)}, Y^{(2)}, \ldots, Y^{(k)}$ be $k+1$ discrete random variables. Consider X as a 'central' variable available for a transmitter T and the $Y^{(i)}$'s as side information available for different stations $S^{(i)}, i = 1, 2, \ldots, k$, that are located at different places. The joint distribution is known for X and $Y^{(i)}$ for every i. The task is that T broadcast a message received by all $S^{(i)}$'s in such a way that learning this message all $S^{(i)}$'s should be able to determine X in an error-free manner. The question is the minimum number of bits that should be used for this per transmission if block coding is allowed. This problem is considered for $k = 1$ by Witsenhausen in [158]. He translated the problem to a graph theoretic one and showed that block coding can indeed help in decreasing the (per transmission) number of possible messages that should be used. The optimal number of bits to be sent per transmission defines a graph parameter that is called *Witsenhausen's zero-error rate* in [6]. (We will write simply *Witsenhausen rate* in the sequel.) In this section we define the Witsenhausen rate of a family of graphs. (When we speak about a family of graphs we always mean a finite family in this section.) Our main result is that the Witsenhausen rate of a family of graphs equals its obvious lower bound: the largest Witsenhausen rate of the graphs in the family. This will easily follow from a powerful result of Gargano, Körner, and Vaccaro [62].

1.3.2 The graph theory model

For each $i = 1, 2, \ldots, k$ we define the following graph G_i. The vertex set $V(G_i) = \mathcal{X}$ is the support set of the variable X for every i. Two elements, a and b of $\mathcal{X}$ form an edge in G_i if and only if there exists some possible value c of the variable $Y^{(i)}$ that is jointly possible with both a and b, i.e., $\text{Prob}(c, a)\text{Prob}(c, b) > 0$. It is already explained in [158] that the minimum number of bits to be sent by T to (one) $S^{(i)}$ for making it learn X (for one instance) in an error-free manner is $\log_2 \chi(G_i)$, where $\chi(F)$ denotes the chromatic number of graph F. Indeed, if T would use fewer bits, than there were some two elements of $\mathcal{X}$ that form an edge in G_i and still T sends the same message when one or the other appears as the actual value of X. Since they form an edge there is some possible value c of $Y^{(i)}$ that is jointly possible with both, thus $S^{(i)}$ would not be able to decide which of them occured if it had c as side information. (As in [158], we use the assumption, that the side information $Y^{(i)}$ is not available at T.) On the other hand, if a proper coloring of G_i is given, then if T sends the color of X this will make $S^{(i)}$ learn X using the side-information contained by $Y^{(i)}$.

Just as before, all subsequent logarithms are on base two.

If block coding is allowed then the minimum number of bits to be transmitted to $S^{(i)}$ (not caring about the other $S^{(j)}$'s for the moment) by T after observing the n-fold variable $(X_1, X_2, \ldots, X_n)$ will be $\log \chi(G_i^{\wedge n})$ where $G_i^{\wedge n}$ is an appropriately defined power of the graph G_i.

Definition 13 *Let $G = (V, E)$ be a graph. The n^{th}* normal power *of G is the graph $G^{\wedge n}$ defined as follows. $V(G^{\wedge n}) = V^n$ and*

$$E(G^{\wedge n}) = \{\{\boldsymbol{u}, \boldsymbol{v}\} : \boldsymbol{u} \neq \boldsymbol{v}, \forall i \ u_i = v_i \text{ or } \{u_i, v_i\} \in E(G)\}.$$

That is the vertices of $G^{\wedge n}$ are the n-length sequences over V and two are adjacent iff they are adjacent at every coordinate where they are not equal.

(In [6] the above graph power is called the AND power. This may explain its notation. In the next chapter of this work we will often deal with another closely related graph exponentiation and adopt its name *co-normal*, while it is called OR power in [6], and disjunctive power in [79], [133]. Though it would be consistent to denote it by $G^{\vee n}$, as later we use the co-normal power quite often, we will denote it simply by G^n.)

It is easy to see that two n-length sequences over $\mathcal{X}$ are jointly possible with some n-fold outcome $(Y_1^{(i)}, Y_2^{(i)}, \ldots, Y_n^{(i)})$ iff they are adjacent in $G_i^{\wedge n}$ (our sources are stationary and memoryless). Thus the previous argument gives that if we cared only about $S^{(i)}$ then T should transmit one of $\chi(G_i^{\wedge n})$ messages for making $S^{(i)}$ learn $(X_1, X_2, \ldots, X_n)$. Thus, in case $k = 1$ (and denoting G_1 by G) the value of interest is the Witsenhausen rate of G defined as

$$R(G) = \lim_{n\to\infty} \frac{1}{n} \log(\chi(G^{\wedge n})).$$

In our problem (when $k > 1$) the message sent by T should be such that learning it should be enough for each $S^{(i)}$ to determine X without error. Thus T cannot send the same message for two n-fold outcome of the variable X if they, as vertices of the graphs $G_i^{\wedge n}$, are adjacent *in any* $G_i^{\wedge n}$. On the other hand, if we color the elements of $V^n = \mathcal{X}^n$ in such a way that elements adjacent in any $G_i^{\wedge n}$ get different color, then transmitting the color of the actual $(X_1, X_2, \ldots, X_n)$ will make all $S^{(i)}$'s able to determine $(X_1, X_2, \ldots, X_n)$. This justifies the following definition.

Definition 14 *Let $\mathcal{G} = (G_1, \ldots, G_k)$ be a family of graphs all of which have the same vertex set V. The Witsenhausen rate of the family $\mathcal{G}$ is defined by*

$$R(\mathcal{G}) = \lim_{n\to\infty} \frac{1}{n} \log(\chi(\cup_i G_i^{\wedge n})),$$

where $\cup_i G_i^{\wedge n}$ is meant to be the graph on the common vertex set V^n of the $G_i^{\wedge n}$ with edge set $\cup_i E(G_i^{\wedge n})$.

It is obvious from the definition that $R(\mathcal{G}) \geq \max_i R(G_i)$. Our main result in this section is that this trivial estimation is sharp.

Theorem 1.3.1 *If $\mathcal{G} = (G_1, \ldots, G_k)$ is a family of graphs on the same vertex set, then*

$$R(\mathcal{G}) = \max_{G_i \in \mathcal{G}} R(G_i).$$

To appreciate the above statement consider the following.

Example: Let $|V| = v$ and $\cup_i G_i = K_v$, i.e, the complete graph on v vertices such that each G_i is bipartite. (This needs $k \geq \log v$.) Now for $n = 1$ we would be obliged to use $\log v$ bits to make sure that each $S^{(i)}$ can decode the outcome of X correctly. However, with block coding, the above theorem states that roughly one bit per source outcome is enough if we let n go to infinity.

For proving Theorem 1.3.1 we have to introduce some other notions. This is done in the next subsection.

1.3.3 Probabilistic graph invariants

The proof of our theorem relies on a result that determines the zero-error capacity of a compound channel. Here we give our definitions already in graph terms, the translation is explained in detail in [33] where these investigations started, in [62], where the powerful result we are going to use was obtained, and also in the survey article [96].

Definition 15 *Let $\mathcal{G} = (G_1, \ldots, G_k)$ be a family of graphs all of which have the same vertex set V. The capacity of the family $\mathcal{G}$ is defined by*

$$C_{\text{Sh}}(\mathcal{G}) = \lim_{n \to \infty} \frac{1}{n} \log(\alpha(\cup_i G_i^{\wedge n})),$$

where α stands for the independence number (size of largest edgeless subgraph) of a graph. If $\mathcal{G} = \{G\}$ we write $C_{\text{Sh}}(G)$ instead of $C_{\text{Sh}}(\mathcal{G})$.

Remark 1.5. If $|\mathcal{G}| = 1$ then $C_{\text{Sh}}(\mathcal{G})$ becomes equivalent to what is usually called the Shannon capacity of the graph G. It is not hard to see that the value $C_{\text{Sh}}(\mathcal{G})$ represents the zero-error capacity of the compound channel the individual channels in which are described by the graphs in the family. For a more detailed explanation, see [33], [62], [96], cf. also [124]. We have to warn the reader, however, that several papers, including the ones just cited, use a complementary language and define $C_{\text{Sh}}(G)$ as our $C_{\text{Sh}}(\bar{G})$, while $C_{\text{Sh}}(\mathcal{G})$ is also defined via cliques instead of independent sets. The language we use here is the more traditional one (cf. [139], [109]), although the earlier cited papers have their good reason to do differently. It has to do with a generalization to oriented graphs that we will not need in this section. In the next chapter of this work, however, we will often deal with this generalization and therefore we will also use the complementary language later. There we will use the notation $C(G) = C_{\mathrm{Sh}}(\bar{G})$, thus the subscript in the present notation shows that we use Shannon's language. On the other hand, with slight abuse of the terminology, we will also refer to the quantity $C(G)$ later as the Shannon capacity of graph G. To lessen confusion we will not use the name Shannon capacity in this section any more. $\Diamond$

It is obvious that $C_{\mathrm{Sh}}(\mathcal{G}) \leq \min_i C_{\mathrm{Sh}}(G_i)$. An easy but somewhat more sophisticated upper bound is obtained in [33]. This needs the following *within a given type* version of the $C_{\mathrm{Sh}}(G)$ invariant introduced by Csiszár and Körner [37]. First we need the concept of (P, ε)-typical sequences, cf. [36].

Definition 16 *Let V be a finite set, P a probability distribution on V, and $\varepsilon > 0$. A sequence $\boldsymbol{x}$ in V^n is said to be (P, ε)-typical if for every $a \in V$ we have $|\frac{1}{n}N(a|\boldsymbol{x}) - P(a)| < \varepsilon$, where $N(a|\boldsymbol{x}) = |\{i : x_i = a\}|$.*

Definition 17 *Given graph G, probability distribution P, and $\varepsilon > 0$, the graph $G^{\wedge n}_{P,\varepsilon}$ is the graph induced in $G^{\wedge n}$ by the (P, ε)-typical sequences.*

Definition 18 ([37]) *The within a given type P version of the $C_{\mathrm{Sh}}(G)$ invariant is $C_{\mathrm{Sh}}(G, P)$ defined as*

$$C_{\mathrm{Sh}}(G, P) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \alpha(G^{\wedge n}_{P,\varepsilon}).$$

In a similar manner we write

$$C_{\mathrm{Sh}}(\mathcal{G}, P) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \alpha(\cup_i G^{\wedge n}_{i;P,\varepsilon}).$$

The upper bound shown in [33] states $C_{\mathrm{Sh}}(\mathcal{G}) \leq \max_P \min_i C_{\mathrm{Sh}}(G_i, P)$. Gargano, Körner, and Vaccaro [62] proved the surprising result that this bound is sharp. This is a corollary of their more general result that we will also need.

Theorem (Gargano, Körner, Vaccaro [62]): *For any family of graphs $\mathcal{G} = \{G_1, \ldots, G_k\}$ and any probability distribution P on the common vertex set of the G_i's we have*

$$C_{\mathrm{Sh}}(\mathcal{G}, P) = \min_i C_{\mathrm{Sh}}(G_i, P).$$

Remark 1.6. The exact statement proven in [62] (cf. also [61] for an important special case) is that $C_{\mathrm{Sh}}(\mathcal{G}_1 \cup \mathcal{G}_2, P) = \min\{C_{\mathrm{Sh}}(\mathcal{G}_1, P), C_{\mathrm{Sh}}(\mathcal{G}_2, P)\}$ holds for any two graph families $\mathcal{G}_1$ and $\mathcal{G}_2$ with common vertex set. This easily implies the above by setting $\mathcal{G}_1 = \{G_1, G_2, \ldots, G_i\}$ and $\mathcal{G}_2 = \{G_{i+1}\}$ iteratively for all $i = 1, 2, \ldots, k-1$. $\Diamond$

To relate $C_{\mathrm{Sh}}(G)$ and $R(G)$ we will use the "within a type version" of $R(G)$ which was already introduced in [91] by Körner and Longo in a different context under the name *complementary graph entropy.* (We use the name *co-entropy* as in [141] and [142].) Marton [115] investigated this functional further, while recent interest in it also occured in [87].

Definition 19 *The* co-entropy $\bar{H}(G,P)$ *of a graph* G *within a given type* P *is the value*

$$\bar{H}(G,P) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\chi(G^{\wedge n}_{P,\varepsilon}).$$

In a similar manner we write

$$\bar{H}(\mathcal{G},P) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log\chi(\cup_i G^{\wedge n}_{i;P,\varepsilon}).$$

Remark 1.7. It would be appropriate to denote the within a type version of $R(G)$ by $R(G,P)$. Here we keep the $\bar{H}(G,P)$ notation only to emphasize that this is not a new concept. $\Diamond$

Remark 1.8. We use the name *co-entropy* to distinguish the above value from the related notion of graph entropy introduced by Körner [88]. (The original definition of graph entropy in [88] differs from that of $\bar{H}(G,P)$ only in the definition of the graph exponentiation involved.) For a detailed account of the relation of these two concepts see [115] or [141], [142]. For further relations of graph entropy and source coding, cf. also [7]. $\Diamond$

In [115] the following relation is proven.

Lemma (Marton [115]): *For any graph* G *and probability distribution* P *on its vertex set*

$$\bar{H}(G,P) = H(P) - C_{\mathrm{Sh}}(G,P)$$

where $H(P)$ *is the Shannon entropy of the distribution* P*.*

We also need the following more general statement the proof of which is exactly the same as that of Marton's Lemma.

Lemma 1.3.2 *For any family of graphs and any probability distribution on their common vertex set* V *one has*

$$\bar{H}(\mathcal{G},P) = H(P) - C_{\mathrm{Sh}}(\mathcal{G},P).$$

We sketch the proof of this lemma for the sake of completeness. Setting $\mathcal{G} = \{G\}$ it also implies Marton's result. The following lemma of Lovász [107] is needed.

Lemma 1.3.3 *For any graph* G

$$\chi(G) \le \chi^*(G)(1+\ln\alpha(G)),$$

where, as before, $\chi^*(G)$ *is the fractional chromatic number of graph* G*.*

Remark 1.9. Lovász's result is formulated in a more general setting for the covering numbers of hypergraphs. The above statement is a straightforward corollary of that. For basic facts about the fractional chromatic number we refer the reader to [133]. One such fact we need is that for a vertex-transitive graph G one always has $\chi^*(G) = \frac{|V(G)|}{\alpha(G)}$ (see [133] Proposition 3.1.1). $\diamondsuit$

Sketch of proof of Lemma 1.3.2. Consider a sequence of probability distributions P_n that can be represented as types of n-length sequences for each n, respectively, and the series of which converges to P in the sense that $\forall\varepsilon > 0\ \exists n_0$ such that $n \geq n_0$ implies $\forall a \in V : |P_n(a) - P(a)| < \varepsilon$. As the number of possible types of an n-length sequence is only a polynomial function of n (cf. Lemma 2.2 of Chapter 1 in [36]) we can write (using also that our lim sup's are actually limits)

$$\bar{H}(\mathcal{G}, P) = \lim_{n\to\infty} \frac{1}{n} \log \chi(\cup_i G_{i;P_n,0}^{\wedge n}).$$

Since sequences of the same type are all permutations of each other, one easily sees that the graph $\cup_i G_{i;P_n,0}^{\wedge n}$ is vertex transitive for any n. Thus we can continue by

$$\lim_{n\to\infty} \frac{1}{n} \log \chi(\cup_i G_{i;P_n,0}^{\wedge n}) \leq \lim_{n\to\infty} \frac{1}{n} \log \frac{|V(\cup_i G_{i;P_n,0}^{\wedge n})|}{\alpha(\cup_i G_{i;P_n,0}^{\wedge n})}(1 + \ln \alpha(\cup_i G_{i;P_n,0}^{\wedge n})) =$$

$$H(P) - C_{\mathrm{Sh}}(\mathcal{G}, P) + \lim_{n\to\infty} \frac{1}{n} \log(1 + \ln \alpha(\cup_i G_{i;P_n,0}^{\wedge n})) = H(P) - C_{\mathrm{Sh}}(\mathcal{G}, P),$$

where we used Lemma I.2.3 of [36] for the first equality.

The opposite inequality is obvious as $\chi(F) \geq \frac{|V(F)|}{\alpha(F)}$ is trivially true for any graph F and applying it for $F = \cup_i G_{i;P_n,0}^{\wedge n}$ we get the above with the inequality reversed.

$\square$

1.3.4 Proof of Theorem 1.3.1

For proving Theorem 1.3.1 we first need an easy lemma.

Lemma 1.3.4

$$R(\mathcal{G}) = \max_P \bar{H}(\mathcal{G}, P).$$

Proof. Using again the Type Counting Lemma (Lemma 2.2 on page 29) from Csiszár and Körner's book [36] we get that

$$\chi(\cup_i G_i^{\wedge n}) \leq (n+1)^{|V|} \max_P \chi(\cup_i G_{i;P,0}^{\wedge n})$$

where the maximization is meant over those P's that can be exact types of sequences of length n. (Equivalently, we can just think of $G_{i;P,0}^{\wedge n}$ as a graph with no vertices for other

P's). Since $\chi(\cup_i G_i^{\wedge n}) \geq \chi(\cup_i G_{i;P,0}^{\wedge n})$ obviously holds for any P, taking the logarithm, dividing by n, and let n go to infinity in the earlier inequality we get the desired result. □

Proof of Theorem 1.3.1. By the previous two lemmas and the Gargano-Körner-Vaccaro theorem we have

$$R(\mathcal{G}) = \max_P \bar{H}(\mathcal{G}, P) = \max_P \{H(P) - C_{\mathrm{Sh}}(\mathcal{G}, P)\} =$$

$$= \max_P \{H(P) - \min_{G_i \in \mathcal{G}} C_{\mathrm{Sh}}(G_i, P)\} = \max_P \max_{G_i} \{H(P) - C_{\mathrm{Sh}}(G_i, P)\} =$$

$$= \max_{G_i} \max_P \bar{H}(G_i, P) = \max_{G_i} R(G_i)$$

giving the desired result. □

Remark 1.10. It seems worth noting that while the proof of Theorem 1.3.1 needed separate investigation of the different types P the statement itself does not contain any reference to types. This is not so in the original Gargano-Körner-Vaccaro result. Though the reason of this is a very simple technical difference (namely that the chromatic number is defined as a minimum, while the clique number and the independence number are appropriate maximums), we feel that this phenomenon makes Theorem 1.3.1 another good example of a result that demonstrates the power of the method of types, cf. [35]. ♢

Chapter 2

Graph Capacities

2.1 Different capacities of a digraph

In this section we will extensively use the co-normal product of graphs already mentioned in the last section of the previous chapter. The co–normal product of graphs generalize to digraphs in a natural way. For every class of digraphs closed under this product, the cardinality of the largest subgraph from the given class, contained as an induced subgraph in the co–normal powers of a graph G, has an exponential growth. The corresponding asymptotic exponent is the capacity of G with respect to said class of digraphs. We derive upper and lower bounds for such and related capacity values and give examples for their tightness.

This section is based on the joint paper [59] with Anna Galluccio, Luisa Gargano, and János Körner.

2.1.1 Introduction

An induced complete subgraph is often called a clique and correspondingly, the cardinality of the largest clique of G is called its clique number. The analysis of its asymptotic growth in large product graphs leads to one of the most formidable problems in modern combinatorics. This problem was posed by Shannon [139] in 1956 in connection with his analysis of the capability of certain noisy communication channels to transmit information in an error–free manner. Shannon associated a graph with every channel. In our notation (which is different from his), the vertex set of the graph represents the symbols that can be transmitted through the channel and two vertices are connected by an edge if the corresponding symbols can never get confused by the receiver. Put in other words this means that our two vertices never lead to the reception of the same output symbol. Any graph can be obtained in this manner.

Shannon's model naturally leads to a product of graphs through the repeated use of the channel for the transmission of symbol sequences of some fixed length n, say. If the graph G has vertex set $V = V(G)$, then G^n denotes the graph with vertex set V^n whose edge set contains those pairs of sequences which can never get confused by the receiver. Formally,

$$\{\boldsymbol{x}, \boldsymbol{y}\} \in E(G^n) \quad \text{if and only if} \quad \exists i \quad \{x_i, y_i\} \in E(G),$$

where we suppose that x_i and y_i are the i'th coordinate of $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively. As usual, $E(G)$ denotes the edge set of the graph G. Following Berge [14], G^n is called the n'th co–normal power of G. (As already mentioned in the previous chapter, it is called OR power in [4], [6] and disjunctive power in [79], [133].) Shannon [139] observed that if K is a clique in G then K^n is a clique in G^n, whence the clique number of G^n is at least as big as the n'th power of the clique number of G. In fact, these two quantities coincide whenever the clique number of G equals its chromatic number. (This observation led Berge [17] to his celebrated concept of perfect graphs.) On the other hand, Shannon [139] noticed that for the smallest graph whose chromatic number exceeds its clique number, the now famous pentagon C_5, the clique number of C_5^2 is 5 while the square of the clique number of C_5 is just 4. Thus it was natural to ask as Shannon did for the determination of the (always existing) limit

$$C(G) = \lim_{n\to\infty} \frac{1}{n} \log \omega(G^n),$$

where $\omega(G)$ is the usual notation for the clique number of G, while $C(G)$ is called the Shannon capacity of G. A very good exposition of this problem and the new important developments around it in the late seventies is in the volume [135].

Notice further that this supermultiplicativity of the clique number is in contrast with the growth of the independence number $\alpha(G^n)$ of the powers of G. Recall that $\alpha(G)$ is the cardinality of the largest induced edge–free subgraph of G. In fact, it is rather trivial that every maximal independent set of G^n is the Cartesian product of n independent sets of G whence

$$\alpha(G^n) = [\alpha(G)]^n.$$

We will observe (cf. Proposition 2.1.11) that the number of graph classes closed under both the co-normal product and taking induced subgraphs is surprisingly small. Namely, apart from trivial ones there are only three such classes: the class of all complete graphs, the class of edgeless graphs, and the class of all complete multipartite graphs. We have already mentioned what can be said about the asymptotic growth of the size of largest graphs of the first two classes in co-normal powers: the first class leads to the famous notion of Shannon capacity while the second grows trivially by the last equality above. In case of the third class we experience non-trivial behavior again leading to a new notion of the flavour of Shannon capacity. We will investigate this new notion under the name cascade capacity in the second part of this section. One of the main results of the section is an upper bound on its value.

Originally we were led to the problem of cascade capacity by looking at an analogous problem for oriented graphs in the first part of the section. (The name "cascade capacity" comes up already there as it is defined for general directed graphs.)

The investigation of capacity type concepts for directed graphs already appears in [20] while a new line of this kind of research was initiated in [98] and further developed by Gargano, Körner, and Vaccaro in [60] where the notion of Sperner capacity was introduced as the natural counterpart of Shannon capacity in case of graphs with directed edges. At this point, a few words about terminology. As usual, we call a graph directed if its edge set is an arbitrary subset of the set of ordered pairs of its vertices. Graphs in which every edge is present with at most one of its possible orientations are called oriented. Finally, a directed graph is called symmetrically directed if each of its edges is present with both of its possible orientations. (In neither of these cases can our graphs contain loops.)

Sperner capacity is a generalization of Shannon capacity to directed graphs and is therefore even harder to determine, cf. Calderbank, Frankl, Graham, Li, and Shepp [28] and Blokhuis [23]. This does not mean, however, that this generalization makes no sense. On the contrary, Sperner capacity became the key to the solution of some important open problems in extremal set theory, [61], [62]. Recently its direct information theoretic relevance was also discovered by Nayak and Rose [124].

The study of the extension of a poset to sequences of its elements leads to the equally justified notion of *antichain capacity* that we will introduce in this paper. Although our concept will be defined for arbitrary oriented graphs, not just those corresponding to posets, it is useful to present it first in the special case of posets where it is probably more intuitive.

Let P be a partial order on the set V and write $(x, y) \in P$ to indicate that $x \in V$ precedes $y \in V$ in the partial order P. Consider for some fixed natural integer n the Cartesian power V^n. Then we can extend P to the sequences $\boldsymbol{x} \in V^n$ and $\boldsymbol{y} \in V^n$ by saying that $\boldsymbol{x}$ precedes $\boldsymbol{y}$ and write $(\boldsymbol{x}, \boldsymbol{y}) \in P^n$ if for some coordinate i, $1 \le i \le n$ we have $(x_i, y_i) \in P$, while $(y_j, x_j) \in P$ never holds. Although this relation of precedence is not necessarily a partial order on V^n, it nevertheless has a clear intuitive meaning. Now there are two different reasons why for two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ none precedes the other. Either in no coordinate i is any of (x_i, y_i) and (y_i, x_i) in P, or there are two coordinates, i and j such that both (x_i, y_i) and (y_j, x_j) are contained in P. In both cases, there is no meaningful way to break symmetry and say that one sequence precedes the other. We will call the set $C \subseteq V^n$ an antichain for P^n if

$$\boldsymbol{x} \in C, \quad \boldsymbol{y} \in C \quad \text{implies} \quad (\boldsymbol{x}, \boldsymbol{y}) \notin P^n.$$

An antichain is a set of sequences on which the partial order implies no relation of precedence at all. Let $M(P, n)$ denote the cardinality of the largest antichain for P^n. We call the always existing limit

$$\lim_{n\to\infty} \frac{1}{n} \log M(P, n)$$

the antichain capacity of the partial order P and introduce for it the notation $A(P)$. The above concept has an immediate extension to the case of an arbitrary oriented graph.

Let G be an oriented graph with vertex set V. We say that two sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ in V^n are G–incomparable, if there are two different coordinates, i and j such that both (x_i, y_i) and (y_j, x_j) are directed edges in G. Following [60], we denote by $N(G, n)$ the maximum cardinality of any set $C \subseteq V^n$ such that any two elements of C are G–incomparable. For the sake of completeness, we should mention that the always existing limit

$$\Sigma(G) = \lim_{n\to\infty} \frac{1}{n} \log N(G, n)$$

is called the Sperner capacity of the oriented graph G introduced by Gargano, Körner, and Vaccaro [60]. (Notice that the definition can be extended to any directed graph and thus Sperner capacity is a formal generalization of Shannon capacity, cf. [62] and also Definition 24 on page 48.)

Now, two sequences, $\boldsymbol{x}$ and $\boldsymbol{y}$ in V^n are called G–independent if x_i and y_i are equal or non-adjacent vertices of G for any i, $1 \le i \le n$. Finally, the sequences $\boldsymbol{x}$ and $\boldsymbol{y}$ in V^n are called G–unrelated if they are either G–incomparable or G–independent. Let $M(G, n)$ denote the largest cardinality of a subset $C \subseteq V^n$ such that any pair of elements of C are G–unrelated. We call the always existing limit

$$A(G) = \lim_{n\to\infty} \frac{1}{n} \log M(G, n)$$

the antichain capacity of the oriented graph G. We shall sometimes call a set as above an antichain defined by G.

In case of an oriented graph G one immediately sees from the definitions that the Sperner capacity $\Sigma(G)$ is always bounded from above by $A(G)$ and that the two notions coincide for tournaments. Since the Sperner capacity of tournaments is unknown except for the trivial case of transitive tournaments and a few other special examples for small vertex sets [28], [23] (cf. also [2]), we immediately understand that determining the antichain capacity of a graph is not easy and we will be happy if we can find meaningful and strong enough lower and upper bounds. This is one of the principal aims of the present section. Incidentally, we will find some interesting problems along the way.

In order to stress the analogy of both Sperner capacity and antichain capacity to the familiar Shannon capacity of graphs we will introduce an analogue of the co–normal power of graphs in the case of oriented (and more generally, directed) graphs. The search for antichains in product graphs leads to the search for induced subgraphs in which the vertices are partitioned into independent sets and vertices belonging to different classes of the partition are adjacent. Moreover, the classes are linearly ordered and every edge is directed in accordance with the order of its two endpoints so as to point in the direction of the class that comes later in the order. Induced subgraphs with this property will be called waterfalls. Their growth in product graphs is one of our central problems in this section.

Initially, we shall be interested in oriented graphs, a case for which our problems are easier. To the other extreme, we will later treat symmetrically directed graphs, which will be more conveniently considered as undirected graphs. Although our problems can be stated more generally, for arbitrary directed graphs, still, we will not discuss these in detail.

2.1.2 Waterfalls

The (logarithm of the) clique number is a simple and altogether not very bad lower bound on the (logarithmic) Shannon capacity of an undirected graph. The core of the problem, however, lies in the fact that in product graphs larger cliques exist than just the Cartesian powers of the cliques in the graph itself. In the present case the situation will be similar in some sense.

Let G be an oriented graph with vertex set V and n be a natural number. An induced subgraph W of G will be called a waterfall if its vertices can be colored by natural numbers in such a way that vertices of the same color are non–adjacent, while if the vertex x gets a smaller color than the vertex y , then this implies that $(x, y) \in E(G)$.

A directed graph G with vertex set V defines a directed graph G^n on V^n in which $\boldsymbol{x} = x_1x_2 \dots x_n$ and $\boldsymbol{y} = y_1y_2 \dots y_n$ are connected by a directed edge pointing from $\boldsymbol{x}$ to $\boldsymbol{y}$ if for at least one i, $1 \le i \le n$, we have $(x_i, y_i) \in E(G)$. Notice that we have called $\boldsymbol{x}$ and $\boldsymbol{y}$ G–incomparable if both $(\boldsymbol{x}, \boldsymbol{y})$ and $(\boldsymbol{y}, \boldsymbol{x})$ form a directed edge in G^n. We call G^n the n'th co–normal power of G. It is the natural generalization to directed graphs of the co–normal power of (undirected) graphs with which it coincides in case the graph is symmetrically directed.

The previous definition of a waterfall carries through literally to the case of directed graphs.

Definition 20 *An induced subgraph W of the directed graph $G = (V(G), E(G))$ is called a* waterfall *if its vertices can be colored with natural numbers in such a way that two vertices x and y getting the same color are non–adjacent, while if the color of x is a smaller integer than that of y then necessarily $(x, y) \in E(G)$. (Notice that the latter does not imply the absence of the reversed edge $(y, x) \in E(G)$.)*

Denote by $W(G)$ the maximum cardinality (number of vertices) of a waterfall in G and call it the waterfall number of G.

Waterfalls give a simple construction and an easy lower bound for the antichain capacity of G which is the content of the following lemma.

Lemma 2.1.1 *For every natural n the antichain capacity $A(G)$ of the oriented graph G is bounded from below by the quantity*

$$\frac{1}{n} \log W(G^n).$$

Proof. Fix some n and a waterfall induced by G^n on some set $W \subseteq [V(G)]^n$ with the property $|W| = W(G^n)$. For every natural m consider those sequences in $W^{m|W|}$ in which every element of W occurs m times. One easily sees that, for every natural m, this set of sequences from $W^{m|W|}$ is an antichain for G with elements from $[V(G)]^{nm|W|}$. The cardinality of this antichain, as $m \to \infty$, gives the lower bound stated in the lemma. □

It is easy to see that any co–normal power of a waterfall is again a waterfall, whence

$$W(G^n) \geq [W(G)]^n$$

for any directed graph G.

As we shall see later equality holds above in several special cases, but not in general. This justifies the following

Definition 21 *For an arbitrary directed graph G let $\Theta(G)$ denote the always existing limit*

$$\Theta(G) = \lim_{n\to\infty} \frac{1}{n} \log W(G^n)$$

and call it the cascade capacity *of G.*

The cascade capacity is an obvious lower bound for the antichain capacity of a graph. The bound given by $W(G)$ is not always tight.

Proposition 2.1.2 *For an arbitrary oriented graph G we have*

$$\log W(G) \leq A(G)$$

where strict inequality can also occur.

Proof. Except for an example for the strict inequality, the assertion follows from Lemma 2.1.1. A case of strict inequality is given by the cyclically oriented pentagon graph C_5. Clearly, we have $W(C_5) = 2$, while we will see in a moment that C_5^2 has an antichain of cardinality 5. To see this, denote by the numbers 0,1,2,3,4 the vertices of the pentagon in the order of the cyclic orientation of the graph. Then C_5^2 induces an antichain on the subset of $\{0, 1, 2, 3, 4\}^2$ the elements of which are (00), (14), (23), (32), (41). $\square$

To obtain upper bounds on the antichain capacity of an oriented graph we introduce an auxiliary graph in the next subsection.

2.1.3 Independence graphs

To an arbitrary oriented graph G we associate an undirected graph G^* as follows. The vertex set of G^* is the set of all pairs (x, A) such that x is a vertex of G, A is an independent set of G and A contains x. The pairs (x, A) and (x', A') are adjacent in G^* if either A and A' are disjoint and G induces a waterfall with non-empty edge set on $A \cup A'$, or $A = A'$ but $x \neq x'$.

Our aim in this subsection is to prove that the logarithm of the fractional chromatic number $\chi^*(G^*)$ of the independence graph of G is an upper bound for its antichain capacity.

For simplicity, we will first prove the following weaker statement.

Theorem 2.1.3 *For an arbitrary oriented graph G we have*

$$A(G) \leq \log \chi(G^*),$$

where $\chi(G^*)$ is the chromatic number of the undirected graph G^*. More precisely, we have

$$M(G,n) \leq [\chi(G^*)]^{n-1}\alpha(G).$$

Proof. Let $C \subseteq V^n$ be an arbitrary antichain of maximum size induced by G^n on some subset of V^n that we consider fixed in the rest of the proof. For an arbitrary $\boldsymbol{x} \in V^{n-1}$ we define

$$J(\boldsymbol{x}) = \{a: \quad a \in V, \quad a\boldsymbol{x} \in C\},$$

where $a\boldsymbol{x}$ is the sequence obtained in V^n by prefixing a to $\boldsymbol{x}$. One easily sees that the elements of $J(\boldsymbol{x})$ form an independent set in G, whatever the sequence $\boldsymbol{x}$ is. For any independent set A in G we define

$$C(A) = \{\boldsymbol{x} : \boldsymbol{x} \in V^{n-1}, \quad J(\boldsymbol{x}) = A\}.$$

Then, clearly, for every independent set A of G the set $C(A)$ is either empty or an antichain of G^{n-1}. Moreover, for different independent sets A the corresponding sets $C(A)$ are disjoint. Denoting the set of independent sets of G by $S(G)$ this immediately implies that

$$M(G,n) = \sum_{A \in S(G)} |A||C(A)| = \sum_{(x,A) \in V(G^*)} |C(A)|. \tag{2.1}$$

However, much more than $C(A)$ being an antichain is true. Consider namely an arbitrary vertex–coloring of G^* and fix some color class in it. For brevity's sake this will be referred to as the first color class of G^*. Let the independent sets $A_1, \dots, A_k$ (k arbitrary) of G have the property that for some vertices $z_1, \dots, z_k$ of G all of $(z_1, A_1), \dots, (z_k, A_k)$ belong to the first color class of G^*. In order to prove the theorem it will be sufficient to show that $\bigcup_{i=1}^k C(A_i) \subseteq V^{n-1}$ is an antichain of G^{n-1}. This is what we will do next.

(The above is in fact sufficient for it will imply that

$$M(G,n) \leq \chi(G^*) M(G, n-1),$$

whence the statement follows by iteratated application of this inequality.)

Let $A_1, \dots, A_k$ be as above. We have to prove that for any $\boldsymbol{x}, \boldsymbol{y} \in \bigcup_{i=1}^k C(A_i)$ the $(n-1)$-length sequences $\boldsymbol{x}, \boldsymbol{y}$ are G-unrelated. If $\boldsymbol{x}, \boldsymbol{y}$ are elements of the same $C(A_i)$ then this follows from the observation above. Otherwise assume without loss of generality that $\boldsymbol{x} \in C(A_1)$ and $\boldsymbol{y} \in C(A_2)$. Since (z_1, A_1) and (z_2, A_2) are not adjacent in G^* either there exists an $a \in A_1$ and a $b \in A_2$ that are non–adjacent (including the possibility of $a = b$) in G, or else one of the two sets, say A_2, has an element b such that for some two different vertices of G, $c \in A_1$, $d \in A_1$ we have

$$(b,c) \in E(G) \qquad (d,b) \in E(G).$$

Consider the first hypothesis first. The sequences $a\boldsymbol{x}$ and $b\boldsymbol{y}$ belong to C and hence are G–unrelated. Since a and b are non–adjacent, this implies that also $\boldsymbol{x}$ and $\boldsymbol{y}$ are G–unrelated as we have claimed. In case of the second hypothesis, suppose indirectly, that $\boldsymbol{x}$ and $\boldsymbol{y}$ are not G–unrelated. How can this happen? One reason might be that $(x_i, y_i) \in E(G)$ for some i, while $(y_j, x_j) \in E(G)$ never occurs for any j. But then $d\boldsymbol{x}$ and $b\boldsymbol{y}$ would not be G–unrelated, a contradiction. The only other possibilty is if $(y_i, x_i) \in E(G)$ for some i, whereas $(x_j, y_j) \in E(G)$ does not occur for any j. In this case $c\boldsymbol{x}$ and $b\boldsymbol{y}$ do not satisfy the condition of being G–unrelated. This contradiction completes the proof that $\bigcup_{i=1}^{k} C(A_i)$ is an antichain of G^{n-1}.

Consider now an optimal coloring of the vertices of G^*. For any color $j \in \{1, 2, \ldots, \chi(G^*)\}$ denote by $\mathcal{R}(j)$ the family of all the independent sets A of G appearing in the color class j for some vertex $x \in A$. Rephrasing what we just proved we see that

$$\sum_{A \in \mathcal{R}(j)} |C(A)| \leq M(G, n-1),$$

for $j = 1, 2, \ldots, \chi(G^*)$. Substituting these inequalities into (2.1) we get

$$M(G, n) \leq \chi(G^*) M(G, n-1)$$

as claimed.

In order to prove our more precise second statement it suffices to notice that any antichain in G itself is an independent set. $\square$

Theorem 2.1.4 *For an arbitrary oriented graph G we have*

$$A(G) \leq \log \chi^*(G^*).$$

Proof. For fixed k and a k–fold covering of the vertices of G^* by independent sets of G^* the previous argument applies. $\square$

Theorem 2.1.3 implies the inequality between the leftmost and rightmost quantities in the following proposition that we state for later reference.

Proposition 2.1.5 *For any oriented graph G we have*

$$W(G) \leq \omega(G^*) \leq \chi(G^*).$$

Proof. The first inequality follows by observing that a waterfall in G gives rise to a clique of the same size in G^*. The second inequality is true and well-known for any graph in place of G^*. $\square$

We do not always have equality even in the first inequality of Proposition 2.1.5. The simplest example for strict inequality is provided by C_3, the triangle graph with cyclically oriented edges. For this particular graph we have $W(C_3) = 2$ while $\omega(C_3^*) = 3$. Going back to our starting point we can show however that equality holds in both inequalities for every oriented graph that describes the precedence relations of a poset. In other words this means that for a poset P the antichain capacity is determined in terms of the waterfall number of the corresponding graph. We can also prove equality for bipartite graphs (irrespective of their orientation). Both of these cases will be discussed in Subsection 2.1.5.

2.1.4 Waterfalls in undirected graphs

Beyond their application to derive lower bounds for antichain capacity, waterfalls interest us on their own. We are interested in the study of the asymptotic growth of waterfalls in undirected graphs and in particular in finding classes of graphs for which the waterfall number is multiplicative for powers of the graph.

A symmetrically directed graph can be identified with an undirected graph on the same vertex set in the obvious manner. Although a waterfall and the waterfall number have already been defined for this case, we find it useful to start from scratch and give an equivalent reformulation of our problems.

Let G be an undirected graph with vertex set V. Any induced subgraph of G which is a complete k–partite graph for some k will be called a *waterfall* in G. As before, the largest cardinality of any waterfall of G is called its waterfall number, denoted by $W(G)$. We are interested in determining the exponential asymptotics of $W(G^n)$, where G^n is the n'th co-normal power of the undirected graph G. Our problem has the following quite natural interpretation. If the absence of an edge is understood as a relation of "sameness" (indistinguishability) then the waterfall number is the cardinality of the largest set on which this relation is an equivalence relation. Thus our question amounts to examine the natural extension of the relation of "sameness" to n–length sequences of symbols from the point of view of the growth of the cardinality of the largest equivalence relation induced on some subset of V^n. In what follows we will derive upper and lower bounds for the cascade capacity

$$\Theta(G) = \lim_{n\to\infty} \frac{1}{n} \log W(G^n)$$

of arbitrary undirected graphs. We would be interested in seeing for what classes of graphs do we have

$$\Theta(G) = \log W(G). \tag{2.2}$$

We will show that the above relation is true for comparability graphs and odd cycles.

Once again, we will bound from above waterfall numbers for the powers of the undirected graph G in terms of what we call its independence graph, yet our previous setup does not have an automatic reformulation for the undirected case. We have to be careful, as it can already be seen from the fact that the symmetrically directed graph corresponding to an undirected graph would have an antichain capacity equal to the logarithm of the cardinality of its vertex set, and thus our Theorem 2.1.3 cannot be saved in the present context. We define the *independence graph* G^* of an arbitrary undirected graph G. The vertex set of G^* is the set of all pairs (x, A) such that x is a vertex of G contained in the independent set A of G. The pairs (x, A), (x', A') are adjacent in G^* if either A and A' are disjoint and every vertex of A is joined by an edge with every vertex of A', or $A = A'$ but $x \neq x'$. We will say that the independent sets A and A' are *isochromatic* if not every vertex of A is adjacent to all the vertices of A'.

Our aim in this subsection is to prove that the chromatic number $\chi(G^*)$ of the independence graph of G is an upper bound of all the renormalized waterfall numbers $\sqrt[n]{W(G^n)}$

of the powers of G. The proof of this is not unlike but more complicated than that of Theorem 2.1.3. The reason of this complication is that in the undirected setting we will need to consider triples of sequences in situations where it was enough to look at pairs in the oriented case. We will start by establishing a few lemmas.

Unless otherwise stated, in what follows we will fix a graph G with vertex set $V = V(G)$, an integer $n > 0$ and a set $C \subseteq V^n$ on which the graph G^n induces a waterfall.

Observe that a waterfall cannot contain an induced subgraph on 3 points with exactly one edge. It is easy to see that this property characterizes waterfalls. In the subsequent lemmas we will make repeated use of this fact and in doing so we will refer to this 3–point graph as the *forbidden configuration*.

Lemma 2.1.6 *For an arbitrary $\boldsymbol{x} \in V^{n-1}$ write*

$$J(\boldsymbol{x}) = \{a : \quad a \in V, \quad a\boldsymbol{x} \in C\},$$

where $a\boldsymbol{x}$ denotes the juxtaposition of a and the sequence $\boldsymbol{x}$. Then $J(\boldsymbol{x})$ is a waterfall in G.

Proof. A forbidden configuration in $J(\boldsymbol{x})$ would generate a forbidden configuration in C. □

To any independent set $A \subseteq V$ of G we associate the set $C(A)$ consisting of those sequences $\boldsymbol{x} \in V^{n-1}$ for which A is a maximal independent set in the waterfall $J(\boldsymbol{x})$. One easily sees that $C(A)$ is a waterfall for every A.

Lemma 2.1.7 *If the distinct independent sets A and B of G are isochromatic then $C(A)$ and $C(B)$ are disjoint and G^{n-1} induces a waterfall on their union.*

Proof. Observe first of all that $C(A)$ and $C(B)$ are disjoint. In fact, if they had a common element $\boldsymbol{y}$, say, then $A \cup B$ would be a subset of $J(\boldsymbol{y})$ and thus a waterfall. If $A \cup B$ were independent, then at least one of A and B would not be a maximal independent set in $J(\boldsymbol{y})$ contradicting $\boldsymbol{y} \in C(A)$ or $\boldsymbol{y} \in C(B)$. Thus $A \cup B$ is not independent but as a subset of the waterfall $J(\boldsymbol{y})$ should form a waterfall itself. Therefore every element of A is adjacent to every element of B contradicting that they are isochromatic. Thus $C(A)$ and $C(B)$ are indeed disjoint.

Suppose now to the contrary that A and B are isochromatic independent sets of G and yet G^{n-1} does not induce a waterfall on $C(A) \cup C(B)$. This means that there is a forbidden configuration in G^{n-1} with vertices $\boldsymbol{x}$, $\boldsymbol{y}$ and $\boldsymbol{z}$ in $C(A) \cup C(B)$. Suppose in particular that $\boldsymbol{x}$ and $\boldsymbol{y}$ form an edge. Further, since A and B are isochromatic, there must be an $a \in A$ and a $b \in B$ such that $(a, b) \notin E(G)$. If in the forbidden configuration the two endpoints of the edge are contained one in $C(A)$ and the other in $C(B)$, i.e., $\boldsymbol{x} \in C(A)$ and $\boldsymbol{y} \in C(B)$, say, then $a\boldsymbol{x}$, $b\boldsymbol{y}$ and $a\boldsymbol{z}$ or $b\boldsymbol{z}$ (depending on whether $\boldsymbol{z}$ belongs to $C(A)$ or $C(B)$) form a forbidden configuration in C, yielding a contradiction. Suppose

therefore that the endpoints $\boldsymbol{x}$ and $\boldsymbol{y}$ of the edge in the forbidden configuration induced by G^{n-1} on $C(A) \cup C(B)$ fall both into the same set; $C(A)$ or $C(B)$. We can suppose without loss of generality that they both fall into $C(A)$. Then if $\boldsymbol{z} \in C(B)$, we find that $a\boldsymbol{x}$, $a\boldsymbol{y}$ and $b\boldsymbol{z}$ form a forbidden configuration in C, once again a contradiction. Finally, if $\boldsymbol{z} \in C(A)$, the desired contradiction is achieved by the forbidden configuration $a\boldsymbol{x}, a\boldsymbol{y}, a\boldsymbol{z}$.
$\square$

Lemma 2.1.8 *If for the independent sets A and B of G there exist $\boldsymbol{x} \in C(A)$ and $\boldsymbol{y} \in C(B)$ such that $\{\boldsymbol{x}, \boldsymbol{y}\} \notin E(G^{n-1})$, then G induces a waterfall on $A \cup B$.*

If further A and B are isochromatic, then $A \cup B$ is an independent set of G.

Proof. By our hypothesis, there exist $\boldsymbol{x} \in C(A)$, $\boldsymbol{y} \in C(B)$ such that $\{\boldsymbol{x}, \boldsymbol{y}\} \notin E(G^{n-1})$. To reason indirectly, suppose that G does not induce a waterfall on $A \cup B$. Then, without loss of generality we can suppose that $a \in A$, $b \in B$, $c \in B$ form a forbidden configuration in which $\{a, b\} \in E(G)$. Then, however, $a\boldsymbol{x}$, $b\boldsymbol{y}$, $c\boldsymbol{y}$ form a forbidden configuration in C, yielding a contradiction.

As for the second statement of the lemma, it is immediate from the foregoing that if A and B are isochromatic independent sets in G and the latter induces a waterfall on $A \cup B$, then A and B cannot form a complete bipartite graph and hence the union of A and B must be an independent set of G.

$\square$

Now we are ready to prove our main lemma:

Lemma 2.1.9 *If each pair of the distinct independent sets A, B, and D of G are isochromatic, then G^{n-1} induces a waterfall on $C(A) \cup C(B) \cup C(D)$.*

Proof. By Lemma 2.1.7 the sets $C(A)$, $C(B)$ and $C(D)$ are disjoint and G^{n-1} induces a waterfall on the union of any two of them. We reason indirectly, and suppose that despite the above, G^{n-1} does not induce a waterfall on $C(A) \cup C(B) \cup C(D)$. By the foregoing, this can only happen if there are 3 sequences falling into the 3 different sets, $\boldsymbol{x} \in C(A)$, $\boldsymbol{y} \in C(B)$ and $\boldsymbol{z} \in C(D)$ on which the graph G^{n-1} induces a forbidden configuration. Without loss of generality, we can suppose that in this configuration $(\boldsymbol{x}, \boldsymbol{y}) \in E(G^{n-1})$. By Lemma 2.1.8 this implies that both $D \cup A$ and $D \cup B$ are independent sets of G. Hence there exist 3 vertices, $d \in D$, $a \in A$ and $b \in B$ such that d is not adjacent to either a or b. But then $d\boldsymbol{z}$, $a\boldsymbol{x}$, $b\boldsymbol{y}$ form a forbidden configuration in C, a contradiction. $\square$

The last three lemmas immediately yield

Theorem 2.1.10

$$W(G^n) \leq [\chi(G^*)]^n.$$

Proof. Clearly, it is enough to show that

$$W(G^n) \leq \chi(G^*) W(G^{n-1}).$$

To prove this inequality, consider a waterfall $C \subseteq G^n$ of maximum cardinality. Fix a coloring of the vertices of the independence graph of G with the minimum number of colors $\chi(G^*)$. Fix also an arbitrary color class of this coloring and consider all the independent sets A of G such that there is a vertex x of A for which the vertex (x, A) of G^* belongs to this color class. By the definition of the independence graph all these independent sets A from the fixed color class will occur just once among these pairs and any two of them will be isochromatic. Hence, by Lemma 2.1.7 the subgraphs induced by G^{n-1} on the corresponding sets $C(A)$ will all be vertex–disjoint. Let us denote, for every color $j \in \{1, 2, \ldots, \chi(G^*)\}$ by $\mathcal{R}(j)$ the family of all the independent sets A for which the color of (x, A) is j for some x. Then, clearly,

$$|C| \leq \sum_{j=1}^{\chi(G^*)} \sum_{A \in \mathcal{R}(j)} |C(A)|. \tag{2.3}$$

Observe, however, that by Lemma 2.1.9 the graph G^{n-1} induces a waterfall on the union of all those sets $C(A)$ for which the corresponding independent set A belongs to the same fixed color class. (In fact, G^{n-1} induces a waterfall on any three of them by said lemma. But any possible forbidden configuration would be contained in the union of at most three among the sets $C(A)$.) Hence our last inequality implies that

$$W(G^n) = |C| \leq \chi(G^*) W(G^{n-1}).$$

To conclude the proof, we notice that for $n = 1$

$$W(G) \leq \omega(G^*) \leq \chi(G^*).$$

□

Remark 2.1. Just as in the oriented case, we can prove the stronger statement

$$W(G^n) \leq [\chi^*(G^*)]^n$$

using the same ideas as above. ♢

We conclude this subsection with a last remark about waterfalls. In general, it is interesting to study the growth of the largest subgraph of a certain type in product graphs. If we restrict ourselves to classes closed under the co–normal product, we get supermultiplicative (including multiplicative) behavior. If we further restrict ourselves to hereditary classes, i.e., those which contain every induced subgraph of their elements then only four infinite classes remain as we will see in the next proposition. One of these classes is the trivial class containing all graphs. Another one is the class of empty graphs for which the growth in products is always multiplicative as mentioned in Subsection 2.1.1. The third class is the class of complete graphs, the corresponding question of asymptotic growth in products is the Shannon capacity problem. The only remaining nontrivial class is the class of waterfalls for which the question of asymptotic growth in products leads to the problem of determination of cascade capacity.

Proposition 2.1.11 *The largest class of graphs that is different from the class of all graphs and has the property of being closed under the two operations of taking co–normal products and taking induced subgraphs is the class of waterfalls. Moreover, the only other nontrivial such classes of graphs are the classes containing all empty graphs and all complete graphs, respectively.*

Proof. We prove the first statement first. It is easy to see that the class of waterfalls is closed under the above two operations. What we really have to prove is that a larger class of graphs with this property should contain every graph. To this end we first show that in the co–normal powers of our forbidden configuration every graph appears as an induced subgraph. Let F be an arbitrary graph and Z be the graph of our forbidden configuration, a graph on vertices 0, 1, and 2, with a single edge between 1 and 2. We build up $|V(F)|$ ternary sequences of length $|V(F)|-1$ over $V(Z)$ with the property of inducing a graph isomorphic to F in $Z^{|V(F)|-1}$. For simplicity of the presentation we assume that we already know which sequence will belong to which vertex of F and we will refer to the sequence belonging to the ith vertex of F (according to some arbitrary order) as the ith sequence. Now let the ith coordinate ($1 \le i \le |V(F)|-1$) of the sequences be given as follows. It is 1 for the ith sequence, 2 for the sequences belonging to vertices that are adjacent with the ith vertex and 0 for the rest. It is easy to check that the jth and kth sequences are adjacent in $Z^{|V(F)|-1}$ if and only if the jth and kth vertices are adjacent in F. Now suppose we have a class of graphs that is closed under the two operations we have in the statement. If it contains a graph that is not a waterfall then it necessarily contains Z and so by the foregoing it should contain every graph.

For the second statement notice that once we have a non-empty but not-complete graph in our class then it also contains the graph with three vertices and two edges. It is easy to verify that the co–normal products of this graph contain every possible waterfall as an induced subgraph.

Finally, we have to add that any class of graphs closed under our two operations and containing a graph with two independent vertices must contain all edgeless graphs. Similarly, if such a class contains a graph with two adjacent vertices it must contain all complete graphs. Thus the only non-empty class not listed in the statement is the class containing only the trivial graph on a single point.

□

2.1.5 When are the bounds tight?

We shall call a directed graph G conservative if every induced subgraph $F \subseteq G$ satisfies

$$\Theta(F) = \log W(F). \tag{2.4}$$

This definition is inspired by the concept of perfect graphs. Our bounds allow us to show that several classes of graphs are conservative. First we will show, as promised at the end of Subsection 2.1.3, that comparability digraphs (see definition below) and oriented bipartite graphs give equality in the inequalities of Proposition 2.1.5. This immediately implies the conservativeness of these graphs. Then we will deal with undirected graphs again, showing that comparability graphs and odd cycles are conservative. An example found by Tomasz Łuczak shows the existence of non-conservative undirected graphs. His example can easily be extended to an example of a non-conservative oriented graph once we have Proposition 2.1.17. The section concludes with an application to extremal set theory.

Let us first recall the definition of a comparability graph and its extension to digraphs.

A graph G is a comparability graph if there exists a poset P on $V(G)$ for which $u, v \in V(G)$ are comparable if and only if they are adjacent in G. We call G a comparability digraph if there exists a poset P on $V(G)$ in which $u \in V(G)$ precedes $v \in V(G)$ in P if and only if there is an edge from u to v in G.

It is well known that every comparability graph is perfect, cf. e.g. [14].

It is straightforward from the definition of the independence graph that

$$\omega(G^*) = W(G) \tag{2.5}$$

for every directed graph G that does not contain a cyclically oriented triangle.

We will associate with an arbitrary digraph G its reduced independence graph $G^{(*)}$ through the following definition. The vertex set of $G^{(*)}$ consists of the independent sets of G and two vertices are adjacent in $G^{(*)}$ if and only if the corresponding independent sets form a waterfall which is not a single independent set in G. (In other words, they form a complete bipartite graph.) Notice that G^* can be obtained from $G^{(*)}$ if we substitute to each vertex a clique of the size of the independent set it represents in $G^{(*)}$.

Theorem 2.1.12 *For any comparability digraph G we have*

$$W(G) = \chi(G^*).$$

Proof. We claim that if G is a comparability digraph then $G^{(*)}$ will be a comparability graph. Indeed, it is easy to check that the following is a partial order on the independent sets of G. Let the independent set A precede the independent set B if every element of A precedes every element of B in the original partial order (cf. [45]). The comparability graph defined by this partial order is just $G^{(*)}$. This implies that $G^{(*)}$ is perfect. Using the result of Lovász [106] saying that the substitution of a vertex of a perfect graph by another perfect graph (in fact, by a clique here) preserves perfectness, we see that G^* is perfect. This implies $\chi(G^*) = \omega(G^*)$ and since a comparability digraph cannot contain a cyclically oriented triangle the statement follows by (2.5). $\square$

Corollary 2.1.13 *The antichain capacity of a partial order equals* $\log W(G)$ *where* G *is the comparability digraph of the partial order.*

Proof. The statement immediately follows from Proposition 2.1.2 and Theorem 2.1.3 by the previous theorem. $\square$

Corollary 2.1.14 *Every comparability digraph is conservative.*

Proof. We already know by the foregoing that for any oriented graph F we have

$$\log W(F) \leq \Theta(F) \leq A(F) \leq \log \chi(F^*) \tag{2.6}$$

and so the previous theorem and the trivial observation that every induced subgraph of a comparability digraph is a comparability digraph implies the statement. $\square$

Another class of oriented graphs for which we can prove the tightness of the upper bound in Theorem 2.1.3 is the class of oriented bipartite graphs (regardless of the actual orientation of the edges).

Theorem 2.1.15 *For any oriented bipartite graph* G *we have*

$$A(G) = \log W(G).$$

Proof. What we will actually prove is again the statement $\chi(G^*) = W(G)$. By Theorem 2.1.3 this implies what we need. Consider the reduced independence graph $G^{(*)}$. We claim that if G is bipartite then so is $G^{(*)}$. To see this consider a covering of the nodes of G with two independent sets, A and B. (Such a covering exists since G is bipartite.) Each vertex of $G^{(*)}$ represents an independent set of G that should have a non-empty intersection with at least one of A and B. Now look at those independent sets whose intersection with A is non-empty. No pair of them can induce a waterfall with two independent sets, since they have non-adjacent (or possibly common) vertices, those that are in A. So all these independent sets are non-adjacent as nodes of $G^{(*)}$. However, all other nodes of $G^{(*)}$ represent independent sets intersecting B and thus they are non-adjacent for the same reason. This proves that $G^{(*)}$ is bipartite. Since a bipartite graph is always perfect and cannot contain any triangle (cyclic or not), we conclude that $\chi(G^*) = W(G)$ as in the proof of Theorem 2.1.12. $\square$

Corollary 2.1.16 *Every oriented bipartite graph is conservative.*

Proof. The proof is identical with that of Corollary 2.1.14 if replacing Theorem 2.1.12 by Theorem 2.1.15. □

Before turning to undirected graphs we make a final observation about the oriented case.

It is also natural to ask whether the growth of the largest transitive tournament (which is a special waterfall) has a multiplicative nature. The answer is no and a counterexample is given by a specific orientation of C_5. The size of the largest transitive tournament in this graph is obviously two.

Proposition 2.1.17 *There exists an orientation of C_5 for which the second power of the resulting graph contains a transitive tournament on five points.*

Proof. Let the vertices be $0, 1, 2, 3, 4$ in a cyclic order. The orientation now is not cyclic, however, and the edges are oriented as follows:$(0, 1), (0, 4), (2, 1), (2, 3), (4, 3)$. One easily checks that the vertices (00), (24), (12), (43) and (31) induce a transitive tournament in the second power of the graph. □

Remark 2.2. The above observation also implies, that the Sperner capacity of the pentagon oriented as in the proof equals its Shannon capacity, $\frac{1}{2}\log 5$. Rob Calderbank has shown us [27] by their method [28] that all the other oriented versions of the five length cycle have Sperner capacity 1. ♢

(We mention that a generalization of the first statement in the above remark will be given in the next section (see Theorem 2.2.3), while we will see a generalization of the statement in the second sentence in Section 2.3 (see Theorem 2.3.3).)

Now we come to the analysis of conservativeness in case of undirected graphs. Although an undirected graph can be identified with a symmetrically directed graph in the usual way, let us be explicit again and repeat the definition for this case. We say that an undirected graph G is conservative if every induced subgraph $F \subseteq G$ satisfies (2.4). ($\Theta(F)$ and $W(F)$ are meant as defined for undirected graphs in the previous subsection.)

Corollary 2.1.18 *Bipartite graphs are conservative.*

Proof. The proof of Theorem 2.1.15 can be repeated literally to prove that $\chi(G^{(*)}) = W(G)$ for any bipartite graph G. (In fact, we did not use anything about the orientation when proving Theorem 2.1.15). Then the statement follows from Theorem 2.1.10. □

The previous corollary is also implied by the following more general statement.

Corollary 2.1.19 *Comparability graphs are conservative.*

Proof. Let G be a comparability graph representing some poset. The comparability digraph $\hat{G}$ of this poset is an oriented version of G and one easily checks that any subset of the vertex set inducing a waterfall in G is also a waterfall in $\hat{G}$. Thus $W(\hat{G}) = W(G)$

and $\hat{G}^*$ is isomorphic to G^*. We have $W(\hat{G}) = \chi(\hat{G}^*)$ from Theorem 2.1.12 and thus $W(G) = \chi(G^*)$ also follows. Since induced subgraphs of comparability graphs are also comparability graphs the proof is then completed by Theorem 2.1.10. □

The above arguments may suggest that all perfect graphs G satisfy equality in the inequality $W(G) \leq \chi(G^*)$. This is, however, not the case. A counterexample is provided by $\bar{C}_8$, the complement of the cycle on 8 vertices. Indeed, while $W(\bar{C}_8) = 5$, one needs 6 colors to color $(\bar{C}_8)^*$. To see that 5 colors do not suffice consider the subgraph of $(\bar{C}_8)^*$ induced by the vertices representing the 8 independent sets of cardinality 2 in $\bar{C}_8$. This subgraph has 16 vertices and its independence number is only 3.

Next we show that all cycles are conservative regardless of the parity of their length.

Corollary 2.1.20 *The cycle C_k is conservative for any $k \geq 3$.*

Proof. It is easy to see what C_k^* looks like. The points related to independent sets of size at least 3 form separate components of their own. Each of these components is a clique of the size of the independent set it belongs to. The rest is a 3–chromatic component except for $k = 4$ but that case is already covered by Corollary 2.1.18. Now if $k > 7$ then the chromatic number of C_k^* is determined by the size of the largest previously mentioned clique component. Since its size is that of the largest independent set, i.e., of a special waterfall in C_k, we are done by Theorem 2.1.10. If $4 \neq k \leq 7$ then both $\chi(C_k^*)$ and $W(C_k)$ are 3, so again by Theorem 2.1.10 we are done. □

The following example found by Tomasz Łuczak [114] proves that not every graph is conservative.

Proposition 2.1.21 *(Łuczak) Let D_{10} denote the graph we obtain by substituting a clique of size two to each vertex of C_5, the cycle of length five. This graph is not conservative.*

Proof. It is easy to check that $W(D_{10}) = 4$. On the other hand it is well known that $\omega(C_5^2) = 5$, cf. [139], [109]. If the vertices of C_5 are $0, 1, 2, 3, 4$ in a cyclic order, such a clique is induced by the sequences $00, 12, 24, 31, 43$. Let the vertices of our D_{10} be $0, 0', 1, 1', .., 4, 4'$ where aa' is the clique substituted in place of a of C_5. Then replacing each sequence ab in the above clique of C_5^2 by the four sequences $ab, a'b, ab', a'b'$ we obtain twenty sequences of length 2 inducing a clique in D_{10}^2. This means that $W(D_{10}^2) \geq \omega(D_{10}^2) \geq 20 > [W(D_{10})]^2$. □

Combining the previous example with that of Proposition 2.1.17 one easily obtains an example of a non-conservative oriented graph. Indeed, let us consider the following orientation of the graph D_{10} of the previous proposition. (The vertices are labelled as in the previous proof.) Any edge joining vertices a and a' is oriented from a to a'. Edges joining vertex a or a' to vertex b or b' ($a \neq b$) are oriented as the edge between a and b in the proof of Proposition 2.1.17. Let the resulting oriented graph be denoted by D_{10}^+.

Proposition 2.1.22 *The oriented graph D_{10}^+ is not conservative.*

Proof. It is easy to check that $W(D_{10}^{+}) = 4$. Consider the construction of Proposition 2.1.17 over the five vertices $0, 1, 2, 3, 4$. The five sequences of length 2 obtained this way can be extended to twenty sequences of length 2 in the same way as in Proposition 2.1.21. (To be explicit these twenty sequences are as follows: $00, 00', 0'0, 0'0', 24, 2'4, 24', 2'4', 12, 1'2, 12', 1'2', 43, 4'3, 43', 4'3', 31, 3'1, 31', 3'1'$.) These twenty sequences induce a transitive tournament in the second power of D_{10}^{+} proving the strict supermultiplicativity of the waterfall number for this graph. □

(We mention that the later paper [2] by Alon gives examples of tournaments for which the largest transitive subtournament behaves strictly supermultiplicatively with respect to the co-normal power. As the waterfall number of a tournament is just the size of its largest transitive subtournament, these oriented graphs are also not conservative.)

Some of the results in this section might have applications in extremal set theory as was the case with Sperner capacity. Let us conclude with a simple example.

Corollary 2.1.23 *Given an n–set X, let the family of pairs $\{(A_l, B_l)\}_{l=1}^{m}$ of subsets of X have the properties*

$$A_i \cap B_j = \emptyset \quad \textit{if and only if} \quad A_j \cap B_i = \emptyset,$$

$$A_i \cap B_i = \emptyset, \quad i = 1, 2, \ldots, m.$$

The maximum $M(n)$ of the number m of such pairs from an n–set is 2^n.

Proof. We start by showing that

$$M(n) \leq 2^n.$$

To this end consider the oriented graph G with vertex set $V(G) = \{0, 1, 2\}$ and edge set $E(G) = \{(1, 2)\}$. Now let us have any family of set pairs as in the statement of the proposition. We can identify any set pair (A_l, B_l) with a ternary sequence, i.e., an element of $\{0, 1, 2\}^n$ in the following manner. We identify X with the numbers from 1 to n and define the ternary sequence $\boldsymbol{x} = x_1 x_2 \ldots x_n$ by setting $x_k = 1$ if $k \in A_i$, $x_k = 2$ if $k \in B_i$ and $x_k = 0$ otherwise. Thus we can see that the family of pairs with our property defines an antichain in $[V(G)]^n$ for G and hence, by Theorem 2.1.3 we see that $M(n) \leq 2^n$.

The next observation giving the lower bound $M(n) \geq 2^n$ is due to Zsolt Tuza [155]. Divide X into two disjoint parts A and B any way you like and then consider the family of pairs

$$\{(Y \cap A, Y \cap B) : Y \subseteq X\}.$$

□

For an exhaustive bibliography of extremal problems for set pairs we refer the reader to Tuza's survey [156].

Acknowledgment: We are grateful to Tomasz Łuczak for finding the example presented in Proposition 2.1.21 thereby saving us from the temptation of nice sounding but wrong conjectures.

2.2 Orientations of self-complementary graphs and the relation of Sperner and Shannon capacities

In this section we prove that the edges of a self-complementary graph and its complement can be oriented in such a way that they remain isomorphic as digraphs and their union is a transitive tournament. This result is used to explore the relation between the Shannon and Sperner capacity of certain graphs. In particular, using results of Lovász, we show that the maximum Sperner capacity over all orientations of the edges of a vertex-transitive self-complementary graph equals its Shannon capacity.

This section is based on the joint paper [132] with Attila Sali.

2.2.1 Introduction

The Shannon capacity $C(G)$ of a graph G was defined by Shannon in [139] (see also [109, 6]). It is easy to determine for many graphs, highly non-trivial but known for some others and not even known for another many. Sperner capacity is a generalization of this notion for directed graphs given by Gargano, Körner, and Vaccaro [60]. The motivation of this generalization was the applicability of the new concept in extremal set theory that was carried out quite successfully in [61] and [62]. In this section we are dealing with the connections of these values, the Shannon and Sperner capacities of a graph. Let us recall the definitions first. (The following graph power is the same what we called co-normal power in the previous section. A small change in our notational conventions is that in this section n is kept for denoting the number of vertices of a graph and exponents will usually be denoted by t.)

Definition 22 *Let G be a directed graph on vertex set V. The t^{th} power of G is defined to be the directed graph G^t on vertex set $V^t = \{\boldsymbol{x} = (x_1 \dots x_t) : x_i \in V\}$ with edge set*

$$E(G^t) = \{(\boldsymbol{x}, \boldsymbol{y}) : \exists i \ (x_i, y_i) \in E(G)\}.$$

Recall that G^t may contain edges in both directions between two vertices even if such a pair of edges is not present in G.

Definition 23 *For a directed graph G let $\mathrm{tr}(G)$ denote the size (number of vertices) of the largest transitive tournament that appears as a subgraph of G. The (logarithmic) Sperner capacity of a digraph G is*

$$\Sigma(G) = \lim_{t\to\infty} \frac{1}{t} \log \mathrm{tr}(G^t).$$

We remark that originally the definition of $\Sigma(G)$ was formulated in a different way and we gave that formulation in the previous section. Nevertheless, the two definitions are equivalent, the one we gave here already appears, e.g., in [50]. (Using the $N(G,t)$ notation from Subsection 2.1.1, here is a hint why the equivalence holds. In one hand, $\mathrm{tr}(G^t) \geq N(G,t)$ is trivially true. On the other hand, it is not hard to show that $N(G, mt) \geq (\mathrm{tr}(G^t))^m/(m+1)^{\mathrm{tr}(G^t)}$, thus the m^{th} root of $N(G, mt)$ tends to $\mathrm{tr}(G^t)$ if t is fixed and m goes to infinity.)

The Shannon capacity of an undirected graph G can be defined as the following special case of Sperner capacity. Let us call (again) a graph symmetrically directed if for each of its edges it also contains the edge going opposite way between the same two endpoints. Just as in the previous section we often identify an undirected graph G with the symmetrically directed graph that has edges (in both directions) between the same endpoints as G has. This digraph is called the symmetrically directed equivalent of G. Note, that powers of a symmetrically directed graph are also symmetrically directed, hence they can be considered as undirected graphs, as well.

Definition 24 *The Shannon capacity $C(G)$ of G is the Sperner capacity of its symmetrically directed equivalent.*

Notice that writing undirected edges instead of directed ones in Definition 22 and the clique number $\omega(G^t)$ in place of $\mathrm{tr}(G^t)$ in Definition 23 we get $C(G)$ in place of $\Sigma(G)$.

It should be clear from the definitions that the Sperner capacity of a digraph is always bounded from above by the Shannon capacity of the underlying undirected graph. The two values are not the same in general. It was first shown in [28] (cf. also [23] for a short and different proof) that the Sperner capacity of a cyclically oriented triangle is $1 (= \log 2)$ while the Shannon capacity of K_n is $\log n$ in general, i.e., $\log 3$ for a triangle.

For an undirected graph G let

$$D(G) = \max_{\hat{G}} \Sigma(\hat{G}),$$

where $\hat{G}$ stands for an oriented version of G, i.e., the maximum is taken over all oriented graphs $\hat{G}$ containing exactly one oriented edge for each edge of G. Clearly $D(G) \leq C(G)$ holds. Our main concern is the question whether this inequality is an equality or not.

2.2.2 $D(G)$ versus $C(G)$

It was already proved by Shannon that $C(G)$ satisfies $\log \omega(G) \leq C(G) \leq \log \chi(G)$ where $\chi(G)$ is the chromatic number of the graph G. (In fact, he proved more, namely, though in different terms, that the logarithm of the fractional chromatic number of G is also an upper bound for $C(G)$. For further details of this, see [96]. We remark again that Shannon and many authors following him used a complementary language, i.e., defined $C(\bar{G})$ as we defined $C(G)$. (Cf. our approach in Section 1.3 and Remark 1.5 therein.) The two approaches are equivalent, our reason to break the tradition is our need to orient edges that would become non-edges in the original language.) It is easy to observe that

$\mathrm{tr}(G^t) \geq (\mathrm{tr}(G))^t$, thus $\Sigma(G) \geq \log \mathrm{tr}(G)$ always holds. (Shannon's $\log \omega(G) \leq C(G)$ is a special case of this.) This inequality is not an equality in general even among tournaments as was shown by Alon [2]. On the other hand, a clique of an undirected graph can always be oriented in an acyclic way, thus giving an induced transitive tournament. Therefore the previous inequalities imply that if $\chi(G) = \omega(G)$ for G then $D(G) = C(G)$ also holds. In particular, this happens for all perfect graphs.

The smallest graph for which Shannon was not able to determine the value of its capacity is C_5, the chordless cycle on five points. He observed that $\omega(C_5^2) \geq 5 > (\omega(C_5))^2 = 4$ implying $C(C_5) \geq \frac{1}{2}\log 5$. The theorem that this lower bound is sharp was proven by Lovász in his celebrated paper [109]. Lovász proved there the more general result that any vertex-transitive self-complementary graph on n points has Shannon capacity $\frac{1}{2}\log n$.

It was already observed in [59] (see Proposition 2.1.17 in the previous section) that C_5 has an orientation for which C_5^2 contains an acyclically oriented clique on five points. This implies $D(C_5) \geq \frac{1}{2}\log 5$ and thus by Lovász' result $D(C_5) = C(C_5)$. It is worth mentioning that this orientation of C_5 is unique (up to isomorphism); for all other orientations the Sperner capacity is strictly smaller. This can be shown using the methods of [28] and [23], as it was shown to us by Rob Calderbank [27]. (See Theorem 2.3.3 of the next section for a more general result.)

The main result of this section is a generalization of the above observation, showing that for any self-complementary graph G on n points the edges can be oriented in such a way that G^2 contains a transitively oriented clique of size n. By the above mentioned result of Lovász this will immediately imply $D(G) = C(G)$ for all vertex-transitive self-complementary graphs. Another implication, due to the work of Alon and Orlitsky [6], is that $D(G)$ can be exponentially larger than $\log \omega(G)$.

The core of our result is a theorem about self-complementary graphs that we think to be interesting in itself. This is the topic of the next subsection.

2.2.3 Self-complementary graphs

A graph $G = (V, E)$ is self-complementary if there exists a complementing permutation of the elements of V. That is, there exists a bijection $\tau\colon V \to V$ with the property that for all $v, w \in V, v \neq w$ we have $\{v, w\} \in E$ if and only if $\{\tau(v), \tau(w)\} \notin E$. A characterization of self-complementary graphs can be found in [130] or [131], cf. also [67].

Let G be a self-complementary graph with complementing permutation τ. Then the set $\{(v, \tau(v)) : v \in V\}$, or equivalently, the set $\{(\tau(v), v)) : v \in V\}$ of pairs (two-length sequences) induces a clique of size $|V| = n$ in G^2. Using these two-length sequences as building blocks, we can find cliques of size $n^{\frac{t}{2}}$ in G^t for every even t. This shows $C(G) \geq \frac{1}{2}\log n$ and by Lovász' result this is sharp if G has the additional property of being vertex-transitive (cf. Theorem 12 in [109]). If we could orient the edges of G in such a way that for the so obtained oriented graph $\hat{G}$ at least one of the cliques of the above type in $\hat{G}^2$ would become a transitive tournament then we would have $D(G) \geq \frac{1}{2}\log n$ and thus $D(G) = C(G)$ for vertex-transitive self-complementary graphs. Therefore we seek for such orientations. The following theorem will imply that this kind of orientation always exists. The proof also gives a construction.

If ρ is a linear order of the elements of the set $\{1,\ldots,n\}$ then $\rho(x)$ denotes the element standing on the x^{th} position in this linear order. Thus $\rho^{-1}(i)$ is the position where element i can be found. We say that j is to the right of i (according to ρ) iff $\rho^{-1}(i) < \rho^{-1}(j)$.

Theorem 2.2.1 *Let $G = (V(G), E)$ be a self-complementary graph on $V(G) = \{1, 2, \ldots, n\}$ with complementing permutation τ. Then there exists a linear order σ on $V(G)$ such that if $\{i,j\} \notin E$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$, then $\sigma^{-1}(\tau^{-1}(i)) < \sigma^{-1}(\tau^{-1}(j))$.*

Proof. The essential part of the argument concerns the case when τ contains only one cycle, the remaining cases can be reduced to this easily. So assume first that τ consists of only one cycle. By results in [130], [131] this implies that n should be even, but this will not be exploited here.

We may assume without loss of generality that $\tau = (123\ldots n)$ and that $\{1,2\}$ is an edge of G. An algorithm will be given, that starting from the identity order, successively rearranges the terms thus generating the linear order σ. We give a more formal description of this algorithm first and explain it afterwards.

The algorithm successively produces the linear orders σ_k ($k = 0, 1, \ldots$) until a certain condition becomes satisfied. Then the lastly produced σ_k will be our final order σ.
We start with σ_0 being the identity order.
Once σ_k is given, σ_{k+1} is defined as follows. (Giving σ_{k+1} from σ_k is the main part of the algorithm that we iterate. Below we refer to it as the "general step".)
Let m be the value of $\sigma_k(n)$ and let $i = \sigma_k^{-1}(m+1)$, where $m+1$ means 1 in case $m = n$, that is in the first step.
Consider the set $A := \{r : i < r \text{ and } \{\sigma_k(r), \sigma_k(n)\} \in E\}$. If $A = \emptyset$ then $\sigma := \sigma_k$ and the algorithm stops.
Otherwise, let $j := \min A$ and define σ_{k+1} as follows:

$$\sigma_{k+1}(s) = \begin{cases} \sigma_k(s) & \text{if } s < j \\ m & \text{if } s = j \\ \sigma_k(s-1) & \text{if } j < s \leq n \end{cases}$$

Then we repeat the foregoing with σ_{k+1} in place of σ_k.
The algorithm stops when the $A = \emptyset$ condition is satisfied. This will certainly happen because $|A|$ becomes smaller in each step as it is explained below.

Now we turn to the less formal description.

In the general step when m stands on the last (rightmost) position in σ_k, we check whether m has a neighbour to the right of $m+1$. If there is one, then m is inserted just in front of

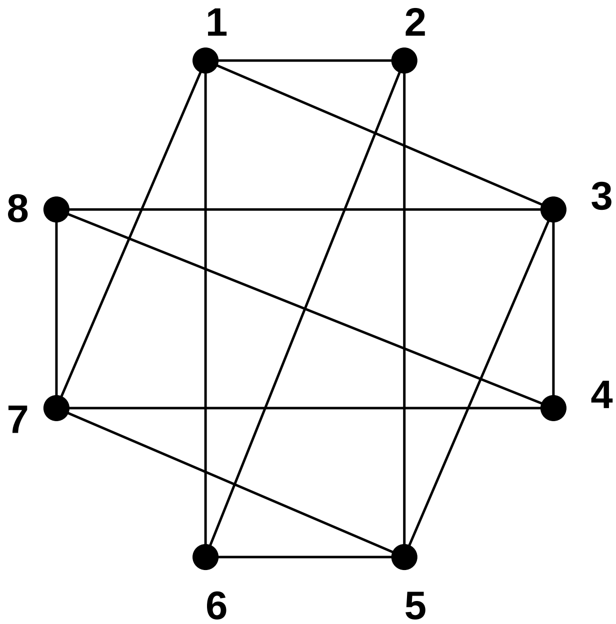

Figure 1.

its leftmost neighbour which is to the right of $m+1$. This step is repeated until, finally, all neighbours of the currently last element m are to the left of the (previously inserted) element $m+1$. Since the number of elements to the right of the previously inserted element is decreasing at every step, the algorithm surely terminates. (As an example, see the graph on Figure 1. In the first iteration of the general step 8 is inserted in front of its leftmost neighbour, which is 3, resulting in the σ_1-sequence 1,2,8,3,4,5,6,7. In the next iteration 7 is inserted in front of 4, its leftmost neighbour to the right of 8 in σ_1. Finally, 6 is inserted in front of 5, thus the resulting σ-sequence is 1,2,8,3,7,4,6,5 for this graph.)

We have to prove that the linear order we obtain satisfies the requirements. Assume that $\sigma(n) = m$, then σ can be viewed as a merge of the two sequences $1, 2, \ldots, m$ and $n, n-1, \ldots, m+1$. Furthermore, if $\sigma^{-1}(m+1) = i$, then $\{\sigma(j), m\} \notin E$ for $i < j < n$, in other words, m has no neighbour between $m+1$ and itself. We have to prove that if $\{i, j\} \notin E$ and j is to the right of i (according to σ), then $\tau^{-1}(j)$ is also to the right of $\tau^{-1}(i)$. Note, that $\tau^{-1}(b) = b - 1$ for $1 < b$ and $\tau^{-1}(1) = n$. The elements $m+1, m+2, \ldots, n$ moved by the algorithm are called *inserted*, while $1, 2, \ldots, m$ are called *original*. Let $\{i, j\} \notin E$ and j be to the right of i. Four cases are distuingished according to which of i and j is inserted.

Case 1. Both i, j are original. Then $i-1$ and $j-1$ are also original, provided $1 < i$. In this case the order of $i-1$ and $j-1$ is the same as the order of i and j. If $i = 1$, then $\tau^{-1}(1) = n$ and $\{\tau^{-1}(1), \tau^{-1}(j)\} \in E$, thus n is put to the left of $j-1$.

Case 2. Assume that i is inserted and j is original. Now $i > 1$ and $i-1$ is to the right of i. Since $\{i, j\} \notin E$ i could not be inserted just in front of j, so $j-1$ is also to the

right of i. However, $\{i-1, j-1\} \in E$, that is $i-1$ must be inserted, otherwise $i-1$ would have remained as $\sigma(n)$ and then it must not have any connection to the right of i. Furthermore it had to be inserted somewhere before $j-1$, thus $\tau^{-1}(j)$ is also to the right of $\tau^{-1}(i)$.
Case 3. Assume that i is original and j is inserted. If $i = 1$, then $\tau^{-1}(i) = n$ and $\{n, j-1\} \in E$, thus n is inserted before $j-1$, i.e., $\tau^{-1}(j)$ is also to the right of $\tau^{-1}(i)$. Otherwise, $i-1$ is to the left of i and $j-1$ is to the right of j.
Case 4. Both i, j are inserted. In this case the larger of i and j is to the left of the other, thus $i > j$. Also, either both $i-1$ and $j-1$ are also inserted or one of them is inserted and the other is the rightmost element in the linear order obtained. In either case, the larger of the elements $i-1$ and $j-1$ is also to the left of the other.
This completes the proof for unicyclic τ.

If τ is not unicyclic then let the number of cycles in τ be d. For $d = 1$ the theorem is proved by the foregoing. If τ decomposes into more than one cycle, then the subgraphs induced by the vertices in each individual cycle are all self-complementary, thus the above argument can be applied to them one by one. The resulting partial order on $V(G)$, which is the union of d total orders, can be extended to one total order by putting the cycles in order. Thus, $\sigma^{-1}(i) < \sigma^{-1}(j)$ if i is in a cycle of τ put before the cycle of j, or, if i and j are in the same cycle of τ, then this is their order given by the algorithm applied to that cycle alone. Let $\{i, j\} \notin E$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$. If i, j are contained in the same cycle of τ, then $\sigma^{-1}(\tau^{-1}(i)) < \sigma^{-1}(\tau^{-1}(j))$ holds by the first part of the proof. On the other hand, if i and j are in different cycles, then one can use that i and $\tau^{-1}(i)$, furthermore j and $\tau^{-1}(j)$ are in the same cycles, respectively, so their order according to σ is the same. □

Note that, taking the left-to-right ordering according to σ, each edge of G is mapped by τ to an edge of $\bar{G}$ of the same orientation. The union of G and $\bar{G}$ is the transitive tournament given by the order $(\sigma(1), \sigma(2), \ldots, \sigma(n))$.

2.2.4 Consequences for Sperner capacity

An immediate implication of Theorem 2.2.1 is a lower bound on the Sperner capacity of appropriately oriented self-complementary graphs.

Corollary 2.2.2 *If G is a self-complementary graph on n vertices then $D(G) \geq \frac{1}{2}\log n$.*

Proof. Let the vertex set of G be $V = \{1, \ldots, n\}$ and a complementing permutation of these vertices be τ. Let σ be the linear order on V satisfying the requirements of Theorem 2.2.1 and orient the edges of G according to σ, that is, the edge $\{i, j\}$ is oriented from i towards j iff $\sigma^{-1}(i) < \sigma^{-1}(j)$. The resulting oriented graph is denoted by $\hat{G}$. Consider the subset U of the vertices of $\hat{G}^2$ defined by

$$U = \{(i, \tau^{-1}(i)) : i = 1, \ldots, n\}.$$

By the properties of σ if $\{i,j\} \notin E(G)$ then $(\tau^{-1}(i), \tau^{-1}(j)) \in E(\hat{G})$ iff $\sigma^{-1}(i) < \sigma^{-1}(j)$. Thus U induces a transitive tournament in $\hat{G}^2$, because each edge is oriented according to the σ-order of the first coordinate of the vertices. Therefore every even power $\hat{G}^{2k}$ of $\hat{G}$ contains a transitive tournament of size $|U|^k = n^k$ implying $\Sigma(\hat{G}) \geq \frac{1}{2}\log n$. Since $\Sigma(\hat{G})$ is a lower bound on $D(G)$, this proves $D(G) \geq \frac{1}{2}\log n$. $\square$

We remark that the above given lower bound is not tight for all self-complementary graphs. It is easy to give, for example, self-complementary graphs on 8 vertices with clique number 4. If such a maximum clique of size 4 is oriented transitively in this graph then the Sperner capacity of the resulting oriented graph is at least $\log 4 > \frac{1}{2}\log 8$. (Since $\log 3 > \frac{1}{2}\log 8$ the same argument applies also for the graph on Figure 1.) If, however, our graph is not only self-complementary but also vertex-transitive, then by Lovász' results the above bound is tight. The next theorem formulates this statement.

Theorem 2.2.3 *For a vertex-transitive self-complementary graph G, the value of $D(G)$ equals the Shannon capacity of G.*

Proof. Lovász proved in [109] that for a vertex-transitive self-complementary graph G on n vertices $C(G) \leq \frac{1}{2}\log n$. This combined with Corollary 2.2.2 and the fact that $D(G) \leq C(G)$ implies the statement. $\square$

A consequence of Corollary 2.2.2 is that the Sperner capacity of a graph can be exponentially larger than the value implied by its clique number. This will follow by using the proof of the analogous result for Shannon capacity by Alon and Orlitsky [6]. First we quote a lemma of theirs (see as Lemma 3 of [6] on page 1282). We remark that [6] uses the complementary language.

Lemma AO: For every integer n that is divisible by 4, there exists a self-complementary graph G on n vertices with $\omega(G) < 2\lceil \log n \rceil$.

Corollary 2.2.4 *For every integer n divisible by 4, there exists a graph G on n vertices such that $D(G) > 2^{\log(\omega(G)-2)-2}$.*

Proof. Let n be an integer divisible by 4 and G be the graph constructed by Alon and Orlitsky proving Lemma AO. Since this graph is self-complementary, Corollary 2.2.2 implies $D(G) \geq \frac{1}{2}\log n > 2^{\log(\omega(G)-2)-2}$. $\square$

We remark that the graphs in the proof of Lemma AO all have complementing permutations containing cycles of length four only. Thus, the proof of Corollary 2.2.4 does not require the full generality of Theorem 1.

2.2.5 Further remarks

According to the relation of $D(G)$ and $C(G)$, we can distinguish among the following three classes of (undirected) graphs. The first class consists of those graphs every oriented version of which has its Sperner capacity equal to the Shannon capacity of the graph. The second class contains the graphs G for which this is not true but still $D(G) = C(G)$ holds. The third class is the class of those graphs for which $D(G) < C(G)$.

Since every graph containing at least one directed edge has its Sperner capacity at least 1, all graphs with Shannon capacity 1 belong to the first class. Using Shannon's observation that $\log \omega(G) \leq C(G) \leq \log \chi(G)$ one knows that all bipartite graphs have this property. The same chain of inequalities imply that all graphs with $\chi(G) = \omega(G)$ belong to one of the first two classes. This can be seen by orienting a largest clique transitively. Clearly, the possibility of such orientations shows $D(G) = C(G)$ for any graph G having $C(G) = \log \omega(G)$ even if the chromatic number and the clique number of G are different. Examples of such graphs are the complements of Kneser graphs of appropriate parameters, as it is shown by Theorem 13 of Lovász [109].

It is not clear whether a graph G with $C(G) > 1$ can belong to the first class. There is no graph identified to belong to the third class and it is not clear at all whether class three is empty or not. (This question is also mentioned in [50].) The main novelty of this section is that many graphs that have a gap between their capacity values and the logarithm of their clique number also satisfy the $D(G) = C(G)$ equality. This may support the guess that perhaps this equality always holds but we have too little evidence to state this as a conjecture.

Finally, let us express our feeling that Theorem 2.2.1, though motivated completely by the capacity questions exposed here, might have rather different applications, too.

Acknowledgment: We would like to express our sincere gratitude to András Gyárfás, for his encouraging remarks and continuing interest in our work.

Addendum: The above acknowledged interest of András Gyárfás culminated in the nice short paper [71] containing a conceptually simpler proof of Theorem 2.2.1.
We also remark that recent results of Bohman and Holzman [24] (that will be mentioned more explicitely in the next section) combined with earlier results from [28, 2] imply that the first class of graphs described in Subsection 2.2.5 does not contain (chordless) odd cycles. Nevertheless, it still does not follow that all graphs in this class are bipartite.

2.3 Local chromatic number and Sperner capacity

In this section we introduce a directed analog of the local chromatic number defined by Erdős et al. in [47] and show that it provides an upper bound for the Sperner capacity of a directed graph. Applications and variants of this result are presented. In particular, we find a special orientation of an odd cycle and show that it achieves the maximum of Sperner capacity among the differently oriented versions of the cycle. We show that apart from this orientation, for all the others an odd cycle has the same Sperner capacity as a single edge graph. We also show that the (undirected) local chromatic number is bounded from below by the fractional chromatic number while for power graphs the two invariants have the same exponential asymptotics (under the co-normal product on which the definition of Sperner capacity is based). We strengthen our bound on Sperner capacity by introducing a fractional relaxation of our directed variant of the local chromatic number.

(We note our slight abuse of terminology that we use the term Sperner capacity here for the non-logarithmic version of the invariant we called Sperner capacity in the previous sections.)

This section is based on the joint paper [97] with János Körner and Concetta Pilotto.

2.3.1 Introduction

Coloring the vertices of a graph so that no adjacent vertices receive identical colors gives rise to many interesting problems and invariants, of which the book [80] gives an excellent survey. The best known among all these invariants is the chromatic number, the minimum number of colors needed for such proper colorings. An interesting variant was introduced by Erdős, Füredi, Hajnal, Komjáth, Rödl, and Seress [47] (cf. also [57]). They define the local chromatic number of a graph as follows.

Definition 25 ([47]) *The* local chromatic number $\psi(G)$ *of a graph G is the maximum number of different colors appearing in the closed neighbourhood of any vertex, minimized over all proper colorings of G. Formally,*

$$\psi(G) := \min_{c:\ V(G)\to\mathbb{N}} \max_{v\in V(G)} |\{c(u) : u \in \Gamma_G(v)\}|,$$

where $\mathbb{N}$ is the set of natural numbers, $\Gamma_G(v)$, the closed neighborhood of the vertex $v \in V(G)$, is the set of those vertices of G that are either adjacent or equal to v and $c : V(G) \to \mathbb{N}$ runs over those functions that are proper colorings of G.

It is clear that $\psi(G)$ is always bounded from above by the chromatic number, $\chi(G)$. It is much less obvious that $\psi(G)$ can be strictly less than $\chi(G)$. Yet this is true; in fact, as proved in [47], there exist graphs with $\psi(G) = 3$ and $\chi(G)$ arbitrarily large.

Throughout this section, we shall be interested in chromatic invariants as upper bounds for the Shannon capacity of undirected graphs and its natural generalization Sperner capacity for directed graphs. We recall that for the sake of unity in the treatment of undirected and directed graphs it is convenient and customary to treat Shannon capacity in terms that are complementary to Shannon's own, (cf. [139], [109] and [60], [96], and see the introductory words of Subsection 2.2.2 as well as Remark 1.5 in Section 1.3). In this language Shannon capacity describes the asymptotic growth of the clique number in the co-normal powers of a graph. As already mentioned in the beginning of Subsection 2.2.2 Shannon proved (although in different terms) that the (non-logarithmic) Shannon capacity $c(G)$ of a graph is bounded from above by the fractional chromatic number.

We show that $\psi(G)$ is bounded from below by the fractional chromatic number of G. This proves, among other things, that $\psi(G)$ is always an upper bound for the Shannon capacity $c(G)$ of G, but it is not a very useful upper bound since it is always weaker than the fractional chromatic number itself. We make this seemingly useless remark only to stress that the situation is rather different in the case of directed graphs.

We introduce an analog of the local chromatic number for directed graphs and show that it is always an upper bound for the Sperner capacity of the digraph at hand. The proof is linear algebraic and generalizes an idea already used for bounding Sperner capacity in [23], [2], [50], cf. also [28]. To illustrate the usefulness of this bound we apply it to show, for example, that an oriented odd cycle with at least two vertices with outdegree and indegree 1 always has its Sperner capacity equal to that of the single-edge graph K_2. We also discuss fractional versions that further strengthen our bounds.

2.3.2 Local chromatic number for directed graphs

The definition of the directed version of $\psi(G)$ is straightforward.

Definition 26 *The* local chromatic number $\psi_d(G)$ of a digraph G *is the maximum number of different colors appearing in the closed out-neighbourhood of any vertex, minimized over all proper colorings of* G*. Formally,*

$$\psi_d(G) := \min_{c:\ V(G)\to\mathbb{N}} \max_{v\in V(G)} |\{c(w) : w \in \Gamma^+_G(v)\}$$

where $\mathbb{N}$ *is the set of natural numbers,* $\Gamma^+_G(v)$*, the closed out-neighborhood of the vertex* $v \in V(G)$*, is the set of those vertices* $w \in V(G)$ *that are either equal to* v *or else are endpoints of directed edges* $(v,w) \in E(G)$*, originated in* v*, and* $c : V(G) \to \mathbb{N}$ *runs over those functions that are proper colorings of* G*.*

Our main goal is to prove that $\psi_d(G)$ is an upper bound for the Sperner capacity of digraph G.

2.3.3 Sperner capacity

Recall that we refer to the (di)graph obtained from G by the exponentiation defined in Definition 22 (see on page 47) as the co-normal power of the graph G.

(An edge (a, b) always means an oriented edge as opposed to undirected edges denoted by $\{a, b\}$.)

Now we recall the definition of Sperner capacity in its non-logarithmic version and via symmetric (rather than transitive) cliques.

Definition 27 *A subgraph of a digraph is called a* symmetric clique *if its edge set contains all ordered pairs of vertices belonging to the subgraph. (In other words, it is a clique with all its edges present in both directions.) For a directed graph G we denote the size (number of vertices) of its largest symmetric clique by $\omega_s(G)$.*

Definition 28 ([60]) *The (non-logarithmic) Sperner capacity of a digraph G is defined as*

$$\sigma(G) = \sup_n \sqrt[n]{\omega_s(G^n)}.$$

Remark 2.3. Denoting the number of vertices in a largest transitive clique of G by $\mathrm{tr}(G)$, it is easy to show that $\sigma(G) = \sup_n \sqrt[n]{\mathrm{tr}(G^n)}$ holds, cf. [60, 50] and also our earlier Definition 23 in Subsection 2.2.1 (page 47). (Recall that by a transitive clique we mean a clique where the edges are oriented transitively, i.e., consistently with some linear order of the vertices. It is allowed that some edges be present also in the reverse direction.) Since $\mathrm{tr}(G^n) \geq [\mathrm{tr}(G)]^n$ this remark implies that $\mathrm{tr}(G) \leq \sigma(G)$ holds for any digraph G. $\Diamond$

Also recall that for an undirected graph G we call the digraph obtained from G by directing all its edges in both ways the symmetrically directed equivalent of G. In Shannon's own language the capacity (cf. [139]) of the complement of a(n undirected) graph G can be defined as the Sperner capacity of its symmetrically directed equivalent. We denote this quantity by $c(G)$.

Thus, as already mentioned in the previous sections, Sperner capacity is a generalization of Shannon capacity. It is a true generalization in the sense that there exist digraphs the Sperner capacity of which is different from the Shannon capacity ($c(G)$ value) of its underlying undirected graph. Denoting by G both an arbitrary digraph and its underlying undirected graph, it follows from the definitions that $\sigma(G) \leq c(G)$ always holds. The smallest example with strict inequality in the previous relation is a cyclically oriented triangle, cf. [28], [23]. (See also [20] for an early and different attempt to generalize Shannon capacity to directed graphs.)

Shannon capacity is known to be a graph invariant that is difficult to determine (not only in the algorithmic but in any sense), and it is unknown for many relatively small and simple graphs, for example, for all odd cycles of length at least 7. This already shows that the more general invariant Sperner capacity cannot be easy to determine either. For a survey on graph invariants defined via powers, including Shannon and Sperner capacities, we refer the reader to [4]. There is an interesting and important connection between Sperner capacity and extremal set theory, introduced in [98] and fully explored in [62]. Several problems of this flavour are also discussed in [96].

2.3.4 Main result

Alon [2] proved that for any digraph G

$$\sigma(G) \leq \min\{\Delta_+(G), \Delta_-(G)\} + 1$$

where $\Delta_+(G)$ is the maximum out-degree of the graph G and similarly $\Delta_-(G)$ is the maximum in-degree. The proof relies on a linear algebraic method similar to the one already used in [23] for a special case (cf. also [50] for a strengthening and cf. [3] for a general setup for this method in case of undirected graphs). We also use this method for proving the following stronger result.

Theorem 2.3.1

$$\sigma(G) \leq \psi_d(G).$$

Proof. Consider a proper coloring $c : V(G) \to \mathbb{N}$ that achieves the value of $\psi_d(G)$. Let $N_c^+(v)$ denote the set of colors each of which appears as the color of some vertex in the (open) out-neighbourhood of v in the coloring c.

For each vertex $\boldsymbol{a} = (a_1, \ldots, a_n) \in V(G^n)$ we define a polynomial

$$P_{\boldsymbol{a},c}(x_1, \ldots, x_n) := \prod_{i=1}^{n} \prod_{j \in N_c^+(a_i)} (x_i - j).$$

Let K be a symmetric clique in G^n. If $\boldsymbol{a} \in K$, $\boldsymbol{b} \in K$ then by definition $P_{\boldsymbol{a},c}(c(b_1), \ldots, c(b_n)) = 0$ if $\boldsymbol{b} \neq \boldsymbol{a}$, while $P_{\boldsymbol{a},c}(c(a_1), \ldots, c(a_n)) \neq 0$ by the properness of coloring c. This implies that the polynomials $\{P_{\boldsymbol{a},c}(x_1, \ldots, x_n)\}_{\boldsymbol{a} \in K}$ are linearly independent over the reals. This can be shown in the usual way: substituting $c(\boldsymbol{b})$ into $\sum_{\boldsymbol{a} \in K} \lambda_{\boldsymbol{a}} P_{\boldsymbol{a},c}(\boldsymbol{x}) = 0$ we obtain $\lambda_{\boldsymbol{b}} = 0$ and this can be done for each $\boldsymbol{b} \in K$.

Since the degree of x_i in $P_{\boldsymbol{a},c}(\boldsymbol{x})$ is at most $\psi_d(G) - 1$, the dimension of the linear space generated by our polynomials is bounded from above by $[\psi_d(G)]^n$. By the previous paragraph, this is also an upper bound for $|K|$. Choosing K to be a symmetric clique of maximum size we obtain $\omega_s(G^n) \leq [\psi_d(G)]^n$ and thus the statement. □

Let G_{rev} denote the "reverse of G", i.e., the digraph we obtain from G by reversing the direction of all of its edges. Since obviously $\sigma(G) = \sigma(G_{\text{rev}})$, Theorem 2.3.1 has the following trivial corollary.

Corollary 2.3.2

$$\sigma(G) \leq \min\{\psi_d(G), \psi_d(G_{\text{rev}})\}.$$

□

In Subsections 2.3.7 and 2.3.8 we will strengthen Theorem 2.3.1 by introducing a fractional version of $\psi_d(G)$.

2.3.5 Application: odd cycles

We call an oriented cycle alternating if it has at most one vertex of outdegree 1. (In stating the following results we follow again the convention that an oriented graph is a graph without oppositely directed edges between the same two points, while a general directed graph may contain such pairs of edges.) Clearly, in any oriented cycle the number of vertices of outdegree 2 equals the number of vertices of outdegree 0. Thus, in particular, an oriented odd cycle of length $2k+1$ is alternating if it has k points of outdegree zero, k points of outdegree 2 and only 1 point of outdegree 1. It takes an easy checking that up to isomorphism there is only one orientation of C_{2k+1} which is alternating.

Theorem 2.3.3 *Let G be an oriented odd cycle that is not alternating. Then*

$$\sigma(G) = 2.$$

Proof. Since any digraph with at least one edge has Sperner capacity at least 2 (see Remark 2.3 after Definition 28), it is enough to prove that 2 is also an upper bound.

Color the vertices of G so that two points receive the same color if and only if they have a common in-neighbour, i.e., a vertex sending an oriented edge to both of them. It is easy to check that this coloring is proper if and only if the odd cycle G is not alternating. In this case, our coloring also has the property that any vertex has only one color in its out-neigbourhood proving $\psi_d(G) = 2$. Then the statement follows by Theorem 2.3.1. □

Remark 2.4. It is easy to see that the following slightly stronger version of the previous theorem can be proven similarly: If G is a directed odd cycle not containing an alternating odd cycle, then $\sigma(G) = 2$. ♢

The Sperner capacity of an alternating odd cycle can indeed be larger than 2. This is obvious for C_3, where the alternating orientation produces a transitive clique of size 3. A construction proving that the Sperner capacity of the alternating C_5 is at least $\sqrt{5}$ is given in [59] (cf. Proposition 2.1.17), and this is further analyzed in [132] (see the previous section). The construction is clearly best possible by the celebrated result of Lovász [109] showing $c(C_5) = \sqrt{5}$.

In [132] (see the previous section) the invariant $D(G) = \max \log \sigma(\hat{G})$ was defined where the maximization is over all orientations $\hat{G}$ of G. Let $\hat{D}(G) = 2^{D(G)}$. It follows from the definitions that $\hat{D}(G) \leq c(G)$, and it is asked in [132] whether one always has equality. No counterexample is known, while equality is trivial if $\chi(G) = \omega(G)$ (just orient a maximum size clique transitively) and it is proven for vertex-transitive self-complementary graphs in [132] (cf. the previous section). Denoting the alternatingly oriented C_{2k+1} by C_{2k+1}^{alt} Theorem 2.3.3 has the following immediate corollary.

Corollary 2.3.4

$$\hat{D}(C_{2k+1}) = \sigma(C_{2k+1}^{\mathrm{alt}})$$

holds for every positive integer k. □

The discussion in this subsection becomes more relevant in the light of a recent result by Bohman and Holzman [24]. Until recently it was not known whether the Shannon capacity (in our complementary sense) of the odd cycle C_{2k+1}, i.e., $c(C_{2k+1})$ is larger than 2 for any value of $k > 2$. In [24] an affirmative answer to this question was given by an ingenious construction, showing that this is always the case, i.e., $c(C_{2k+1}) > 2$ for every positive integer k. This means that the bound provided by $\psi_d(G)$ goes beyound the obvious upper bound $c(G)$ of Sperner capacity in case of non-alternatingly oriented odd cycles, i.e., the following consequence of Theorem 2.3.3 can also be formulated.

Corollary 2.3.5 *If k is any positive integer and $C_{2k+1}^{\rightarrow}$ is a non-alternatingly oriented C_{2k+1}, then*

$$\sigma(C_{2k+1}^{\rightarrow}) < c(C_{2k+1}).$$

□

It is a natural idea to try to use the Bohman-Holzman construction for alternatingly oriented odd cycles and check whether the so obtained sets of vertices inducing cliques in the appropriate power graphs will form transitive cliques in the oriented case. (If the answer were yes it would prove $\hat{D}(C_{2k+1}) > 2$ for every k strengthening the result $c(C_{2k+1}) > 2$ of [24].) This idea turned out to work in the case of C_7, thus showing $\hat{D}(C_7) > 2$. (To record this we list the 17 vertices of C_7^4 that form a transitive clique defined by their ordering on this list. The labels of the vertices of C_7 are the first 7 non-negative integers as in [24] and the unique point with outdegree and indegree 1 is the point labelled 5. Here we give the vertices simply as sequences. Thus the list is: 4444, 0520, 2030, 2051, 0605, 1205, 1320, 3006, 3012, 5106, 5112, 0561, 0613, 1213, 6130, 6151, 1361.) Strangely, however, the same construction did not work for C_9: after our unsuccesful attempts to prove a similar statement, Attila Sali wrote a computer program to check whether the clique of Bohman and Holzman in the 8^{th} power of an alternating C_9 contains a transitive clique of the same size and the answer turned out to be negative. (Again, to record more than just this fact, we give six vertices of C_9^8 that form a directed cycle without inversely oriented edges in the clique of Bohman and Holzman whenever the path obtained after deleting vertex 5 of C_9 is oriented alternatingly. The existence of this cycle shows that the Bohman-Holzman clique does not contain a transitive clique whenever the only outdegree 1, indegree 1 point of the alternatingly oriented C_9 is 4 or 6 (or 5, but this case is less important), that is one of the neighbours of 5, the point the construction distinguishes. So the promised cycle is: 20302040, 12072040, 12140720, 40121207, 20401320, 07204012.) In spite of this, we believe that the Sperner capacity of alternating odd cycles will achieve the corresponding Shannon capacity value $c(C_{2k+1})$.

One more remark is in order. It is easy to check that the vertices of C_{2k+3}^{alt} can be mapped to those of C_{2k+1}^{alt} in an edge-preserving manner. This immediately implies that $\sigma(C_{2k+3}^{\text{alt}}) \leq \sigma(C_{2k+1}^{\text{alt}})$, i.e., if there were any odd cycle C_{2k+1} with $\hat{D}(C_{2k+1}) = 2$, then the same must hold for all longer odd cycles as well.

2.3.6 The undirected case

Since identifying with any undirected graph G its symmetrically directed equivalent gives both $\sigma(G) = c(G)$ and $\psi_d(G) = \psi(G)$, it is immediate from Theorem 2.3.1 that $c(G) \le \psi(G)$. We will show, however, that $\psi(G)$ is always bounded from below by the fractional chromatic number of G, which in turn is a well known upper bound for $c(G)$, cf. [139], [109]. Thus, unlike in the directed case, the local chromatic number does not give us new information about Shannon capacity. Looking at it from another perspective, this relation tells us something about the behaviour of the local chromatic number. (For more on this other perspective, see the follow up paper [144], cf. the next chapter.)

One of the main points in the investigations of the local chromatic number in [47] is the recognition of the relevance of the universal graphs $U(m,k)$ defined as follows. (From now on we will use the notation $[m] = \{1,\ldots,m\}$).

Definition 29 ([47]) *Let the graph $U(m,k)$ for positive integers $k \le m$ be defined as follows.*

$$V(U(m,k)) := \{(x,A) : x \in [m], A \subseteq [m], |A| = k-1, x \notin A\}$$

and

$$E(U(m,k)) := \{\{(x,A),(y,B)\} : x \in B, y \in A\}.$$

The relevance of these graphs is expressed by the following lemma. Recall that a homomorphism from some graph F to a graph G is an edge-preserving mapping of $V(F)$ to $V(G)$.

Lemma 2.3.6 ([47]) *A graph G admits a proper coloring c with m colors and $\max_{v\in V(G)} |\{c(u) : u \in \Gamma_G(v)\}| \le k$ if and only if there exists a homomorphism of G to $U(m,k)$. In particular, $\psi(G) \le k$ if and only if there exists an m such that G admits a homomorphism to $U(m,k)$.*

We use these graphs to prove the relation between the fractional chromatic number and the local chromatic number.

Recall that the fractional chromatic number is $\chi^*(G) = \min \sum_{A\in S(G)} w(A)$ where $S(G)$ denotes the family of independent sets of graph G and the minimization is over all non-negative weightings $w : S(G) \to \mathbb{R}$ satisfying $\sum_{A\ni x} w(A) \ge 1$ for every $x \in V(G)$. (Such a non-negative weighting is called a fractional coloring.) It is straightforward from the definition that $\chi^*(G) \ge \omega(G)$ holds for any graph G. Another important fact we will use (as we already did in Section 1.3) is that if G is vertex-transitive, then $\chi^*(G) = \frac{|V(G)|}{\alpha(G)}$. For a proof of this fact and for further information about the fractional chromatic number we refer to the books [133], [68].

Theorem 2.3.7 *For any graph G*

$$\psi(G) \ge \chi^*(G).$$

The proof relies on the following simple observation.

Lemma 2.3.8 *For all $m \geq k \geq 2$ we have $\chi^*(U(m,k)) = k$.*

Proof. It is easy to check that $\chi^*(U(m,k)) \geq \omega(U(m,k)) = k$ thus we only have to prove that k is also an upper bound. It is straightforward from their definition that the graphs $U(m,k)$ are vertex-transitive. (Any permutation of $[m]$ gives an isomorphism, and any vertex can be mapped to any other by such a permutation.) Consider those vertices (x, A) for which $x \leq a_i$ for all $a_i \in A$. These form an independent set S. Thinking about the vertices (x, A) as k-tuples with one distinguished element and the elements of S as those k-tuples whose distinguished element is the smallest one, we immediately get $\chi^*(U(m,k)) = \frac{|V(U(m,k))|}{\alpha(U(m,k))} \leq \frac{|V(U(m,k))|}{|S|} = k$ proving the statement. □

Proof of Theorem 2.3.7. Let us have $\psi(G) = k$. This means that there is a homomorphism from G to $U(m,k)$ for some m (cf. Lemma 2.3.6). Since a homomorphism cannot decrease the fractional chromatic number, from Lemma 2.3.8 we obtain $\chi^*(G) \leq \chi^*(U(m,k)) = k = \psi(G)$.

□

In the rest of this subsection we formulate a consequence of Theorem 2.3.7 for the asymptotic behaviour of the local chromatic number with respect to the co-normal power of graphs.

It is a well-known theorem of McEliece and Posner [120] (cf. also Berge and Simonovits [19] and, for this particular formulation, [133]) that

$$\lim_{n\to\infty} \sqrt[n]{\chi(G^n)} = \chi^*(G).$$

It is equally well-known (cf., e. g., Corollary 3.4.2 in [133]) that $\chi^*(G^n) = [\chi^*(G)]^n$. These two statements and Theorem 2.3.7 immediately imply the following.

Corollary 2.3.9

$$\lim_{n\to\infty} \sqrt[n]{\psi(G^n)} = \chi^*(G).$$

Proof. By $\chi^*(G^n) \leq \psi(G^n) \leq \chi(G^n)$ we have $\chi^*(G) = \sqrt[n]{\chi^*(G^n)} = \lim_{n\to\infty} \sqrt[n]{\chi^*(G^n)} \leq \lim_{n\to\infty} \sqrt[n]{\psi(G^n)} \leq \lim_{n\to\infty} \sqrt[n]{\chi(G^n)} = \chi^*(G)$ where the last equality is by the McEliece-Posner theorem mentioned above. □

2.3.7 Fractional colorings

Now we define the fractional version of the local chromatic number. For $v \in V(G)$ let $\Gamma_G^+(v)$ denote, as before, the closed out-neighbourhood of v, i.e., the set containing v and its out-neighbours.

Definition 30 *For a digraph G its* (directed) fractional local chromatic number $\psi_d^*(G)$ *is defined as follows:*

$$\psi_d^*(G) := \min_w \max_{v \in V(G)} \sum_{\Gamma_G^+(v) \cap A \neq \emptyset} w(A),$$

where the minimization is over all fractional colorings w of G.

The fractional local chromatic number $\psi^*(G)$ of an undirected graph G is just $\psi_d^*(\check{G})$ where $\check{G}$ is the symmetrically directed equivalent of G.

An r-fold coloring of a graph G is a coloring of each of its vertices with r distinct colors with the property that the sets of colors assigned to adjacent vertices are disjoint. More formally, an r-fold coloring is a set-valued function $f : V(G) \to \binom{\mathbb{N}}{r}$ satisfying $f(u) \cap f(v) = \emptyset$ whenever $(u, v) \in E(G)$.

Definition 31 *Let $\psi_d(G, r)$ denote the r-fold (directed) local chromatic number of digraph G defined as*

$$\psi_d(G, r) := \min_f \max_{u \in V(G)} |\cup_{v \in \Gamma_G^+(u)} f(v)|,$$

where the minimization is over all r-fold colorings f of G.

The r-fold local chromatic number $\psi(G, r)$ of an undirected graph G is just $\psi_d(\check{G}, r)$ where $\check{G}$ is the symmetrically directed equivalent of G.

It is obvious that

$$\psi_d^*(G) = \inf_r \frac{\psi_d(G, r)}{r}$$

for every digraph G. This includes the equality

$$\psi^*(G) = \inf_r \frac{\psi(G, r)}{r}$$

for undirected graphs, too.

For a digraph G let $G[K_r]$ denote the graph obtained by substituting a symmetric clique of size r into each of its vertices. Formally this means

$$V(G[K_r]) = \{(v, i) : v \in V(G), i \in \{1, \dots, r\}\}$$

and

$$E(G[K_r]) = \{((u, i), (v, j)) : (u, v) \in E(G) \text{ or } u = v \text{ and } i \neq j\}.$$

It is easy to see that $\psi_d(G[K_r]) = \psi_d(G, r)$ for every digraph G and positive integer r. It is also not difficult to see that $\omega_s((G[K_r])^n) = r^n \omega_s(G^n)$ for every n. Indeed, any vertex of G^n can be substituted by r^n vertices of $(G[K_r])^n$ in the natural way and a symmetric clique K of G^n becomes a symmetric clique of size $r^n|K|$ in $(G[K_r])^n$ this way proving $\omega_s((G[K_r])^n) \geq r^n \omega_s(G^n)$. To see that equality holds let us denote by $\boldsymbol{a}(\boldsymbol{x})$ the unique vertex of G^n from which $\boldsymbol{x} \in (G[K_r])^n$ can be obtained by the previous substitution.

(Thus the set $A_{\boldsymbol{x}} := \{\boldsymbol{y} : \boldsymbol{a}(\boldsymbol{y}) = \boldsymbol{a}(\boldsymbol{x})\}$ has r^n elements for every $\boldsymbol{x} \in V((G[K_r])^n)$.) The crucial observation is that if K is a symmetric clique in $(G[K_r])^n$ and $\boldsymbol{x} \in K$, then $K \cup A_{\boldsymbol{x}}$ is still a symmetric clique (it may be identical to K but may also be larger). Thus maximal symmetric cliques of $(G[K_r])^n$ can always be obtained as the union of some sets $A_{\boldsymbol{x}}$, which means that they can be obtained as "blown up" versions of symmetric cliques of G^n. This proves our claim that $\omega_s((G[K_r])^n) = r^n \omega_s(G^n)$. This equality implies $\sigma(G[K_r]) = r\sigma(G)$ for every digraph G and positive integer r.

The observations of the previous paragraph provide the following strengthening of Theorem 2.3.1.

Theorem 2.3.10 *For every digraph G*

$$\sigma(G) \leq \psi_d^*(G)$$

holds.

Proof. By Theorem 2.3.1 and the previous observations we have

$$\sigma(G) = \frac{\sigma(G[K_r])}{r} \leq \frac{\psi_d(G[K_r])}{r} = \frac{\psi_d(G, r)}{r}.$$

Since this holds for every r we can write

$$\sigma(G) \leq \inf_r \frac{\psi_d(G, r)}{r} = \psi_d^*(G).$$

□

We can formulate again the following trivial corollary.

Corollary 2.3.11

$$\sigma(G) \leq \min\{\psi_d^*(G), \psi_d^*(G_{\mathrm{rev}})\}.$$

□

To illustrate the usefulness of Theorem 2.3.10 we consider the complement of a 7-cycle with its only orientation in which all triangles are oriented cyclically. We denote this graph by D_7 (abbreviating double 7-cycle). None of the earlier bounds we know give a better upper bound for the Sperner capacity of D_7 than 3. Now we can improve on this.

Proposition 2.3.12

$$\sqrt{5} \leq \sigma(D_7) \leq \frac{5}{2}\ .$$

Proof. The lower bound follows by observing that D_7 contains an alternating 5-cycle. The upper bound is a consequence of Theorem 2.3.10 since $\psi_d^*(D_7) = \frac{5}{2}$. We actually need here only $\psi_d^*(D_7) \leq \frac{5}{2}$ and this can be seen by giving weight $\frac{1}{2}$ to each 2-element stable set of D_7. □

This example can be further generalized as follows. Let D_{2k+1} denote the following oriented graph.

$$V(D_{2k+1}) = \{0, 1, \ldots, 2k\},$$

and

$$E(D_{2k+1}) = \{(u, v) : v \equiv u + j \pmod{2k+1},\ j \in \{2, 3, \ldots, k\}\}.$$

Observe that this definition is consistent with the earlier definition of D_7 and that the underlying undirected graph of D_{2k+1} is the complement of the odd cycle C_{2k+1}. Now we can state the following

Proposition 2.3.13

$$\left\lceil \frac{k-1}{2} \right\rceil + 1 \leq \sigma(D_{2k+1}) \leq \frac{k}{2} + 1.$$

In particular, $\sigma(D_{2k+1}) = \frac{k}{2} + 1$ *if* k *is even.*

Proof. It is easy to verify for the transitive clique number that $\mathrm{tr}(D_{2k+1}) = \lceil \frac{k-1}{2} \rceil + 1$ and this gives the lower bound. The upper bound is proven by assigning weight $\frac{1}{2}$ to every 2-element independent set of D_{2k+1} which clearly gives a fractional coloring. The weight thus assigned to any closed out-neighbourhood is $\frac{k}{2} + 1$ giving the upper bound by Theorem 2.3.10.

If k is even, the two bounds coincide. □

We remark that while the upper bound in Proposition 2.3.13 generalizes that of Proposition 2.3.12, the lower bound does not; it is weaker in case $k = 3$ than that of Proposition 2.3.12. Therefore we consider the oriented graph D_7 a particularly interesting instance of the problem.

As it was the case without fractionalization, Theorem 2.3.10 does not give us new information in the undirected case, i.e., about Shannon capacity. The reason for this is the following relation.

Theorem 2.3.14 *Let* G *be an undirected graph. Then*

$$\psi^*(G) = \chi^*(G).$$

To prove Theorem 2.3.14 we need the following generalization of the universal graphs $U(m, k)$.

Definition 32 *We define the graph $U_r(m,k)$ for positive integers $2r \le k \le m$ as follows.*

$$V(U_r(m,k)) := \{(X,A) : X, A \subseteq [m], X \cap A = \emptyset, |X| = r, |A| = k - r\}$$

and

$$E(U_r(m,k)) := \{\{(X,A),(Y,B)\} : X \subseteq B, Y \subseteq A\}.$$

Remark 2.5. Note that $U_1(m,k) = U(m,k)$, while $U_r(m,m) = \mathrm{KG}(m,r)$, the Kneser graph with parameters m and r. Thus the graphs we just defined provide a common generalization of Kneser graphs and the universal graphs $U(m,k)$ of [47]. $\Diamond$

The following lemma is the general version of Lemma 2.3.6 for multicolorings.

Lemma 2.3.15 *A graph G admits a proper r-fold coloring f with m colors in which the closed neighbourhood of every vertex contains at most k colors if and only if there exists a homomorphism from G to $U_r(m,k)$. In particular, $\psi(G,r) \le k$ if and only if there exists an m alongside with a homomorphism from G to $U_r(m,k)$.*

Proof. The proof is more or less identical to that of Lemma 2.3.6 (cf. [47]). If the required coloring f exists then assign to each vertex v a pair of sets of colors (X,A) with $X = f(v)$ and $\cup_{\{u,v\}\in E(G)} f(u) \subseteq A$, $|A| = k - r$. If f has the required properties then this assignment is possible and is indeed a homomorphism to $U_r(m,k)$.

On the other hand, if the required homomorphism h exists then the r-fold coloring f defined by the X-part of $h(v) = (X,A)$ as $f(v)$ satisfies the requirements. $\Box$

The following lemma is a generalization of Lemma 2.3.8.

Lemma 2.3.16 *For all feasible parameters m, k, r*

$$\chi^*(U_r(m,k)) = \frac{k}{r}.$$

Proof. Think of the vertices of $U_r(m,k)$ as k-sets of the set $[m]$ with r elements of the k-set distinguished. The number of vertices is thus $\binom{m}{k}\binom{k}{r}$, while the number of those vertices in which the smallest element of the chosen k-set is among the distinguished ones is $\binom{m}{k}\binom{k-1}{r-1}$. Since the latter kind of vertices form an independent set in $U_r(m,k)$, we have $\alpha(U_r(m,k)) \ge \binom{m}{k}\binom{k-1}{r-1}$. The reverse inequality $\alpha(U_r(m,k)) \le \binom{m}{k}\binom{k-1}{r-1}$ follows from the Erdős-Ko-Rado theorem: once the chosen k-set is fixed, we can have at most $\binom{k-1}{r-1}$ vertices (X_i, A_i) with the property that if $i \ne j$ then $X_i \cap X_j \ne \emptyset$. If $X_i \cup A_i = X_j \cup A_j$, then the latter is the very same condition as non-adjacency in $U_r(m,k)$. Thus we know $\alpha(U_r(m,k)) = \binom{m}{k}\binom{k-1}{r-1}$.

Since $U_r(m,k)$ is vertex-transitive (because any permutation of the elements of $[m]$ gives an automorphism), we have $\chi^*(U_r(m,k)) = \frac{|V(U_r(m,k))|}{\alpha(U_r(m,k))} = \frac{k}{r}$. $\Box$

Proof of Theorem 2.3.14. We know by Lemma 2.3.15 that $\psi(G,r) = k$ implies the existence, for some m, of a homomorphism from G to $U_r(m,k)$. Since a homomorphism cannot decrease the value of the fractional chromatic number, this implies $\chi^*(G) \leq \chi^*(U_r(m,k)) = \frac{k}{r} = \frac{\psi(G,r)}{r}$, where, in particular, the first equality holds by Lemma 2.3.16.

On the other hand, denoting by $\chi(G,r)$ the minimum number of colors needed for a proper r-fold coloring of G, $\inf_r \frac{\psi(G,r)}{r} \leq \chi^*(G)$ follows from $\inf_r \frac{\chi(G,r)}{r} = \chi^*(G)$ (cf. Theorem 7.4.5 in [68]) and the obvious inequality $\psi(G,r) \leq \chi(G,r)$. $\square$

We note that universal graphs can also be defined for the directed version of the local chromatic number. Denoting these graphs by $U_d(m,k)$ they have $V(U_d(m,k)) = V(U(m,k))$ while

$$E(U_d(m,k)) = \{((x,A),(y,B)) : y \in A\}.$$

To show the analog of Lemma 2.3.6 is straightforward. Comparing $U_d(m,k)$ to $U(m,k)$ one can see that the symmetrically directed edges of $U_d(m,k)$ are exactly the (undirected) edges present in $U(m,k)$. This means (but the same can be seen also directly) that $\omega_s(U_d(m,k)) = k$. On the other hand, naturally, $\psi_d(U_d(m,k)) = k$, thus for these graphs we have $\sigma(U_d(m,k)) = \omega_s(U_d(m,k))$ by Theorem 2.3.1 and the obvious inequality $\omega_s(G) \leq \sigma(G)$.

2.3.8 Fractional covers

A non-negative real valued function $g : 2^{V(G)} \to \mathbb{R}$ is called a fractional cover of $V(G)$ if $\sum_{U \ni v} g(U) \geq 1$ holds for all $v \in V(G)$.

The most general upper bound on $\sigma(G)$ we prove in this section is given by the following inequality that generalizes Theorem 2.3.10 along the lines of a result (Theorem 2) of [50].

Theorem 2.3.17 *For any digraph G we have*

$$\sigma(G) \leq \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G[U]),$$

where the minimization is over all fractional covers g of $V(G)$ and $G[U]$ denotes the digraph induced by G on $U \subseteq V(G)$.

By $\sigma(G) = \sigma(G_{\mathrm{rev}})$ we again have the following immediate corollary (cf. Corollary 2.3.2 of Theorem 2.3.1).

Corollary 2.3.18

$$\sigma(G) \leq \min \left\{ \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G[U]),\ \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G_{\mathrm{rev}}[U]) \right\}.$$

$\square$

The proof of Theorem 2.3.17 is almost identical to that of Theorem 2 of [50]. Yet, we give the details for the sake of completeness.

We need some lemmas. Following [3], we can speak about the *representation* of a (di)graph $G = (V, E)$ over a subspace $\mathcal{F}$ of polynomials in m variables over a field F. Such a representation is an assignment of a polynomial f_v in $\mathcal{F}$ and a vector $\boldsymbol{a}_v \in F^m$ to each vertex $v \in V$ such that the following two conditions hold:
i) for each $v \in V$, $f_v(\boldsymbol{a}_v) \neq 0$,
and
ii) if $(u, v) \in E(G)$ then $f_u(\boldsymbol{a}_v) = 0$.

Notice that we adapted the description of a representation given in [3] to our terminology (where capacities are defined via cliques instead of stable sets) and to digraphs.

The following two lemmas are from [3]. Their proofs are essentially identical to those of Lemma 2.2 and Lemma 2.3 in [3] (after some trivial changes caused by the different language).

Lemma 2.3.19 ([3]) *Let $G = (V, E)$ be a digraph and let $\mathcal{F}$ be a subspace of polynomials in m variables over a field F. If G has a representation over $\mathcal{F}$ then $\omega_s(G) \leq \dim(\mathcal{F})$.*

Lemma 2.3.20 ([3]) *If G and H are two digraphs, G has a representation over $\mathcal{F}$ and H has a representation over $\mathcal{H}$, where $\mathcal{F}$ and $\mathcal{H}$ are spaces of polynomials over the same field F, then $\omega_s(G \cdot H) \leq \dim(\mathcal{F}) \cdot \dim(\mathcal{H})$.*

Remark 2.6. Lemmas 2.3.19 and 2.3.20 imply that if G and $\mathcal{F}$ are as in Lemma 2.3.19 then $\sigma(G) \leq \dim(\mathcal{F})$ (cf. Theorem 2.4 in [3]). Notice that our Theorem 2.3.1 is a specialized version of this statement where the subspace $\mathcal{F}$ of polynomials is defined via a proper coloring of the vertices attaining the value of $\psi_d(G)$. $\diamondsuit$

Our next Lemma is analogous to Proposition 1 of [50].

Lemma 2.3.21 *Let $F_1, F_2, \ldots, F_n$ be digraphs. Then*

$$\omega_s(F_1 \cdot F_2 \cdot \ldots \cdot F_n) \leq \prod_{i=1}^{n} \psi_d^*(F_i).$$

Proof. First observe that the argument for $\omega_s((G[K_r])^n) = r^n \omega_s(G^n)$ that led us to state Theorem 2.3.10 generalizes to

$$\omega_s(F_1[K_r] \cdot F_2[K_r] \cdot \ldots \cdot F_n[K_r]) = r^n \omega_s(F_1 \cdot F_2 \cdot \ldots \cdot F_n).$$

(This is simply by realizing that in the argument mentioned above we have not used anywhere that in the n-fold product in question all the graphs were the same whereby we dealt with the n'th power of a fixed graph.)

Take the representation (over subspaces of polynomials) given in the proof of Theorem 2.3.1 now for $F_1[K_r], F_2[K_r], \ldots, F_n[K_r]$, i.e., represent $F_i[K_r]$ for each i by the polynomials $\{P_{a,c_i}(x_i) := \prod_{j\in N^+_{c_i}(a)}(x_i - j)\}_{a\in V(F_i[K_r])}$, where c_i is a coloring of $V(F_i[K_r])$ that attains the value of $\psi_d(F_i[K_r])$. The dimension of this representation of $F_i[K_r]$ is bounded from above by $\psi_d(F_i[K_r])$. Now applying Lemma 2.3.20 we obtain

$$\omega_s(F_1[K_r] \cdot F_2[K_r] \cdot \ldots \cdot F_n[K_r]) \leq \prod_{i=1}^{n} \psi_d(F_i[K_r]) = \prod_{i=1}^{n} \psi_d(F_i, r).$$

Thus

$$\omega_s(F_1 \cdot F_2 \cdot \ldots \cdot F_n) = \frac{\omega_s(F_1[K_r] \cdot F_2[K_r] \cdot \ldots \cdot F_n[K_r])}{r^n} \leq \prod_{i=1}^{n} \frac{\psi_d(F_i, r)}{r}.$$

Since this last inequality is true for every positive integer r we can also write

$$\omega_s(F_1 \cdot F_2 \cdot \ldots \cdot F_n) \leq \inf_r \prod_{i=1}^{n} \frac{\psi_d(F_i, r)}{r} = \liminf_r \prod_{i=1}^{n} \frac{\psi_d(F_i, r)}{r} =$$

$$= \prod_{i=1}^{n} \liminf_r \frac{\psi_d(F_i, r)}{r} = \prod_{i=1}^{n} \psi^*_d(F_i).$$

□

Proof of Theorem 2.3.17. We call a function h assigning non-negative integer values to the elements of $2^{V(G)}$ a q-cover (q is a positive integer) of $V(G)$ if $\sum_{U\ni v} h(U) \geq q$ holds for all $v \in V(G)$.

It is clear that

$$\min_g \sum_{U\subseteq V(G)} g(U)\psi^*_d(G[U]) = \inf_q \frac{1}{q} \min_h \sum_{U\subseteq V(G)} h(U)\psi^*_d(G[U]),$$

where the minimization on the left hand side is over all fractional covers g while the minimization on the right hand side is over all q-covers h.

Let us fix a q and let h be the q-cover achieving the minimum on the right hand side. Let $\mathcal{U}$ be the multiset of those subsets of $V(G)$ that are assigned a positive value by h and let the multiplicity of $U \in V(G)$ in $\mathcal{U}$ be $h(U)$.

Fixing any natural number n denote by $\mathcal{U}^n$ the multiset of all n-fold Cartesian products of sets from $\mathcal{U}$. (The multiplicity of some $A = U_1 \times U_2 \times \ldots \times U_n \in \mathcal{U}^n$ is thus given by $h(U_1) \cdot h(U_2) \cdot \ldots \cdot h(U_n)$.)

We consider a maximum size symmetric clique K in G^n and observe that

$$q^n|K| \leq \sum_{\times_{i=1}^n U_i \in \mathcal{U}^n} \omega_s(G^n[\times_{i=1}^n U_i]).$$

Each summand in this last inequality satisfies by Lemma 2.3.21

$$\omega_s(G^n[\times_{i=1}^n U_i]) = \omega_s(\prod_{i=1}^n G[U_i]) \leq \prod_{i=1}^n \psi_d^*(G[U_i]).$$

Substituting this into the previous inequality we get

$$q^n|K| \leq \sum_{\times_{i=1}^n U_i \in \mathcal{U}^n} \prod_{i=1}^n \psi_d^*(G[U_i]) = \left[\sum_{U_i \in \mathcal{U}} \psi_d^*(G[U_i])\right]^n,$$

where the summations are meant with multiplicities.

Since K is a maximum size symmetric clique of G^n and the multiplicity of U_i in $\mathcal{U}$ is $h(U_i)$, we obtained

$$\omega_s(G^n) \leq \frac{1}{q^n}\left[\sum_{U \subseteq V(G)} h(U)\psi_d^*(G[U])\right]^n.$$

This implies

$$\sigma(G) \leq \inf_q \frac{1}{q}\left(\sum_{U \subseteq V(G)} h(U)\psi_d^*(G[U])\right) = \min_g \sum_{U \subseteq V(G)} g(U)\psi_d^*(G[U]),$$

where the minimization is over all fractional covers g of $V(G)$, i.e., we arrived at the statement. □

To illustrate that the bound of Theorem 2.3.17 may indeed give an improvement over that of Theorem 2.3.10 (or, in fact, over that of Corollary 2.3.11) consider the following digraph G. Let $V(G) = \{1, 2, \ldots, 2k+1, a, b\}$ and $E(G) = E(\vec{C}_{2k+1}) \cup \{(a,i),(i,b) : i \in \{1, \ldots, 2k+1\}\}$, where $\vec{C}_{2k+1}$ is an arbitrary non-alternatingly oriented cycle on $2k+1$ vertices. It is easy to check that $\psi_d^*(G) = 3 + \frac{1}{k}$ (and also $\psi_d^*(G_{\text{rev}}) = 3 + \frac{1}{k}$), i.e., Theorem 2.3.10 gives $\sigma(G) \leq 3 + \frac{1}{k}$ only, while Theorem 2.3.17 gives $\sigma(G) \leq 3$. Indeed, using the fractional cover (which is also an integer cover) $g(V_1) = g(V_2) = 1$, where $V_1 = \{1, 2, \ldots, 2k+1\}, V_2 = \{a, b\}$ (and $g(U) = 0$ for all other $U \subseteq V(G)$) we get $\sigma(G) \leq \psi_d^*(\vec{C}_{2k+1}) + \psi_d^*(\bar{K}_2) \leq \psi_d(\vec{C}_{2k+1}) + 1 = 3$. This bound is sharp since G contains transitive triangles.

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Chapter 3

Graph Colorings

3.1 Local chromatic number, Ky Fan's theorem, and circular colorings

In this section we will further investigate the local chromatic number of a graph that was introduced in [47]. As we have seen in the last section of the previous chapter it is in between the chromatic and fractional chromatic numbers. This motivates the study of the local chromatic number of graphs for which the former two quantities are far apart. Such graphs include Kneser graphs, their vertex color-critical subgraphs, the Schrijver (or stable Kneser) graphs; Mycielski graphs, and their generalizations; and Borsuk graphs. We give more or less tight bounds for the local chromatic number of many of these graphs.

We use an old topological result of Ky Fan [52] which generalizes the Borsuk-Ulam theorem. It implies the existence of a multicolored copy of the complete bipartite graph $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ in every proper coloring of many graphs whose chromatic number t is determined via a topological argument. (This was in particular noted for Kneser graphs by Ky Fan [53].) This yields a lower bound of $\lceil t/2\rceil + 1$ for the local chromatic number of these graphs. We show this bound to be tight or almost tight in many cases.

As another consequence of the above we prove that the graphs considered here have equal circular and ordinary chromatic numbers if the latter is even. This partially proves a conjecture of Johnson, Holroyd, and Stahl [81] and was independently attained by F. Meunier [121] for some of the most important special cases. We also show that odd chromatic Schrijver graphs behave differently, their circular chromatic number can be arbitrarily close to the other extreme.

This section is based on the joint paper [144] with Gábor Tardos.

3.1.1 Introduction

Recall that the local chromatic number $\psi(G)$ of a graph G is defined in [47] as the minimum number of colors that must appear within distance 1 of a vertex, see Definition 25 in the previous chapter (page 55). (When referring to the neighbors of a vertex v in graph G we will use the notation $N(v) = N_G(v)$. If c is a coloring of the vertices, then $c(N(v))$ will denote the set of those colors that appear on some vertex in $N(v)$, i.e., $c(N(v)) = \{c(u) : u \in N(v)\}$.)

Also recall the fact, that the local chromatic number of a graph G cannot be more than the chromatic number $\chi(G)$. If G is properly colored with $\chi(G)$ colors then each color class must contain a vertex, whose neighborhood contains all other colors. Thus a value $\psi(G) < \chi(G)$ can only be attained with a coloring in which more than $\chi(G)$ colors are used. Therefore it is somewhat surprising, that the local chromatic number can be arbitrarily less than the chromatic number, cf. [47], [57].

On the other hand, it was shown in [97] (see Theorem 2.3.7 in the previous chapter) that

$$\psi(G) \geq \chi^*(G)$$

holds for any graph G, where, as before, $\chi^*(G)$ denotes the fractional chromatic number of G.

This suggests to investigate the local chromatic number of graphs for which the chromatic number and the fractional chromatic number are far apart. This is our main goal in this section.

Prime examples of graphs with a large gap between the chromatic and the fractional chromatic numbers are Kneser graphs and Mycielski graphs, cf. [133]. Other, closely related examples are provided by Schrijver graphs, that are vertex color-critical induced subgraphs of Kneser graphs, and many of the so-called generalized Mycielski graphs. In this introductory subsection we focus on Kneser graphs and Schrijver graphs, Mycielski graphs and generalized Mycielski graphs will be treated in detail in Subsection 3.1.4.

We recall that the Kneser graph $\mathrm{KG}(n,k)$ is defined for parameters $n \geq 2k$ as the graph with all k-subsets of an n-set as vertices where two such vertices are connected if they represent disjoint k-sets. It is a celebrated result of Lovász [108] (see also [11, 69]) proving the earlier conjecture of Kneser [86], that $\chi(\mathrm{KG}(n,k)) = n - 2k + 2$. For the fractional chromatic number one has $\chi^*(\mathrm{KG}(n,k)) = n/k$ as easily follows from the vertex-transitivity of $\mathrm{KG}(n,k)$ via the Erdős-Ko-Rado theorem, see [133, 68].

Bárány's proof [11] of the Lovász-Kneser theorem was generalized by Schrijver [134] who found a fascinating family of subgraphs of Kneser graphs that are vertex-critical with respect to the chromatic number.

Let $[n]$ denote again the set $\{1, 2, \ldots, n\}$.

Definition 33 ([134]) *The stable Kneser graph or* Schrijver graph $\mathrm{SG}(n,k)$ *is defined as follows.*

$$\begin{aligned} V(\mathrm{SG}(n,k)) &= \{A \subseteq [n] : |A| = k, \forall i : \ \{i, i+1\} \nsubseteq A \ \text{ and } \ \{1, n\} \nsubseteq A\}, \\ E(\mathrm{SG}(n,k)) &= \{\{A, B\} : A \cap B = \emptyset\}. \end{aligned}$$

Thus $\mathrm{SG}(n,k)$ is the subgraph induced by those vertices of $\mathrm{KG}(n,k)$ that contain no neighboring elements in the cyclically arranged basic set $\{1, 2, \ldots, n\}$. These are sometimes called *stable k-subsets*. The result of Schrijver in [134] is that $\chi(\mathrm{SG}(n,k)) = n-2k+2 (= \chi(\mathrm{KG}(n,k))$, but deleting any vertex of $\mathrm{SG}(n,k)$ the chromatic number drops, i.e., $\mathrm{SG}(n,k)$ is vertex-critical with respect to the chromatic number. Recently Talbot [151] proved an Erdős-Ko-Rado type result, conjectured by Holroyd and Johnson [78], which implies that the ratio of the number of vertices and the independence number in $\mathrm{SG}(n,k)$ is n/k. This gives $n/k \le \chi^*(\mathrm{SG}(n,k))$ and equality follows by $\chi^*(\mathrm{SG}(n,k)) \le \chi^*(\mathrm{KG}(n,k)) = n/k$. Notice that $\mathrm{SG}(n,k)$ is not vertex-transitive in general. See more on Schrijver graphs in [22, 105, 116, 151, 160].

Concerning the local chromatic number it was observed by several people [58, 90], that $\psi(\mathrm{KG}(n,k)) \ge n-3k+3$ holds, since the neighborhood of any vertex in $\mathrm{KG}(n,k)$ induces a $\mathrm{KG}(n-k,k)$ with chromatic number $n-3k+2$. Thus for n/k fixed but larger than 3, $\psi(G)$ goes to infinity with n and k. In fact, the results of [47] have a similar implication also for $2 < n/k \le 3$. Namely, it follows from those results, that if a series of graphs $G_1, \dots, G_i, \dots$ is such that $\psi(G_i)$ is bounded, while $\chi(G_i)$ goes to infinity, then the number of colors to be used in colorings attaining the local chromatic number grows at least doubly exponentially in the chromatic number. However, Kneser graphs with n/k fixed and n (therefore also the chromatic number $n-2k+2$) going to infinity cannot satisfy this, since the total number of vertices grows simply exponentially in the chromatic number.

The estimates mentioned in the previous paragraph are elementary. On the other hand, all known proofs for $\chi(\mathrm{KG}(n,k)) \ge n-2k+2$ use topology or at least have a topological flavor (see [108, 11, 69, 117] to mention just a few such proofs). They use (or at least, are inspired by) the Borsuk-Ulam theorem.

Here we use a stronger topological result due to Ky Fan [52] to establish that all proper colorings of a t-chromatic Kneser, Schrijver or generalized Mycielski graph contain a multicolored copy of a balanced complete bipartite graph. This was noticed by Ky Fan for Kneser graphs [53]. We also show that the implied lower bound of $\lceil t/2 \rceil + 1$ on the local chromatic number is tight or almost tight for many Schrijver graphs and generalized Mycielski graphs.

In the following subsection we summarize our main results in more detail.

3.1.2 Results

In this subsection we summarize our results without introducing the topological notions needed to state the results in their full generality. We will introduce the phrase that a graph G is *topologically t-chromatic* meaning that $\chi(G) \ge t$ and this fact can be shown by a specific topological method, see Definition 34 in Subsection 3.1.3 (page 79). Here we use this phrase only to emphasize the generality of the corresponding statements, but the reader can always substitute the phrase "a topologically t-chromatic graph" by "a t-chromatic Kneser graph" or "a t-chromatic Schrijver graph" or by "a generalized Mycielski graph of chromatic number t".

Our general lower bound for the local chromatic number proven in Subsection 3.1.3 is the following.

Theorem 3.1.1 *If G is topologically t-chromatic for some $t \geq 2$, then*

$$\psi(G) \geq \left\lceil \frac{t}{2} \right\rceil + 1.$$

This result on the local chromatic number is the immediate consequence of the Zig-zag theorem in Subsection 3.1.3 that we state here in a somewhat weaker form:

Theorem 3.1.2 *Let G be a topologically t-chromatic graph and let c be a proper coloring of G with an arbitrary number of colors. Then there exists a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ of G all vertices of which receive a different color in c.*

We use Ky Fan's generalization of the Borsuk-Ulam theorem [52] for the proof. The Zig-zag theorem was previously established for Kneser graphs by Ky Fan [53].

We remark that János Körner [90] suggested to introduce a graph invariant $b(G)$ which is the size (number of points) of the largest completely multicolored complete bipartite graph that should appear in any proper coloring of graph G. It is obvious from the definition that this parameter is bounded from above by $\chi(G)$ and bounded from below by the local chromatic number $\psi(G)$. An obvious consequence of Theorem 3.1.2 is that if G is topologically t-chromatic, then $b(G) \geq t$.

In Subsection 3.1.4 we show that Theorem 3.1.1 is essentially tight for several Schrijver and generalized Mycielski graphs. In particular, this is always the case for a topologically t-chromatic graph that has a *wide* t-coloring as defined in Definition 35 in Subsection 3.1.4 (page 82).

As the first application of our result on wide colorings we show, that if the chromatic number is fixed and odd, and the size of the Schrijver graph is large enough, then Theorem 3.1.1 is exactly tight:

Theorem 3.1.3 *If $t = n - 2k + 2 > 2$ is odd and $n \geq 4t^2 - 7t$ then*

$$\psi(\mathrm{SG}(n,k)) = \left\lceil \frac{t}{2} \right\rceil + 1.$$

See Remark 3.4 in Subsection 3.1.4 for a relaxed bound on n. The proof of Theorem 3.1.3 is combinatorial. It will also show that the claimed value of $\psi(\mathrm{SG}(n,k))$ can be attained with a coloring using $t+1$ colors and avoiding the appearance of a totally multicolored $K_{\lceil \frac{t}{2} \rceil, \lceil \frac{t}{2} \rceil}$. To appreciate the latter property, cf. Theorem 3.1.2.

Since $\mathrm{SG}(n,k)$ is an induced subgraph of $\mathrm{SG}(n+1,k)$ Theorem 3.1.3 immediately implies that for every fixed even $t = n - 2k + 2$ and n, k large enough

$$\psi(\mathrm{SG}(n,k)) \in \left\{ \frac{t}{2} + 1, \frac{t}{2} + 2 \right\}.$$

To demonstrate that requiring large n and k in Theorem 3.1.3 is crucial we prove the following statement.

Proposition 3.1.4 $\psi(\mathrm{SG}(n,2)) = n-2 = \chi(\mathrm{SG}(n,2))$ *for every* $n \geq 4$.

As a second application of wide colorings we prove in Subsection 3.1.4 that Theorem 3.1.1 is also tight for several generalized Mycielski graphs. These graphs will be denoted by $M_{\boldsymbol{r}}^{(d)}(K_2)$ where $\boldsymbol{r} = (r_1, \ldots, r_d)$ is a vector of positive integers. See Subsection 3.1.4 for the definition. Informally, d is the number of iterations and r_i is the number of "levels" in iteration i of the generalized Mycielski construction. $M_{\boldsymbol{r}}^{(d)}(K_2)$ is proven to be $(d+2)$-chromatic "because of a topological reason" by Stiebitz [149]. This topological reason implies that these graphs are topologically $(d+2)$-chromatic. Thus Theorem 3.1.1 applies and gives the lower bound part of the following result.

Theorem 3.1.5 *If* $\boldsymbol{r} = (r_1, \ldots, r_d)$, *$d$ is odd, and* $r_i \geq 7$ *for all* i, *then*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) = \left\lceil \frac{d}{2} \right\rceil + 2.$$

It will be shown in Theorem 3.1.17 that relaxing the $r_i \geq 7$ condition to $r_i \geq 4$ an only slightly weaker upper bound is still valid. As a counterpart we also show (see Proposition 3.1.13 in Subsection 3.1.4) that for the ordinary Mycielski construction, which is the special case of $\boldsymbol{r} = (2, \ldots, 2)$, the local chromatic number behaves just like the chromatic number.

The Borsuk-Ulam Theorem in topology is known to be equivalent (see Lovász [111]) to the validity of a tight lower bound on the chromatic number of graphs defined on the n-dimensional sphere, called Borsuk graphs. In Subsection 3.1.4 we prove that the local chromatic number of Borsuk graphs behaves similarly as that of the graphs already mentioned above. In this subsection we also formulate a topological consequence of our results on the tightness of Ky Fan's theorem [52]. We also give a direct proof for the same tightness result.

The circular chromatic number $\chi_c(G)$ of a graph G was introduced by Vince [157], see Definition 38 in Subsection 3.1.5 (page 95). It satisfies $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$. In Subsection 3.1.5 we prove the following result using the Zig-zag theorem.

Theorem 3.1.6 *If G is topologically t-chromatic and t is even, then* $\chi_c(G) \geq t$.

This theorem implies that $\chi_c(G) = \chi(G)$ if the chromatic number is even for Kneser graphs, Schrijver graphs, generalized Mycielski graphs, and certain Borsuk graphs. The result on Kneser and Schrijver graphs gives a partial solution of a conjecture by Johnson, Holroyd, and Stahl [81] and a partial answer to a question of Hajiabolhassan and Zhu [73]. These results were independently obtained by Meunier [121]. The result on generalized Mycielski graphs answers a question of Chang, Huang, and Zhu [29] and partially solves a conjecture of theirs.

We will also discuss the circular chromatic number of odd chromatic Borsuk and Schrijver graphs showing that they can be close to one less than the chromatic number. We will use a a similar result for generalized Mycielski graphs proven by Lam, Lin, Gu, and Song [102].

3.1.3 Lower bound

Topological preliminaries

The following is a brief overview of some of the topological concepts we need. We refer to [21, 75] and [116] for basic concepts and also for a more detailed discussion of the notions and facts given below.

A $\mathbb{Z}_2$*-space* (or *involution space*) is a pair (T,ν) of a topological space T and the involution $\nu : T \to T$, which is continuous and satisfies that ν^2 is the identity map. The points $x \in T$ and $\nu(x)$ are called *antipodal*. The involution ν and the $\mathbb{Z}_2$-space (T,ν) are *free* if $\nu(x) \neq x$ for all points x of T. If the involution is understood from the context we speak about T rather than the pair (T,ν). This is the case, in particular, for the unit sphere $\mathbb{S}^d$ in $\mathbb{R}^{d+1}$ with the involution given by the central reflection $\boldsymbol{x} \mapsto -\boldsymbol{x}$. A continuous map $f : S \to T$ between $\mathbb{Z}_2$-spaces (S,ν) and (T,π) is a $\mathbb{Z}_2$*-map* (or an *equivariant map*) if it respects the respective involutions, that is $f \circ \nu = \pi \circ f$. If such a map exists we write $(S,\nu) \to (T,\pi)$. If $(S,\nu) \to (T,\pi)$ does not hold we write $(S,\nu) \not\to (T,\pi)$. If both $S \to T$ and $T \to S$ we call the $\mathbb{Z}_2$-spaces S and T $\mathbb{Z}_2$-equivalent and write $S \leftrightarrow T$.

We try to avoid using homotopy equivalence and $\mathbb{Z}_2$-homotopy equivalence (i.e., homotopy equivalence given by $\mathbb{Z}_2$-maps), but we will have to use two simple observations. First, if the $\mathbb{Z}_2$-spaces S and T are $\mathbb{Z}_2$-homotopy equivalent, then $S \leftrightarrow T$. Second, if the space S is homotopy equivalent to a sphere $\mathbb{S}^h$ (this relation is between topological spaces, not $\mathbb{Z}_2$-spaces), then for any involution ν we have $\mathbb{S}^h \to (S,\nu)$.

The $\mathbb{Z}_2$-index of a $\mathbb{Z}_2$-space (T,ν) is defined (see e.g. [118, 116]) as

$$\mathrm{ind}(T,\nu) := \min\{d \geq 0 : (T,\nu) \to \mathbb{S}^d\},$$

where $\mathrm{ind}(T,\nu)$ is set to be ∞ if $(T,\nu) \not\to \mathbb{S}^d$ for all d.

The $\mathbb{Z}_2$-coindex of a $\mathbb{Z}_2$-space (T,ν) is defined as

$$\mathrm{coind}(T,\nu) := \max\{d \geq 0 : \mathbb{S}^d \to (T,\nu)\}.$$

If such a map exists for all d, then we set $\mathrm{coind}(T,\nu) = \infty$. Notice that if (T,ν) is not free, we have $\mathrm{ind}(T,\nu) = \mathrm{coind}(T,\nu) = \infty$.

Note that $S \to T$ implies $\mathrm{ind}(S) \leq \mathrm{ind}(T)$ and $\mathrm{coind}(S) \leq \mathrm{coind}(T)$. In particular, $\mathbb{Z}_2$-equivalent spaces have equal index and also equal coindex.

The celebrated Borsuk-Ulam Theorem can be stated in many equivalent forms. Here we state three of them. For more equivalent versions and several proofs we refer to [116]. Here (i) and (ii) are standard forms of the Borsuk-Ulam Theorem, while (iii) is clearly equivalent to (ii).

Borsuk-Ulam Theorem.

(i) *(Lyusternik-Shnirel'man version) Let $d \geq 0$ and let $\mathcal{H}$ be a collection of open (or closed) sets covering $\mathbb{S}^d$ with no $H \in \mathcal{H}$ containing a pair of antipodal points. Then $|\mathcal{H}| \geq d+2$.*

(ii) $\mathbb{S}^{d+1} \not\to \mathbb{S}^d$ *for any* $d \geq 0$.

(iii) *For a* $\mathbb{Z}_2$*-space* T *we have* $\mathrm{ind}(T) \geq \mathrm{coind}(T)$.

The suspension $\mathrm{susp}(S)$ of a topological space S is defined as the factor of the space $S \times [-1, 1]$ that identifies all the points in $S \times \{-1\}$ and identifies also the points in $S \times \{1\}$. If S is a $\mathbb{Z}_2$-space with the involution ν, then the suspension $\mathrm{susp}(S)$ is also a $\mathbb{Z}_2$-space with the involution $(x, t) \mapsto (\nu(x), -t)$. Any $\mathbb{Z}_2$-map $f : S \to T$ naturally extends to a $\mathbb{Z}_2$-map $\mathrm{susp}(f) : \mathrm{susp}(S) \to \mathrm{susp}(T)$ given by $(x, t) \mapsto (f(x), t)$. We have $\mathrm{susp}(\mathbb{S}^n) \cong \mathbb{S}^{n+1}$ with a $\mathbb{Z}_2$-homeomorphism. These observations show the well known inequalities below.

Lemma 3.1.7 *For any* $\mathbb{Z}_2$*-space* S $\mathrm{ind}(\mathrm{susp}(S)) \leq \mathrm{ind}(S) + 1$ *and* $\mathrm{coind}(\mathrm{susp}(S)) \geq \mathrm{coind}(S) + 1$.

A(n abstract) simplicial complex K is a non-empty, hereditary set system. That is, $F \in K$, $F' \subseteq F$ implies $F' \in K$ and we have $\emptyset \in K$. In this work we consider only finite simplicial complexes. The non-empty sets in K are called *simplices*. We call the set $V(K) = \{x : \{x\} \in K\}$ the set of *vertices* of K. In a *geometric realization* of K a vertex x corresponds to a point $||x||$ in a Euclidean space, a simplex σ corresponds to its *body*, the convex hull of its vertices: $||\sigma|| = \mathrm{conv}(\{||x|| : x \in \sigma\})$. We assume that the points $||x||$ for $x \in \sigma$ are affine independent, and so $||\sigma||$ is a geometric simplex. We also assume that disjoint simplices have disjoint bodies. The body of the complex K is $||K|| = \cup_{\sigma \in K} ||\sigma||$, it is determined up to homeomorphism by K. Any point in $p \in ||K||$ has a unique representation as a convex combination $p = \sum_{x \in V(K)} \alpha_x ||x||$ such that $\{x : \alpha_x > 0\} \in K$.

A map $f : V(K) \to V(L)$ is called simplicial if it maps simplices to simplices, that is $\sigma \in K$ implies $f(\sigma) \in L$. In this case we define $||f|| : ||K|| \to ||L||$ by setting $||f||(||x||) = ||f(x)||$ for vertices $x \in V(K)$ and taking an affine extension of this function to the bodies of each of the simplices in K. If $||K||$ and $||L||$ are $\mathbb{Z}_2$-spaces (usually with an involution also given by simplicial maps), then we say that f is a $\mathbb{Z}_2$*-map* if $||f||$ is a $\mathbb{Z}_2$-map. If $||K||$ is a $\mathbb{Z}_2$-space we use $\mathrm{ind}(K)$ and $\mathrm{coind}(K)$ for $\mathrm{ind}(||K||)$ and $\mathrm{coind}(||K||)$, respectively.

Following the papers [5, 101, 118] we introduce the *box complex* $B_0(G)$ for any finite graph G. See [118] for several similar complexes. We define $B_0(G)$ to be a simplicial complex on the vertices $V(G) \times \{1, 2\}$. For subsets $S, T \subseteq V(G)$ we denote the set $S \times \{1\} \cup T \times \{2\}$ by $S \uplus T$, following the convention of [116, 118]. For $v \in V(G)$ we denote by $+v$ the vertex $(v, 1) \in \{v\} \uplus \emptyset$ and $-v$ denotes the vertex $(v, 2) \in \emptyset \uplus \{v\}$. We set $S \uplus T \in B_0(G)$ if $S \cap T = \emptyset$ and the complete bipartite graph with sides S and T is a subgraph of G. Note that $V(G) \uplus \emptyset$ and $\emptyset \uplus V(G)$ are simplices of $B_0(G)$.

The $\mathbb{Z}_2$-map $S \uplus T \mapsto T \uplus S$ acts simplicially on $B_0(G)$. It makes the body of the complex a free $\mathbb{Z}_2$-space.

We define the *hom space* $H(G)$ of G to be the subspace consisting of those points $p \in ||B_0(G)||$ that, when written as a convex combination $p = \sum_{x \in V(B_0(G))} \alpha_x ||x||$ with $\{x : \alpha_x > 0\} \in B_0(G)$ give $\sum_{x \in V(G) \uplus \emptyset} \alpha_x = 1/2$.

Notice that $H(G)$ can also be obtained as the body of a *cell complex* $Hom(K_2, G)$, see [9], or of a simplicial complex $B_{chain}(G)$, see [118].

A useful connection between $B_0(G)$ and $H(G)$ follows from a combination of results of Csorba [39] and Matoušek and Ziegler [118].

Proposition 3.1.8 $||B_0(G)|| \leftrightarrow \mathrm{susp}(H(G))$

Proof. Csorba [39] proves the $\mathbb{Z}_2$-homotopy equivalence of $||B_0(G)||$ and the suspension of the body of yet another box complex $B(G)$ of G. As we mentioned, $\mathbb{Z}_2$-homotopy equivalence implies $\mathbb{Z}_2$-equivalence. Matoušek and Ziegler [118] prove the $\mathbb{Z}_2$-equivalence of $||B(G)||$ and $H(G)$. Finally for $\mathbb{Z}_2$-spaces S and T if $S \to T$, then $\mathrm{susp}(S) \to \mathrm{susp}(T)$, therefore $||B(G)|| \leftrightarrow H(G)$ implies $\mathrm{susp}(||B(G)||) \leftrightarrow \mathrm{susp}(H(G))$. $\square$

Note that Csorba [39] proves, cf. also Živaljević [161], the $\mathbb{Z}_2$-homotopy equivalence of $||B(G)||$ and $H(G)$, and therefore we could also claim $\mathbb{Z}_2$-homotopy equivalence in Proposition 3.1.8.

Some earlier topological bounds

Recall that a graph homomorphism is an edge preserving map from the vertex set of a graph F to the vertex set of another graph G. If there is a homomorphism f from F to G, then it generates a simplicial map from $B_0(F)$ to $B_0(G)$ in the natural way. This map is a $\mathbb{Z}_2$-map and thus it shows $||B_0(F)|| \to ||B_0(G)||$. One can often prove $||B_0(F)|| \not\to ||B_0(G)||$ using the indexes or coindexes of these complexes and this relation implies the non-existence of a homomorphism from F to G. A similar argument applies with the spaces $H(\cdot)$ in place of $||B_0(\cdot)||$.

Coloring a graph G with m colors can be considered as a graph homomorphism from G to the complete graph K_m. The box complex $B_0(K_m)$ is the boundary complex of the m-dimensional *cross-polytope* (i.e., the convex hull of the basis vectors and their negatives in $\mathbb{R}^m$), thus $||B_0(K_m)|| \cong \mathbb{S}^{m-1}$ with a $\mathbb{Z}_2$-homeomorphism and $\mathrm{coind}(B_0(G)) \leq \mathrm{ind}(B_0(G)) \leq m-1$ is necessary for G being m-colorable. Similarly, $\mathrm{coind}(H(G)) \leq \mathrm{ind}(H(G)) \leq m-2$ is also necessary for $\chi(G) \leq m$ since $H(K_m)$ can be obtained from intersecting the boundary of the m-dimensional cross-polytope with the hyperplane $\sum x_i = 0$, and therefore $H(K_m) \cong \mathbb{S}^{m-2}$ with a $\mathbb{Z}_2$-homeomorphism. These four lower bounds on $\chi(G)$ can be arranged in a single line of inequalities using Lemma 3.1.7 and Proposition 3.1.8:

$$\chi(G) \geq \mathrm{ind}(H(G)) + 2 \geq \mathrm{ind}(B_0(G)) + 1 \geq \mathrm{coind}(B_0(G)) + 1 \geq \mathrm{coind}(H(G)) + 2 \quad (3.1)$$

In fact, many of the known proofs of Kneser's conjecture can be interpreted as a proof of an appropriate lower bound on the (co)index of one of the above complexes. In particular, Bárány's simple proof [11] exhibits a map showing $\mathbb{S}^{n-2k} \to H(\mathrm{KG}(n,k))$ to conclude that $\mathrm{coind}(H(\mathrm{KG}(n,k))) \geq n-2k$ and thus $\chi(\mathrm{KG}(n,k)) \geq n-2k+2$. The even simpler proof of Greene [69] exhibits a map showing $\mathbb{S}^{n-2k+1} \to B_0(\mathrm{KG}(n,k))$ to conclude that $\mathrm{coind}(B_0(\mathrm{KG}(n,k))) \geq n-2k+1$ and thus $\chi(\mathrm{KG}(n,k)) \geq n-2k+2$. Schrijver's proof [134] of $\chi(\mathrm{SG}(n,k)) \geq n-2k+2$ is a generalization of Bárány's and it also can be interpreted as a proof of $\mathbb{S}^{n-2k} \to H(\mathrm{SG}(n,k))$. We remark that the same kind of technique is used with other complexes related to graphs, too. In particular, Lovász's original proof [108] can also be considered as exhibiting a $\mathbb{Z}_2$-map from $\mathbb{S}^{n-2k}$ to such a complex, different from the ones we consider here. For a detailed discussion of several such complexes and their usefulness in bounding the chromatic number we refer the reader to [118].

The above discussion gives several possible "topological reasons" that can force a graph to be at least t-chromatic. Here we single out two such reasons. We would like to stress that these two reasons are just two out of many and refer to the paper [8] for some that are not even mentioned above. In this sense, our terminology is somewhat arbitrary. The statement of our results in Subsection 3.1.2 becomes precise by applying the conventions given by the following definition.

Definition 34 *We say that a graph G is* topologically t-chromatic *if*

$$\mathrm{coind}(B_0(G)) \geq t-1.$$

We say that a graph G is strongly topologically t-chromatic *if*

$$\mathrm{coind}(H(G)) \geq t-2.$$

By inequality (3.1) if a graph is strongly topologically t-chromatic, then it is topologically t-chromatic, and if G is topologically t-chromatic, then $\chi(G) \geq t$.

The notion that a graph is (strongly) topologically t-chromatic is useful, as it applies to many widely studied classes of graphs. As we mentioned above, Bárány [11] and Schrijver [134] establish this for t-chromatic Kneser and Schrijver graphs. For the reader's convenience we recall the proof here. See the analogous statement for generalized Mycielski graphs and (certain finite subgraphs of the) Borsuk graphs after we introduce those graphs.

Proposition 3.1.9 (Bárány; Schrijver) *The t-chromatic Kneser and Schrijver graphs are strongly topologically t-chromatic.*

Proof. We need to prove that $\mathrm{SG}(n,k)$ is strongly topologically $(n-2k+2)$-chromatic, i.e., that $\mathrm{coind}(H(\mathrm{SG}(n,k))) \geq n-2k$. The statement for Kneser graphs follows. For $\boldsymbol{x} \in \mathbb{S}^{n-2k}$ let $H_{\boldsymbol{x}}$ denote the open hemisphere in $\mathbb{S}^{n-2k}$ around $\boldsymbol{x}$. Consider an arrangement of the elements of $[n]$ on $\mathbb{S}^{n-2k}$ so that each open hemisphere contains a stable k-subset, i.e., a vertex of $\mathrm{SG}(n,k)$. It is not hard to check that identifying $i \in [n]$ with $\boldsymbol{v}_i/|\boldsymbol{v}_i|$

for $\boldsymbol{v}_i = (-1)^i(1, i, i^2, \ldots, i^{n-2k}) \in \mathbb{R}^{n-2k+1}$ provides such an arrangement. (See [134] or [116] for details of this.) For each vertex v of $\mathrm{SG}(n,k)$ and $\boldsymbol{x} \in \mathbb{S}^{n-2k}$ let $D_v(\boldsymbol{x})$ denote the smallest distance of a point in v from the set $\mathbb{S}^{n-2k} \setminus H_{\boldsymbol{x}}$ and let $D(\boldsymbol{x}) = \sum_{v \in V(\mathrm{SG}(n,k))} D_v(\boldsymbol{x})$. Note that $D_v(\boldsymbol{x}) > 0$ if v is contained in $H_{\boldsymbol{x}}$ and therefore $D(\boldsymbol{x}) > 0$ for all $\boldsymbol{x}$. Let $f(\boldsymbol{x}) := \frac{1}{2D(\boldsymbol{x})} \sum_{v \in V(\mathrm{SG}(n,k))} D_v(\boldsymbol{x}) ||+v|| + \frac{1}{2D(-\boldsymbol{x})} \sum_{v \in V(\mathrm{SG}(n,k))} D_v(-\boldsymbol{x}) ||-v||$. This f is a $\mathbb{Z}_2$-map $\mathbb{S}^{n-2k} \to H(\mathrm{SG}(n,k))$ proving the proposition. □

Ky Fan's result on covers of spheres and the Zig-Zag theorem

The following result of Ky Fan [52] implies the Lyusternik-Shnirel'man version of the Borsuk-Ulam theorem. Here we state two equivalent versions of the result, both in terms of sets covering the sphere. See the original paper for another version generalizing another standard form of the Borsuk-Ulam theorem.

Ky Fan's Theorem.

(i) *Let $\mathcal{A}$ be a system of open (or a finite system of closed) subsets of $\mathbb{S}^k$ covering the entire sphere. Assume a linear order $<$ is given on $\mathcal{A}$ and all sets $A \in \mathcal{A}$ satisfy $A \cap (-A) = \emptyset$. Then there are sets $A_1 < A_2 < \ldots < A_{k+2}$ of $\mathcal{A}$ and a point $\boldsymbol{x} \in \mathbb{S}^k$ such that $(-1)^i \boldsymbol{x} \in A_i$ for all $i = 1, \ldots, k+2$.*

(ii) *Let $\mathcal{A}$ be a system of open (or a finite system of closed) subsets of $\mathbb{S}^k$ such that $\cup_{A \in \mathcal{A}}(A \cup (-A)) = \mathbb{S}^k$. Assume a linear order $<$ is given on $\mathcal{A}$ and all sets $A \in \mathcal{A}$ satisfy $A \cap (-A) = \emptyset$. Then there are sets $A_1 < A_2 < \ldots < A_{k+1}$ of $\mathcal{A}$ and a point $\boldsymbol{x} \in \mathbb{S}^k$ such that $(-1)^i \boldsymbol{x} \in A_i$ for all $i = 1, \ldots, k+1$.*

The Borsuk-Ulam theorem is easily seen to be implied by version (i), that shows in particular, that $|\mathcal{A}| \geq k+2$. We remark that [52] contains the above statements only for closed sets. The statements on open sets can be deduced by a standard argument using the compactness of the sphere. We also remark that version (ii) is formulated a little differently in [52]. A place where one finds exactly the above formulation (for closed sets, but for any $\mathbb{Z}_2$-space) is Bacon's paper [10].

Zig-zag Theorem *Let G be a topologically t-chromatic finite graph and let c be an arbitrary proper coloring of G by an arbitrary number of colors. We assume the colors are linearly ordered. Then G contains a complete bipartite subgraph $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ such that c assigns distinct colors to all t vertices of this subgraph and these colors appear alternating on the two sides of the bipartite subgraph with respect to their order.*

Proof. We have $\mathrm{coind}(B_0(G)) \geq t-1$, so there exists a $\mathbb{Z}_2$-map $f : \mathbb{S}^{t-1} \to ||B_0(G)||$. For any color i we define a set $A_i \subset \mathbb{S}^{t-1}$ letting $\boldsymbol{x} \in A_i$ if and only if for the minimal simplex $U_{\boldsymbol{x}} \uplus V_{\boldsymbol{x}}$ containing $f(\boldsymbol{x})$ there exists a vertex $z \in U_{\boldsymbol{x}}$ with $c(z) = i$. These sets are open, but they do not necessarily cover the entire sphere $\mathbb{S}^{t-1}$. Notice that $-A_i$ consists of the points $\boldsymbol{x} \in \mathbb{S}^{t-1}$ with $-\boldsymbol{x} \in A_i$, which happens if and only if there exists a vertex $z \in U_{-\boldsymbol{x}}$

with $c(z) = i$. Here $U_{-\boldsymbol{x}} = V_{\boldsymbol{x}}$. For every $\boldsymbol{x} \in \mathbb{S}^{t-1}$ either $U_{\boldsymbol{x}}$ or $V_{\boldsymbol{x}}$ is not empty, therefore we have $\cup_i(A_i \cup (-A_i)) = \mathbb{S}^{t-1}$. Assume for a contradiction that for a color i we have $A_i \cap (-A_i) \neq \emptyset$ and let $\boldsymbol{x}$ be a point in the intersection. We have a vertex $z \in U_{\boldsymbol{x}}$ and a vertex $z' \in V_{\boldsymbol{x}}$ with $c(z) = c(z') = i$. By the definition of $B_0(G)$ the vertices z and z' are connected in G. This contradicts the choice of c as a proper coloring. The contradiction shows that $A_i \cap (-A_i) = \emptyset$ for all colors i.

Applying version (ii) of Ky Fan's theorem we get that for some colors $i_1 < i_2 < \ldots < i_t$ and a point $\boldsymbol{x} \in \mathbb{S}^{t-1}$ we have $(-1)^j\boldsymbol{x} \in A_{i_j}$ for $j = 1, 2, \ldots t$. This implies the existence of vertices $z_j \in U_{(-1)^j\boldsymbol{x}}$ with $c(z_j) = i_j$. Now $U_{(-1)^j\boldsymbol{x}} = U_{\boldsymbol{x}}$ for even j and $U_{(-1)^j\boldsymbol{x}} = V_{\boldsymbol{x}}$ for odd j. Therefore the complete bipartite graph with sides $\{z_j | j \text{ is odd}\}$ and $\{z_j | j \text{ is even}\}$ is a subgraph of G with the required properties. □

This result was previously established for Kneser graphs in [53].

Remark 3.1. Since for any fixed coloring we are allowed to order the colors in an arbitrary manner, the Zig-zag theorem implies the existence of several totally multicolored copies of $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$. For a uniform random order any fixed totally multicolored $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$ satisfies the zig-zag rule with probability $1/\binom{t}{\lfloor t/2\rfloor}$ if t is odd and with probability $2/\binom{t}{t/2}$ if t is even. Thus the Zig-zag theorem implies the existence of many differently colored totally multicolored subgraphs $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$ in G: $\binom{t}{\lfloor t/2\rfloor}$ copies for odd t and $\binom{t}{t/2}/2$ copies for even t.

If the coloring uses only t colors we get a totally multicolored $K_{\lceil\frac{t}{2}\rceil,\lfloor\frac{t}{2}\rfloor}$ subgraph with all possible colorings, and the number of these different subgraphs is exactly the lower bound stated. ♢

Proof of Theorems 3.1.1 and 3.1.2.

Theorems 3.1.1 and 3.1.2 are direct consequences of the Zig-zag theorem. For Theorem 3.1.2 this is obvious. To prove Theorem 3.1.1 consider any vertex of the $\lfloor t/2\rfloor$ side of a multicolored complete bipartite graph. It has $\lceil t/2\rceil$ differently colored neighbors on the other side, thus at least $\lceil t/2\rceil$ different colors in its neighborhood. □

Remark 3.2. Theorem 3.1.1 gives tight lower bounds for the local chromatic number of topologically t-chromatic graphs for odd t as several examples of the next subsection will show. In [146] we present examples that show that the situation is similar for even values of t. However, the graphs establishing this fact are *not* strongly topologically t-chromatic, whereas the graphs showing tightness of Theorem 3.1.1 for odd t are. This leaves open the question whether $\psi(G) \geq t/2 + 2$ holds for all strongly topologically t-chromatic graphs G and even $t \geq 4$. While we prove this statement in [146] for $t = 4$ we do not know the answer for higher values of t. ♢

3.1.4 Upper bound

In this subsection we present the combinatorial constructions that prove Theorems 3.1.3 and 3.1.5. In both cases general observations on wide colorings (to be defined below) prove useful. The upper bound in either of Theorems 3.1.3 or 3.1.5 implies the existence of certain open covers of spheres. These topological consequences and the local chromatic number of Borsuk graphs are discussed in the last part of this subsection.

Wide colorings

We start here with a general method to alter a t-coloring and get a $(t+1)$-coloring showing that $\psi \leq t/2+2$. It works if the original coloring was wide as defined below.

Definition 35 *A vertex coloring of a graph is called* wide *if the end vertices of all walks of length* 5 *receive different colors.*

Note that any wide coloring is proper, furthermore any pair of vertices of distance 3 or 5 receive distinct colors. Moreover, if a graph has a wide coloring it does not contain a cycle of length 3 or 5. For graphs that do not have cycles of length 3, 5, 7, or 9 any coloring is wide that assigns different colors to vertices of distance 1, 3 or 5 apart. Another equivalent definition (considered in [72]) is that a proper coloring is wide if the neighborhood of any color class is an independent set and so is the second neighborhood.

Lemma 3.1.10 *If a graph G has a wide coloring using t colors, then $\psi(G) \leq \lfloor t/2 \rfloor + 2$.*

Proof. Let c_0 be the wide t-coloring of G. We alter this coloring by switching the color of the neighbors of the troublesome vertices to a new color. We define a vertex x to be *troublesome* if $|c_0(N(x))| > t/2$. Assume the color β is not used in the coloring c_0. For $x \in V(G)$ we let

$$c(x) = \begin{cases} \beta & \text{if } x \text{ has a troublesome neighbor} \\ c_0(x) & \text{otherwise.} \end{cases}$$

The color class β in c is the union of the neighborhoods of troublesome vertices. To see that this is an independent set consider any two vertices z and z' of color β. Let y be a troublesome neighbor of z and let y' be a troublesome neighbor of z'. Both $c_0(N(y))$ and $c_0(N(y'))$ contain more than half of the t colors in c_0, therefore these sets are not disjoint. We have a neighbor x of y and a neighbor x' of y' satisfying $c_0(x) = c_0(x')$. This shows that z and z' are not connected, as otherwise the walk $xyzz'y'x'$ of length 5 would have two end vertices in the same color class.

All other color classes of c are subsets of the corresponding color classes in c_0, and are therefore independent. Thus c is a proper coloring.

Any troublesome vertex x has now all its neighbors recolored, therefore $c(N(x)) = \{\beta\}$. For the vertices of G that are not troublesome one has $|c_0(N(x))| \leq t/2$ and $c(N(x)) \subseteq c_0(N(x)) \cup \{\beta\}$, therefore $|c(N(x))| \leq t/2+1$. Thus the coloring c shows $\psi(G) \leq t/2+2$ as claimed. $\square$

We note that the coloring c found in the proof uses $t+1$ colors and any vertex that sees the maximal number $\lfloor t/2 \rfloor + 1$ of the colors in its neighborhood must have a neighbor of color β. In particular, for odd t one will always find two vertices of the same color in any $K_{(t+1)/2,(t+1)/2}$ subgraph.

Schrijver graphs

Here we prove Theorem 3.1.3 which shows that the local chromatic number of Schrijver graphs with certain parameters are as low as allowed by Theorem 3.1.1. We also prove Proposition 3.1.4 to show that for some other Schrijver graphs the local chromatic number agrees with the chromatic number.

For the proof of Theorem 3.1.3 we will use the following simple lemma.

Lemma 3.1.11 *Let $u, v \subseteq [n]$ be two vertices of $\mathrm{SG}(n, k)$. If there is a walk of length $2s$ between u and v in $\mathrm{SG}(n, k)$ then $|v \setminus u| \leq s(t-2)$, where $t = n - 2k + 2 = \chi(\mathrm{SG}(n, k))$.*

Proof. Let xyz be a 2-length walk in $\mathrm{SG}(n, k)$. Since y is disjoint from x, it contains all but $n - 2k = t - 2$ elements of $[n] \setminus x$. As z is disjoint from y it can contain at most $t - 2$ elements not contained in x. This proves the statement for $s = 1$.

Now let $x_0x_1 \ldots x_{2s}$ be a $2s$-length walk between $u = x_0$ and $v = x_{2s}$ and assume the statement is true for $s - 1$. Since $|v \setminus u| \leq |v \setminus x_{2s-2}| + |x_{2s-2} \setminus u| \leq (t-2) + (s-1)(t-2)$ we can complete the proof by induction. □

We remark that Lemma 3.1.11 remains true for $\mathrm{KG}(n, k)$ with literally the same proof, but we will need it for $\mathrm{SG}(n, k)$, this is why it is stated that way.

Theorem 3.1.3 (restated) *If $t = n - 2k + 2 > 2$ is odd and $n \geq 4t^2 - 7t$, then*

$$\psi(\mathrm{SG}(n, k)) = \left\lceil \frac{t}{2} \right\rceil + 1.$$

Proof. We need to show that $\psi(\mathrm{SG}(n, k)) = (t+3)/2$. Note that the $t = 3$ case is trivial as all 3-chromatic graphs have local chromatic number 3. The lower bound for the local chromatic number follows from Theorem 3.1.1 and Proposition 3.1.9.

We define a wide coloring c_0 of $\mathrm{SG}(n, k)$ using t colors. From this Lemma 3.1.10 gives the upper bound on $\psi(\mathrm{SG}(n, k))$.

Let $[n] = \{1, \ldots, n\}$ be partitioned into t sets, each containing an odd number of consecutive elements of $[n]$. More formally, $[n]$ is partitioned into disjoint sets $A_1, \ldots, A_t$, where each A_i contains consecutive elements and $|A_i| = 2p_i - 1$. We need $p_i \geq 2t - 3$ for the proof, this is possible as long as $n \geq t(4t - 7)$ as assumed.

Notice, that $\sum_{i=1}^{t}(p_i - 1) = k - 1$, and therefore any k-element subset x of $[n]$ must contain more than half (i.e., at least p_i) of the elements in some A_i. We define our coloring c_0 by arbitrarily choosing such an index i as the color $c_0(x)$. This is a proper coloring even for the graph $\mathrm{KG}(n, k)$ since if two sets x and y both contain more than half of the elements of A_i, then they are not disjoint.

As a coloring of $\mathrm{KG}(n, k)$ the coloring c_0 is not wide. We need to show that the coloring c_0 becomes wide if we restrict it to the subgraph $\mathrm{SG}(n, k)$.

The main observation is the following: A_i contains a single subset of cardinality p_i that does not contain two consecutive elements. Let C_i be this set consisting of the first, third, etc. elements of A_i. A vertex of $\mathrm{SG}(n,k)$ has no two consecutive elements, thus a vertex x of $\mathrm{SG}(n,k)$ of color i must contain C_i.

Consider a walk $x_0x_1\dots x_5$ of length 5 in $\mathrm{SG}(n,k)$ and let $i=c_0(x_0)$. Thus the set x_0 contains C_i. By Lemma 3.1.11 $|x_4\setminus x_0|\le 2(t-2)$. In particular, x_4 contains all but at most $2t-4$ elements of C_i. As $p_i=|C_i|\ge 2t-3$, this means $x_4\cap C_i\ne\emptyset$. Thus the set x_5, which is disjoint from x_4, cannot contain all elements of C_i, showing $c_0(x_5)\ne i$. This proves that the coloring c_0 is wide, thus Lemma 3.1.10 completes the proof of the theorem. $\square$

Note that the smallest Schrijver graph for which the above proof gives $\psi(\mathrm{SG}(n,k))<\chi(\mathrm{SG}(n,k))$ is $G=\mathrm{SG}(65,31)$ with $\chi(G)=5$ and $\psi(G)=4$. In Remark 3.4 below we show how the lower bound on n can be lowered somewhat. After that we show that some lower bound is needed as $\psi(\mathrm{SG}(n,2))=\chi(\mathrm{SG}(n,2))$ for every n.

Remark 3.3. In the previous chapter (cf. Definition 29) we have already seen the universal graphs $U(m,r)$ defined in [47] for which it is shown that a graph G can be colored with m colors such that the neighborhood of every vertex contains fewer than r colors if and only if a homomorphism from G to $U(m,r)$ exists. The proof of Theorem 3.1.3 gives, for odd t, a $(t+1)$-coloring of $\mathrm{SG}(n,k)$ (for appropriately large n and k that give chromatic number t) for which no neighborhood contains more than $(t+1)/2$ colors, thus establishing the existence of a homomorphism from $\mathrm{SG}(n,k)$ to $U(t+1,(t+3)/2)$. This, in particular, proves that $\chi(U(t+1,(t+3)/2))\ge t$, which is a special case of Theorem 2.6 in [47]. It is not hard to see that this inequality is actually an equality. Further, by the composition of the appropriate maps, the existence of this homomorphism also proves that $U(t+1,(t+3)/2)$ is strongly topologically t-chromatic. $\Diamond$

Remark 3.4. For the price of letting the proof be a bit more complicated one can improve upon the bound given on n in Theorem 3.1.3. In particular, one can show that the same conclusion holds for odd t and $n\ge 2t^2-4t+3$. More generally, we can show $\psi(\mathrm{SG}(n,k))\le\chi(\mathrm{SG}(n,k))-m=n-2k+2-m$ provided that $\chi(\mathrm{SG}(n,k))\ge 2m+3$ and $n\ge 8m^2+16m+9$ or $\chi(\mathrm{SG}(n,k))\ge 4m+3$ and $n\ge 20m+9$. The smallest Schrijver graph for which we can prove that the local chromatic number is smaller than the ordinary chromatic number is $\mathrm{SG}(33,15)$ with 1496 vertices and $\chi=5$ but $\psi=4$. (In general, one has $|V(\mathrm{SG}(n,k))|=\frac{n}{k}\binom{n-k-1}{k-1}$, cf. Lemma 1 in [151].) The smallest n and k for which we can prove $\psi(\mathrm{SG}(n,k))<\chi(\mathrm{SG}(n,k))$ is for the graph $\mathrm{SG}(29,12)$ for which $\chi=7$ but $\psi\le 6$.

We only sketch the proof. For a similar and more detailed proof see Theorem 3.1.17. The idea is again to take a basic coloring c_0 of $\mathrm{SG}(n,k)$ and obtain a new coloring c by recoloring to a new color some neighbors of those vertices v for which $|c_0(N(v))|$ is too large. The novelty is that now we do not recolor all such neighbors, just enough of them, and also the definition of the basic coloring c_0 is a bit different. Partition $[n]$ into $t=n-2k+2$ intervals $A_1,\dots,A_t$, each of odd length as in the proof of Theorem 3.1.3 and also define C_i similarly to be the unique largest subset of A_i not containing consecutive elements. For a vertex x we define $c_0(x)$ to be the *smallest* i for which $C_i\subseteq x$. Note that such an i must exist. Now we define when to recolor a vertex to the new color β if our goal is to prove $\psi(\mathrm{SG}(n,k)) \le b := t-m$, where $m > 0$. We let $c(y) = \beta$ iff y is the neighbor of a vertex x having at least $b-2$ different colors *smaller* than $c_0(y)$ in its neighborhood. Otherwise, $c(y) = c_0(y)$. It is clear that $|c(N(x))| \le b-1$ is satisfied, the only problem we face is that c may not be a proper coloring. To avoid this problem we only need that the recolored vertices form an independent set. For each vertex v define the index set $I(v) := \{j : v \cap C_j = \emptyset\}$. If y and y' are recolored vertices then they are neighbors of some x and x', respectively, where $I(x)$ contains $c_0(y)$ and at least $b-2$ indices smaller than $c_0(y)$ and $I(x')$ contains $c_0(y')$ and at least $b-2$ indices smaller than $c_0(y')$. Since $|[n] \setminus (x \cup y)| = t-2$, there are at most $t-2$ elements in $\cup_{j \in I(x)} C_j$ not contained in y. The definition of c_0 also implies that at least one element of C_j is missing from y for every $j < c_0(y)$. Similarly, there are at most $t-2$ elements in $\cup_{j \in I(x')} C_j$ not contained in y' and at least one element of C_j is missing from y' for every $j < c_0(y')$. These conditions lead to $y \cap y' \neq \emptyset$ if the sizes $|A_i| = 2|C_i|-1$ are appropriately chosen. In particular, if $t \ge 2m+3$ and $|A_t| \ge 1$, $|A_{t-1}| \ge 2m+3$, $|A_{t-2}| \ge \ldots \ge |A_{t-(2m+2)}| \ge 4m+5$, or $t \ge 4m+3$ and $|A_t| \ge 1$, $|A_{t-1}| \ge 3$, $|A_{t-2}| \ge \ldots \ge |A_{t-(4m+2)}| \ge 5$, then the above argument leads to a proof of $\psi(\mathrm{SG}(n,k)) \le t-m$. (It takes some further but simple argument why the last two intervals A_i can be chosen smaller than the previous ones.) These two possible choices of the interval sizes give the two general bounds on n we claimed sufficient for attaining $\psi(\mathrm{SG}(n,k)) \le t-m$. The strengthening of Theorem 3.1.3 is obtained by the $m = (t-3)/2$ special case of the first bound. $\Diamond$

Proposition 3.1.4 (restated) $\psi(\mathrm{SG}(n,2)) = n-2 = \chi(\mathrm{SG}(n,2))$ *for every* $n \ge 4$.

Proof. In the $n = 4$ case $\mathrm{SG}(n,2)$ consists of a single edge and the statement of the proposition is trivial. Assume for a contradiction that $\psi(\mathrm{SG}(n,2)) \le n-3$ for some $n \ge 5$ and let c be a proper coloring of $\mathrm{SG}(n,2)$ showing this with the minimal number of colors. As $\chi(\mathrm{SG}(n,2)) = n-2$ and a coloring of a graph G with exactly $\chi(G)$ colors cannot show $\psi(G) < \chi(G)$ the coloring c uses at least $n-1$ colors.

It is worth visualizing the vertices of $\mathrm{SG}(n,2)$ as diagonals of an n-gon (cf. [22]). In other words, $\mathrm{SG}(n,2)$ is the complement of the line graph of D_n, where D_n is the complement of the cycle C_n. The color classes are independent sets in $\mathrm{SG}(n,2)$, so they are either stars or triangles in D_n.

We say that a vertex x *sees* the color classes of its neighbors. By our assumption every vertex sees at most $n-4$ color classes.

Assume a color class consists of a single vertex x. As x sees at most $n-4$ of the at least $n-1$ color classes we can choose a different color for x. The resulting coloring attains the same local chromatic number with fewer colors. This contradicts the choice of c and shows that no color class is a singleton.

A triangle color class is seen by all other edges of D_n. A star color class with center i and at least three elements is seen by all vertices that, as edges of D_n, are not incident to i. For star color classes of two edges there can be one additional vertex not seeing the class. So every color class is seen by all but at most $n-2$ vertices. We double count the pairs of a vertex x and a color class C seen by x. On one hand every vertex sees at most $n-4$ classes. On the other hand all the color classes are seen by at least $\left(\binom{n}{2}-n\right)-(n-2)$ vertices. We have

$$(n-1)\left(\binom{n}{2}-2n+2\right) \leq \left(\binom{n}{2}-n\right)(n-4),$$

and this contradicts our $n \geq 5$ assumption. The contradiction proves the statement. □

Generalized Mycielski graphs

Another class of graphs for which the chromatic number is known only via the topological method is formed by generalized Mycielski graphs, see [72, 116, 149]. They are interesting for us also for another reason: there is a big gap between their fractional and ordinary chromatic numbers (see [103, 152]), therefore the local chromatic number can take its value from a large interval.

Recall that the Mycielskian $M(G)$ of a graph G is the graph defined on $(\{0,1\}\times V(G))\cup \{z\}$ with edge set $E(M(G)) = \{\{(0,v),(i,w)\} : \{v,w\} \in E(G), i \in \{0,1\}\} \cup \{\{(1,v),z\} : v \in V(G)\}$. Mycielski [123] used this construction to increase the chromatic number of a graph while keeping the clique number fixed: $\chi(M(G)) = \chi(G)+1$ and $\omega(M(G)) = \omega(G)$.

Following Tardif [152], the same construction can also be described as the direct (also called categorical) product of G with a path on three vertices having a loop at one end and then identifying all vertices that have the other end of the path as their first coordinate. Recall that the direct product of F and G is a graph on $V(F)\times V(G)$ with an edge between (u,v) and (u',v') if and only if $\{u,u'\} \in E(F)$ and $\{v,v'\} \in E(G)$. The generalized Mycielskian of G (called a cone over G by Tardif [152]) $M_r(G)$ is then defined by taking the direct product of P and G, where P is a path on $r+1$ vertices having a loop at one end, and then identifying all the vertices in the product with the loopless end of the path as their first coordinate. With this notation $M(G) = M_2(G)$. These graphs were considered by Stiebitz [149], who proved that if G is k-chromatic "for a topological reason" then $M_r(G)$ is $(k+1)$-chromatic for a similar reason. (Gyárfás, Jensen, and Stiebitz [72] also consider these graphs and quote Stiebitz's argument a special case of which is also presented in [116].) The topological reason of Stiebitz is in different terms than those we use in this work but using results of [9] they imply strong topological $(t+d)$-chromaticity for graphs obtained by d iterations of the generalized Mycielski construction starting, e.g, from K_t or from a t-chromatic Schrijver graph. More precisely, Stiebitz proved that the body of the so-called neighborhood complex $\mathcal{N}(M_r(G))$ of $M_r(G)$, introduced in [108] by Lovász, is homotopy equivalent to the suspension of $||\mathcal{N}(G)||$. Since $\mathrm{susp}(\mathbb{S}^n) \cong \mathbb{S}^{n+1}$ this implies that whenever $||\mathcal{N}(G)||$ is homotopy equivalent to an n-dimensional sphere, then $||\mathcal{N}(M_r(G))||$ is homotopy equivalent to the $(n+1)$-dimensional sphere. This happens, for example, if G is a complete graph, or an odd cycle. By a recent result of Björner and de Longueville [22] we also have a similar situation if G is isomorphic to any Schrijver graph $\mathrm{SG}(n,k)$. Notice that the latter include complete graphs and odd cycles.

It is known, that $||\mathcal{N}(F)||$ is homotopy equivalent to $H(F)$ for every graph F, see Proposition 4.2 in [9]. All this implies that $\mathrm{coind}(H(M_r(G))) = \mathrm{coind}(H(G))+1$ whenever $H(G)$ is homotopy equivalent to a sphere, in particular, whenever G is a complete graph or an odd cycle, or, more generally, a Schrijver graph. In the first version of the paper this section is based on we wrote that it is very likely that Stiebitz's proof can be generalized to show that $H(M_r(G)) \leftrightarrow \mathrm{susp}(H(G))$ and therefore $\mathrm{coind}(H(M_r(G))) \geq \mathrm{coind}(H(G)) + 1$ always holds. Since then Csorba [40, 41] succeeded to prove this generalization. In fact, he proved $\mathbb{Z}_2$-homotopy equivalence of $H(M_r(G))$ and $\mathrm{susp}(H(G))$. Nevertheless, here we restrict attention to graphs G with $H(G)$ homotopy equivalent to a sphere.

For an integer vector $\boldsymbol{r} = (r_1, \ldots, r_d)$ with $r_i \geq 1$ for all i we let $M_{\boldsymbol{r}}^{(d)}(G) = M_{r_d}(M_{r_{d-1}}(\ldots M_{r_1}(G)\ldots))$ denote the graph obtained by a d-fold application of the generalized Mycielski construction with respective parameters $r_1, \ldots, r_d$. Now we state Stiebitz's result with this notation for later reference.

Proposition 3.1.12 (Stiebitz) *If G is a graph for which $H(G)$ is homotopy equivalent to a sphere $\mathbb{S}^h$ with $h = \chi(G) - 2$ (in particular, G is a complete graph or an odd cycle, or, more generally, a Schrijver graph) and $\boldsymbol{r} = (r_1, \ldots, r_d)$ is arbitrary, then $M_{\boldsymbol{r}}^{(d)}(G)$ is strongly topologically t-chromatic for $t = \chi(M_{\boldsymbol{r}}^{(d)}(G)) = \chi(G) + d$.* □

It is interesting to remark that $\chi(M_r(G)) > \chi(G)$ does not hold in general if $r \geq 3$, e.g., for $\overline{C}_7$, the complement of the 7-cycle, one has $\chi(M_3(\overline{C}_7)) = \chi(\overline{C}_7) = 4$ (cf. [152]). Still, the result of Stiebitz implies that the sequence $\{\chi(M_{\boldsymbol{r}}^{(d)}(G))\}_{d=1}^{\infty}$ may avoid to increase only a finite number of times.

The fractional chromatic number of Mycielski graphs were determined by Larsen, Propp, and Ullman [103], who proved that $\chi^*(M(G)) = \chi^*(G) + \frac{1}{\chi^*(G)}$ holds for every G. This already shows that there is a large gap between the chromatic and the fractional chromatic numbers of $M_{\boldsymbol{r}}^{(d)}(G)$ if d is large enough and $r_i \geq 2$ for all i, since obviously, $\chi^*(M_r(F)) \leq \chi^*(M(F))$ holds if $r \geq 2$. The previous result was generalized by Tardif [152] who showed that $\chi^*(M_r(G))$ can also be expressed by $\chi^*(G)$ as $\chi^*(G) + \frac{1}{\sum_{i=0}^{r-1}(\chi^*(G)-1)^i}$ whenever G has at least one edge.

First we show that for the original Mycielski construction the local chromatic number behaves similarly to the chromatic number.

Proposition 3.1.13 *For any graph G we have*

$$\psi(M(G)) = \psi(G) + 1.$$

Proof. We proceed similarly as one does in the proof of $\chi(M(G)) = \chi(G) + 1$. Recall that $V(M(G)) = \{0,1\} \times V(G) \cup \{z\}$.

For the upper bound consider a coloring c' of G establishing its local chromatic number and let α and β be two colors not used by c'. We define $c((0,x)) = c'(x)$, $c((1,x)) = \alpha$ and $c(z) = \beta$. This proper coloring shows $\psi(M(G)) \leq \psi(G) + 1$.

For the lower bound consider an arbitrary proper coloring c of $M(G)$. We have to show that some vertex must see at least $\psi(G)$ different colors in its neighborhood.

We define the coloring c' of G as follows:

$$c'(x) = \begin{cases} c((0,x)) & \text{if } c((0,x)) \neq c(z) \\ c((1,x)) & \text{otherwise.} \end{cases}$$

It follows from the construction that c' is a proper coloring of G. Note that c' does not use the color $c(z)$.

By the definition of $\psi(G)$, there is some vertex x of G that has at least $\psi(G) - 1$ different colors in its neighborhood $N_G(x)$. If $c'(y) = c(0, y)$ for all vertices $y \in N_G(x)$, then the vertex $(1, x)$ has all these colors in its neighborhood, and also the additional color $c(z)$. If however $c'(y) \neq c(0, y)$ for a neighbor y of x, then the vertex $(0, x)$ sees all the colors in $c'(N_G(x))$ in its neighborhood $N_{M(G)}(0, x)$, and also the additional color $c(0, y) = c(z)$. In both cases a vertex has $\psi(G)$ different colors in its neighborhood as claimed. □

We remark that $M_1(G)$ is simply the graph G with a new vertex connected to every vertex of G, therefore the following trivially holds.

Proposition 3.1.14 *For any graph G we have*

$$\psi(M_1(G)) = \chi(G) + 1.$$

□

For our first upper bound we apply Lemma 3.1.10. We use the following result of Gyárfás, Jensen, and Stiebitz [72]. The lemma below is an immediate generalization of the $l = 2$ special case of Theorem 4.1 in [72]. We reproduce the simple proof from [72] for the sake of completeness.

Lemma 3.1.15 ([72]) *If G has a wide coloring with t colors and $r \geq 7$, then $M_r(G)$ has a wide coloring with $t + 1$ colors.*

Proof. As there is a homomorphism from $M_r(G)$ to $M_7(G)$ if $r > 7$ it is enough to give the coloring for $r = 7$. We fix a wide t-coloring c_0 of G and use the additional color γ. The coloring of $M_7(G)$ is given as

$$c((v,x)) = \begin{cases} \gamma & \text{if } v \text{ is the vertex at distance 3, 5 or 7 from the loop} \\ c_0(x) & \text{otherwise.} \end{cases}$$

It is straightforward to check that c is a wide coloring. □

We can apply the results of Stiebitz and Gyárfás et al. recursively to give tight or almost tight bounds for the local chromatic number of the graphs $M_{\boldsymbol{r}}^{(d)}(G)$ in many cases:

Corollary 3.1.16 *If G has a wide t-coloring and $\boldsymbol{r} = (r_1, \dots, r_d)$ with $r_i \geq 7$ for all i, then $\psi(M_{\boldsymbol{r}}^{(d)}(G)) \leq \frac{t+d}{2} + 2$.*

If $H(G)$ is homotopy equivalent to a sphere $\mathbb{S}^h$, then $\psi(M_{\boldsymbol{r}}^{(d)}(G)) \geq \frac{h+d}{2} + 2$.

Proof. For the first statement we apply Lemma 3.1.15 recursively to show that $M_{\boldsymbol{r}}^{(d)}(G)$ has a wide $(t+d)$-coloring and then apply Lemma 3.1.10.

For the second statement we apply the result of Stiebitz recursively to show that $H(M_{\boldsymbol{r}}^{(d)}(G))$ is homotopy equivalent to $\mathbb{S}^{h+d}$. As noted in the preliminaries of the present subsection this implies $\mathrm{coind}(H(M_{\boldsymbol{r}}^{(d)}(G))) \geq h + d$. By Theorem 3.1.1 the statement follows. □

Theorem 3.1.5 (restated) *If $\boldsymbol{r} = (r_1, \dots, r_d)$, d is odd, and $r_i \geq 7$ for all i, then*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) = \left\lceil \frac{d}{2} \right\rceil + 2.$$

Proof. Notice that for $\boldsymbol{r} = (r_1, \dots, r_d)$ with d odd and $r_i \geq 7$ for all i the lower and upper bounds of Corollary 3.1.16 give the exact value for the local chromatic number $\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) = (d+5)/2$. This proves the theorem. □

Notice that a similar argument gives the exact value of $\psi(G)$ for the more complicated graph $G = M_{\boldsymbol{r}}^{(d)}(\mathrm{SG}(n,k))$ whenever $n+d$ is odd, $r_i \geq 7$ for all i, and $n \geq 4t^2 - 7t$ for $t = n - 2k + 2$. This follows from Corollary 3.1.16 via the wide colorability of $\mathrm{SG}(n,k)$ for $n \geq 4t^2 - 7t$ shown in the proof of Theorem 3.1.3 and Björner and de Longueville's result [22] about the homotopy equivalence of $H(\mathrm{SG}(n,k))$ to $\mathbb{S}^{n-2k}$. (Instead of the latter we can also use Csorba's result [40, 41] mentioned above and refer to the strong topological t-chromaticity of $\mathrm{SG}(n,k)$.)

We summarize our knowledge on $\psi(M_{\boldsymbol{r}}^{(d)}(K_2))$ after proving the following theorem, which shows that almost the same upper bound as in Corollary 3.1.16 is implied from the relaxed condition $r_i \geq 4$.

Theorem 3.1.17 *For $\boldsymbol{r} = (r_1, \dots, r_d)$ with $r_i \geq 4$ for all i one has*

$$\psi(M_{\boldsymbol{r}}^{(d)}(G)) \leq \psi(G) + \left\lfloor \frac{d}{2} \right\rfloor + 2.$$

Moreover, for $G \cong K_2$, the following slightly sharper bound holds:

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2)) \leq \left\lceil \frac{d}{2} \right\rceil + 3.$$

Proof. We denote the vertices of $Y := M_{\boldsymbol{r}}^{(d)}(G)$ in accordance to the description of the generalized Mycielski construction via graph products. That is, a vertex of Y is a sequence $a_1a_2\dots a_du$ of length $(d+1)$, where $\forall i:\ a_i \in \{0,1,\dots,r_i\}\cup\{*\}$, $u\in V(G)\cup\{*\}$ and if $a_i = r_i$ for some i then necessarily $u=*$ and $a_j=*$ for every $j>i$, and this is the only way $*$ can appear in a sequence. To define adjacency we denote by $\hat{P}_{r_i+1}$ the path on $\{0,1,\dots,r_i\}$ where the edges are of the form $\{i-1,i\}, i\in\{1,\dots,r_i\}$ and there is a loop at vertex 0. Two vertices $a_1a_2\dots a_du$ and $a'_1a'_2\dots a'_du'$ are adjacent in Y if and only if

$$u = * \text{ or } u' = * \text{ or } \{u,u'\}\in E(G) \text{ and}$$
$$\forall i:\quad a_i = * \text{ or } a'_i = * \text{ or } \{a_i,a'_i\}\in E(\hat{P}_{r_i+1}).$$

Our strategy is similar to that used in Remark 3.4. Namely, we give an original coloring c_0, identify the set of "troublesome" vertices for this coloring, and recolor most of the neighbors of these vertices to a new color.

Let us fix a coloring c_G of G with at most $\psi(G)-1$ colors in the neighborhood of a vertex. Let the colors we use in this coloring be called $0,-1,-2$, etc. Now we define c_0 as follows.

$$c_0(a_1\dots a_du) = \begin{cases} c_G(u) & \text{if } \forall i: a_i\le 2\\ i & \text{if } a_i\ge 3 \text{ is odd and } a_j\le 2 \text{ for all } j<i\\ 0 & \text{if } \exists i: a_i\ge 4 \text{ is even and } a_j\le 2 \text{ for all } j<i\end{cases}$$

It is clear that vertices having the same color form independent sets, i.e., c_0 is a proper coloring. Notice that if a vertex has neighbors of many different "positive" colors, then it must have many coordinates that are equal to 2. Now we recolor most of the neighbors of these vertices.

Let β be a color not used by c_0 and set $c(a_1\dots a_du)=\beta$ if $|\{i: a_i \text{ is odd}\}|>d/2$. (In fact, it would be enough to give color β only to those of the above vertices, for which the first $\lfloor\frac{d}{2}\rfloor$ odd coordinates are equal to 1. We recolor more vertices for the sake of simplicity.) Otherwise, let $c(a_1\dots a_du)=c_0(a_1\dots a_du)$.

First, we have to show that c is proper. To this end we only have to show that no pair of vertices getting color β can be adjacent. If two vertices, $\boldsymbol{x}=x_1\dots x_dv_x$ and $\boldsymbol{y}=y_1\dots y_dv_y$ are colored β then both have more than $d/2$ odd coordinates (among their first d coordinates). Thus there is some common coordinate i for which x_i and y_i are both odd. This implies that $\boldsymbol{x}$ and $\boldsymbol{y}$ are not adjacent.

Now we show that for any vertex $\boldsymbol{a}$ we have $|c(N(\boldsymbol{a}))\cap\{1,\dots,d\}|\le d/2$. Indeed, if $|c_0(N(\boldsymbol{a}))\cap\{1,\dots,d\}|>d/2$ then we have $\boldsymbol{a}=a_1\dots a_du$ with more than $d/2$ coordinates a_i that are even and positive. Furthermore, the first $\lfloor d/2\rfloor$ of these coordinates should be 2. Let I be the set of indices of these first $\lfloor d/2\rfloor$ even and positive coordinates. We claim that $c(N(\boldsymbol{a}))\cap\{1,\dots,d\}\subseteq I$. This is so, since if a neighbor has an odd coordinate somewhere outside I, then it cannot have $*$ at the positions of I, therefore it has more than $d/2$ odd coordinates and it is recolored by c to the color β.

It is also clear that no vertex can see more than $\psi(G)-1$ "negative" colors in its neighborhood in either coloring c_0 or c. Thus the neighborhood of any vertex can contain at most $\lfloor d/2\rfloor+(\psi(G)-1)+2$ colors, where the last 2 is added because of the possible appearance of colors β and 0 in the neighborhood. This proves $\psi(Y)\le d/2+\psi(G)+2$ proving the first statement in the theorem.

For $G\cong K_2$ the above gives $\psi(M_{\boldsymbol{r}}^{(d)}(K_2))\le\lfloor d/2\rfloor+4$ which implies the second statement for odd d. For even d the bound of the second statement is 1 less. We can gain 1 as follows. When defining c let us recolor to β those vertices $\boldsymbol{a}=a_1\dots a_d u$, too, for which the number of odd coordinates a_i is exactly $\frac{d}{2}$ and $c_G(u)=-1$. The proof proceeds similarly as before but we gain 1 by observing that those vertices who see -1 can see only $\frac{d}{2}-1$ "positive" colors. □

We collect the implications of Theorems 3.1.5, 3.1.17 and Propositions 3.1.13 and 3.1.14. It would be interesting to estimate the value $\psi(M_{\boldsymbol{r}}^{(d)}(K_2))$ for the missing case $\boldsymbol{r}=(3,\dots,3)$. What we know then is $\lceil d/2\rceil+2\le\psi\le d+2$.

Corollary 3.1.18 *For $\boldsymbol{r}=(r_1,\dots,r_d)$ we have*

$$\psi(M_{\boldsymbol{r}}^{(d)}(K_2))=\begin{cases}(d+5)/2 & \textit{if } d \textit{ is odd and } \forall i: r_i\ge 7\\ \lceil d/2\rceil+2 \textit{ or } \lceil d/2\rceil+3 & \textit{if } \forall i: r_i\ge 4\\ d+2 & \textit{if } r_d=1 \textit{ or } \forall i: r_i=2.\end{cases}$$

□

Remark 3.5. The improvement for even d given in the last paragraph of the proof of Theorem 3.1.17 can also be obtained in a different way we explain here. Instead of changing the rule for recoloring, we can enforce that a vertex can see only $\psi(G)-2$ negative colors. This can be achieved by setting the starting graph G to be $M_4(K_2)\cong C_9$ instead of K_2 itself and coloring this C_9 with the pattern $-1,0,-1,-2,0,-2,-3,0,-3$ along the cycle. One can readily check that every vertex can see only one non-0 color in its neighborhood.

The same trick can be used also if the starting graph is not K_2 or C_9, but some large enough Schrijver graph of odd chromatic number. Coloring it as in the proof of Lemma 3.1.10 (using the wide coloring as given in the proof of Theorem 3.1.3), we arrive to the same phenomenon if we let the new color (of the proof of Lemma 3.1.10) be 0. ◊

Remark 3.6. Gyárfás, Jensen, and Stiebitz [72] use generalized Mycielski graphs to show that another graph they denote by G_k is k-chromatic. The way they prove it is that they exhibit a homomorphism from $M_{\boldsymbol{r}}^{(k-2)}(K_2)$ to G_k for $\boldsymbol{r}=(4,\dots,4)$. The existence of this homomorphism implies that G_k is strongly topologically k-chromatic, thus its local chromatic number is at least $k/2+1$. We do not know any non-trivial upper bound for $\psi(G_k)$. Also note that [72] gives universal graphs for the property of having a wide t-coloring. By Lemma 3.1.10 this graph has $\psi\le t/2+2$. On the other hand, since any graph with a wide t-coloring admits a homomorphism to this graph, and we have seen the wide t-colorability of some strongly topologically t-chromatic graphs, it is strongly topologically t-chromatic, as well. This gives $\psi\ge t/2+1$. ◊

Borsuk graphs and the tightness of Ky Fan's theorem

The following definition goes back to Erdős and Hajnal [48], see also [111].

Definition 36 *The Borsuk graph $B(n,\alpha)$ of parameters n and $0<\alpha<2$ is the infinite graph whose vertices are the points of the unit sphere in $\mathbb{R}^n$ (i.e., $\mathbb{S}^{n-1}$) and its edges connect the pairs of points with distance at least α.*

The Borsuk-Ulam theorem implies that $\chi(B(n,\alpha))\ge n+1$, and, as Lovász [111] remarks, these two statements are in fact equivalent. For α large enough (depending on n) this lower bound on the chromatic number is sharp as shown by the standard $(n+1)$-coloring of the sphere $\mathbb{S}^{n-1}$ (see [111, 116] or cf. the proof of Corollary 3.1.20 below).

The local chromatic number of Borsuk graphs for large enough α (and n even) can also be determined by our methods. First we want to argue that Theorem 3.1.1 is applicable for this infinite graph. Lovász gives in [111] for any n and α a finite graph $G_P=G_P(n,\alpha)\subseteq B(n,\alpha)$ which has the property that its neighborhood complex $\mathcal{N}(G_P)$ is homotopy equivalent to $\mathbb{S}^{n-1}$. Now we can continue the argument the same way as in the previous subsection: Proposition 4.2 in [9] states that $\mathcal{N}(F)$ is homotopy equivalent to $H(F)$ for every graph F, thus $\mathrm{coind}(H(G_P))\ge n-1$, i.e., G_P is strongly topologically $(n+1)$-chromatic. As $G_P\subseteq B(n,\alpha)$ we have $\lceil\frac{n+3}{2}\rceil\le\psi(G_P)\le\psi(B(n,\alpha))$ by Theorem 3.1.1.

The following lemma shows the special role of Borsuk graphs among strongly topologically t-chromatic graphs. It will also show that our earlier upper bounds on the local chromatic number have direct implications for Borsuk graphs.

Lemma 3.1.19 *A finite graph G is strongly topologically $(n+1)$-chromatic if and only if for some $\alpha<2$ there is a graph homomorphism from $B(n,\alpha)$ to G.*

Proof. For the if part consider the finite graph $G_P\subseteq B(n,\alpha)$ given by Lovász [111] satisfying $\mathrm{coind}(H(G_P))\ge n-1$. If there is a homomorphism from $B(n,\alpha)$ to G, it clearly gives a homomorphism also from G_P to G which further generates a $\mathbb{Z}_2$-map from $H(G_P)$ to $H(G)$. This proves $\mathrm{coind}(H(G))\ge n-1$.

For the only if part, let $f:\mathbb{S}^{n-1}\to H(G)$ be a $\mathbb{Z}_2$-map. For a point $\boldsymbol{x}\in\mathbb{S}^{n-1}$ write $f(\boldsymbol{x})\in H(G)$ as the convex combination $f(\boldsymbol{x})=\sum\alpha_v(\boldsymbol{x})||{+v}||+\sum\beta_v(\boldsymbol{x})||{-v}||$ of the vertices of $||B_0(G)||$. Here the summations are for the vertices v of G, $\sum\alpha_v(\boldsymbol{x})=\sum\beta_v(\boldsymbol{x})=1/2$, and $\{v:\alpha_v(\boldsymbol{x})>0\}\uplus\{v:\beta_v(\boldsymbol{x})>0\}\in B_0(G)$. Note that α_v and β_v are continuous as f is continuous and $\beta_v(\boldsymbol{x})=\alpha_v(-\boldsymbol{x})$ by the equivariance of f. Set $\varepsilon=1/(2|V(G)|)$. For $\boldsymbol{x}\in\mathbb{S}^{n-1}$ select an arbitrary vertex $v=g(\boldsymbol{x})$ of G with $\alpha_v\ge\varepsilon$. We claim that g is a graph homomorphism from $B(n,\alpha)$ to G if α is close enough to 2. By compactness it is enough to prove that if we have vertices v and w of G and sequences $\boldsymbol{x}_i\to\boldsymbol{x}$ and $\boldsymbol{y}_i\to-\boldsymbol{x}$ of points in $\mathbb{S}^{n-1}$ with $g(\boldsymbol{x}_i)=v$ and $g(\boldsymbol{y}_i)=w$ for all i, then v and w are connected in G. But since α_v is continuous we have $\alpha_v(\boldsymbol{x})\ge\varepsilon$ and similarly

$\beta_w(\boldsymbol{x}) = \alpha_w(-\boldsymbol{x}) \geq \varepsilon$ and so $+v$ and $-w$ are contained in the smallest simplex of $B_0(G)$ containing $f(\boldsymbol{x})$ proving that v and w are connected. $\square$

By Lemma 3.1.19 either of Theorems 3.1.3 or 3.1.5 implies that the above given lower bound on $\psi(B(n,\alpha))$ is tight whenever $\chi(B(n,\alpha))$ is odd, that is, n is even, and $\alpha < 2$ is close enough to 2. In the following corollary we give an explicit bound on α by proving for that value of α that the standard coloring is wide.

Corollary 3.1.20 *If n is even and $2\cos\left(\frac{\arcsin(n+2)^{-1/2}}{5}\right) < \alpha < 2$, then*

$$\psi(B(n,\alpha)) = \frac{n}{2} + 2.$$

Proof. The lower bound on $\psi(B(n,\alpha))$ follows from the discussion preceding Lemma 3.1.19. The upper bound follows from Lemma 3.1.10 as long as we can give a wide $(n+1)$-coloring of the graph $B(n,\alpha)$.

To this end we use the standard $(n+1)$-coloring of $B(n,\alpha)$ (see, e.g., [111, 116]). Consider a regular simplex R inscribed into the unit sphere $\mathbb{S}^{n-1}$ and color a point $\boldsymbol{x} \in \mathbb{S}^{n-1}$ by the facet of R intersected by the segment from the origin to $\boldsymbol{x}$. If this segment meets a lower dimensional face then we arbitrarily choose a facet containing this face. To see for what α gives this a proper coloring we have to find the maximal distance α_0 between pairs of points that we can color the same. Calculation shows that projections from the origin of the middle points of two disjoint $(n/2-1)$-dimensional faces of R are farthest apart, thus $\alpha_0 = 2\sqrt{1-1/(n+2)}$. (Notice that [111] gives a different treshold value for α. We were informed by László Lovász [113], however, that it was noticed by several researchers that the correct value is larger than the one given in [111].)

We let $\varphi = 2\arccos(\alpha/2)$. Clearly, $\boldsymbol{x}$ and $\boldsymbol{y}$ is connected if and only if the length of the shortest arc on $\mathbb{S}^{n-1}$ connecting $-\boldsymbol{x}$ and $\boldsymbol{y}$ is at most φ. Therefore $\boldsymbol{x}$ and $\boldsymbol{y}$ are connected by a walk of length 5 if and only if the length of this same minimal arc is at most 5φ. For the standard coloring the length of the shortest arc between $-\boldsymbol{x}$ and $\boldsymbol{y}$ for two vertices $\boldsymbol{x}$ and $\boldsymbol{y}$ colored with the same color is at least $2\arccos(\alpha_0/2) = 2\arcsin(n+2)^{-1/2}$. Therefore the standard coloring is wide as long as $\alpha > 2\cos\left(\frac{\arcsin(n+2)^{-1/2}}{5}\right)$. $\square$

Our investigations of the local chromatic number led us to consider the following function $Q(h)$. The question of its values was independently asked by Micha Perles motivated by a related question of Matatyahu Rubin[1].

Definition 37 *For a nonnegative integer parameter h let $Q(h)$ denote the minimum l for which $\mathbb{S}^h$ can be covered by open sets in such a way that no point of the sphere is contained in more than l of these sets and none of the covering sets contains an antipodal pair of points.*

[1] We thank Imre Bárány [12] and Gil Kalai [84] for this information.

Ky Fan's theorem implies $Q(h) \geq \frac{h}{2}+1$. Either of Theorems 3.1.3 or 3.1.5 implies the upper bound $Q(h) \leq \frac{h}{2}+2$. Using the concepts of Corollary 3.1.20 and Lemma 3.1.10 one can give an explicit covering of the sphere $\mathbb{S}^{2l-3}$ by open subsets where no point is contained in more than l of the sets and no set contains an antipodal pair of points. In fact, the covering we give satisfies a stronger requirement and proves that version (ii) of Ky Fan's theorem is tight, while version (i) is almost tight.

Corollary 3.1.21 *There is a configuration $\mathcal{A}$ of $k+2$ open (closed) sets such that $\cup_{A\in\mathcal{A}}(A \cup (-A)) = \mathbb{S}^k$, all sets $A \in \mathcal{A}$ satisfy $A \cap (-A) = \emptyset$, and no $\boldsymbol{x} \in \mathbb{S}^k$ is contained in more than $\lceil \frac{k+1}{2} \rceil$ of these sets.*

Furthermore, for every $\boldsymbol{x}$ the number of sets in $\mathcal{A}$ containing either $\boldsymbol{x}$ or $-\boldsymbol{x}$ is at most $k+1$.

Proof. First we construct closed sets. Consider the unit sphere $\mathbb{S}^k$ in $\mathbb{R}^{k+1}$. Let R be a regular simplex inscribed in the sphere. Let $B_1, \ldots, B_{k+2}$ be the subsets of the sphere obtained by the central projection of the facets of R. These closed sets cover $\mathbb{S}^k$. Let C_0 be the set of points covered by at least $\lceil \frac{k+3}{2} \rceil$ of the sets B_i. Notice that C_0 is the union of the central projections of the $\lfloor \frac{k-1}{2} \rfloor$-dimensional faces of R. For odd k let $C = C_0$, while for even k let $C = C_0 \cup C_1$, where C_1 is the set of points in B_1 covered by exactly $k/2+1$ of the sets B_i. Thus C_1 is the union of the central projections of the $\frac{k}{2}$-dimensional faces of a facet of R. Observe that $C \cap (-C) = \emptyset$. Take $0 < \delta < \mathrm{dist}(C, -C)/2$ and let D be the open δ-neighborhood of C in $\mathbb{S}^k$. For $1 \leq i \leq k+2$ let $A_i = B_i \setminus D$. These closed sets cover $\mathbb{S}^k \setminus D$ and none of them contains a pair of antipodal points. As $D \cap (-D) = \emptyset$ we have $\cup_{i=1}^{k+2}(A_i \cup (-A_i)) = \mathbb{S}^k$. It is clear that every point of the sphere is covered by at most $\lceil \frac{k+1}{2} \rceil$ of the sets A_i proving the first statement of the corollary.

For the second statement note that if each set B_i contains at least one of a pair of antipodal points, then one of these points belongs to C and is therefore not covered by any of the sets A_i. Note also, that for odd k the second statement follows also from the first.

To construct open sets as required we can simply take the open ε-neighborhoods of A_i. For small enough $\varepsilon > 0$ they maintain the properties required in the corollary. $\square$

Corollary 3.1.22 *There is a configuration of $k+3$ open (closed) sets covering $\mathbb{S}^k$ none of which contains a pair of antipodal points, such that no $\boldsymbol{x} \in \mathbb{S}^k$ is contained in more than $\lceil \frac{k+3}{2} \rceil$ of these sets and for every $\boldsymbol{x} \in \mathbb{S}^k$ the number of sets that contain one of $\boldsymbol{x}$ and $-\boldsymbol{x}$ is at most $k+2$.*

Proof. For closed sets consider the sets A_i in the proof of Corollary 3.1.21 together with the closure of D. For open sets consider the open ε-neighborhoods of these sets for suitably small $\varepsilon > 0$. $\square$

Note that covering with $k+3$ sets is optimal in Corollary 3.1.22 if $k \geq 3$. By the Borsuk-Ulam Theorem (form (i)) fewer than $k+2$ open (or closed) sets not containing antipodal pairs of points is not enough to cover $\mathbb{S}^k$. If we cover with $k+2$ sets (open or closed), then it gives rise to a proper coloring of $B(k+1,\alpha)$ for large enough α in a natural way. This coloring uses the optimal number $k+2$ of colors, therefore it has a vertex with $k+1$ different colors in its neighborhood. A compactness argument establishes from this that there is a point in $\mathbb{S}^k$ covered by $k+1$ sets. A similar argument gives that $k+2$ in Corollary 3.1.21 is also optimal if $k \geq 3$.

Corollary 3.1.23

$$\frac{h}{2}+1 \leq Q(h) \leq \frac{h}{2}+2.$$

Proof. The lower bound is implied by Ky Fan's theorem. The upper bound follows from Corollary 3.1.22. □

Notice that for odd h Corollary 3.1.23 gives the exact value $Q(h) = \frac{h+3}{2}$. For h even we either have $Q(h) = \frac{h}{2}+1$ or $Q(h) = \frac{h}{2}+2$. It is trivial that $Q(0) = 1$. In [146] we show $Q(2) = 3$. This was independently proved by Imre Bárány [12]. For $h > 2$ even it remains open whether the lower or the upper bound of Corollary 3.1.23 is tight.

3.1.5 Circular colorings

In this subsection we show an application of the Zig-zag theorem for the circular chromatic number of graphs. This will result in the partial solution of a conjecture by Johnson, Holroyd, and Stahl [81] and in a partial answer to a question of Hajiabolhassan and Zhu [73] concerning the circular chromatic number of Kneser graphs and Schrijver graphs, respectively. We also answer a question of Chang, Huang, and Zhu [29] concerning the circular chromatic number of iterated Mycielskians of complete graphs.

The circular chromatic number of a graph was introduced by Vince [157] under the name star chromatic number as follows.

Definition 38 *For positive integers p and q a coloring $c : V(G) \to [p]$ of a graph G is called a* (p,q)-coloring *if for all adjacent vertices u and v one has $q \leq |c(u)-c(v)| \leq p-q$. The* circular chromatic number *of G is defined as*

$$\chi_c(G) = \inf\left\{\frac{p}{q} : \text{ there is a } (p,q)\text{-coloring of } G\right\}.$$

It is known that the above infimum is always attained for finite graphs. An alternative description of $\chi_c(G)$, explaining its name, is that it is the minimum length of the perimeter of a circle on which we can represent the vertices of G by arcs of length 1 in such a way that arcs belonging to adjacent vertices do not overlap. For a proof of this equivalence and for an extensive bibliography on the circular chromatic number we refer to Zhu's survey article [159].

It is known that for every graph G one has $\chi(G)-1 < \chi_c(G) \leq \chi(G)$. Thus $\chi_c(G)$ determines the value of $\chi(G)$ while this is not true the other way round. Therefore the circular chromatic number can be considered as a refinement of the chromatic number.

Our main result on the circular chromatic number is Theorem 3.1.6. Here we restate the theorem with the explicit meaning of being topologically t-chromatic.

Theorem 3.1.6 (restated) *For a finite graph G we have $\chi_c(G) \geq \mathrm{coind}(B_0(G)) + 1$ if $\mathrm{coind}(B_0(G))$ is odd.*

Proof. Let $t = \mathrm{coind}(B_0(G)) + 1$ be an even number and let c be a (p, q)-coloring of G. By the Zig-zag theorem there is a $K_{\frac{t}{2},\frac{t}{2}}$ in G which is completely multicolored by colors appearing in an alternating manner in its two sides. Let these colors be $c_1 < c_2 < \ldots < c_t$. Since the vertex colored c_i is adjacent to that colored c_{i+1}, we have $c_{i+1} \geq c_i + q$ and $c_t \geq c_1 + (t-1)q$. Since t is even, the vertices colored c_1 and c_t are also adjacent, therefore we must have $c_t - c_1 \leq p - q$. The last two inequalities give $p/q \geq t$ as needed. □

This result has been independently obtained by Meunier [121] for Schrijver graphs.

Circular chromatic number of even chromatic Kneser and Schrijver graphs

Johnson, Holroyd, and Stahl [81] considered the circular chromatic number of Kneser graphs and formulated the following conjecture. (See also as Conjecture 7.1 and Question 8.27 in [159].)

Conjecture (Johnson, Holroyd, Stahl [81]): For any $n \geq 2k$

$$\chi_c(\mathrm{KG}(n, k)) = \chi(\mathrm{KG}(n, k)).$$

It is proven in [81] that the above conjecture holds if $k = 2$ or $n = 2k + 1$ or $n = 2k + 2$.

Lih and Liu [105] investigated the circular chromatic number of Schrijver graphs and proved that $\chi_c(\mathrm{SG}(n, 2)) = n - 2 = \chi(\mathrm{SG}(n, 2))$ whenever $n \neq 5$. (For $n = 2k + 1$ one always has $\chi_c(\mathrm{SG}(2k + 1, k)) = 2 + \frac{1}{k}$.) It was conjectured in [105] and proved in [73] that for every fixed k there is a threshold $l(k)$ for which $n \geq l(k)$ implies $\chi_c(\mathrm{SG}(n, k)) = \chi(\mathrm{SG}(n, k))$. This clearly implies the analogous statement for Kneser graphs, for which the explicit threshold $l(k) = 2k^2(k - 1)$ is given in [73]. At the end of their paper [73] Hajiabolhassan and Zhu ask what is the minimum $l(k)$ for which $n \geq l(k)$ implies $\chi_c(\mathrm{SG}(n, k)) = \chi(\mathrm{SG}(n, k))$. We show that no such threshold is needed if n is even.

Corollary 3.1.24 *The Johnson-Holroyd-Stahl conjecture holds for every even n. Moreover, if n is even, then the stronger equality*

$$\chi_c(\mathrm{SG}(n, k)) = \chi(\mathrm{SG}(n, k))$$

also holds.

Proof. As t-chromatic Kneser graphs and Schrijver graphs are topologically t-chromatic, Theorem 3.1.6 implies the statement of the corollary. □

As mentioned above this result has been obtained independently by Meunier [121].

Later, in Theorem 3.1.27 we show that for odd n the situation is different.

Circular chromatic number of Mycielski graphs and Borsuk graphs

The circular chromatic number of Mycielski graphs was also studied extensively, cf. [29, 51, 74, 159]. Chang, Huang, and Zhu [29] formulated the conjecture that $\chi_c(M^d(K_n)) = \chi(M^d(K_n)) = n + d$ whenever $n \geq d + 2$. Here $M^d(G)$ denotes the d-fold iterated Mycielskian of graph G, i.e., using the notation of Subsection 3.1.4 we have $M^d(G) = M_{\boldsymbol{r}}^{(d)}(G)$ with $\boldsymbol{r} = (2, \ldots, 2)$. The above conjecture was verified for the special cases $d = 1, 2$ in [29], where it was also shown that $\chi_c(M^d(G)) \leq \chi(M^d(G)) - 1/2$ if $\chi(G) = d + 1$. A simpler proof for the above special cases of the conjecture was given (for $d = 2$ with the extra condition $n \geq 5$) in [51]. Recently Hajiabolhassan and Zhu [74] proved that $n \geq 2^d + 2$ implies $\chi_c(M^d(K_n)) = \chi(M^d(K_n)) = n + d$. Our results show that $\chi_c(M^d(K_n)) = \chi(M^d(K_n)) = n + d$ always holds if $n + d$ is even. This also answers the question of Chang, Huang, and Zhu asking the value of $\chi_c(M^n(K_n))$ (Question 2 in [29]). The stated equality is given by the following immediate consequence of Theorem 3.1.6.

Corollary 3.1.25 *If $H(G)$ is homotopy equivalent to the sphere $\mathbb{S}^h$, $\boldsymbol{r}$ is a vector of positive integers, and $h + d$ is even, then $\chi_c(M_{\boldsymbol{r}}^{(d)}(G)) \geq d + h + 2$.*
In particular, $\chi_c(M_{\boldsymbol{r}}^{(d)}(K_n)) = n + d$ whenever $n + d$ is even.

Proof. The condition on G implies $\mathrm{coind}(H(M_{\boldsymbol{r}}^{(d)}(G))) = h + d$ by Stiebitz's result [149] (cf. the discussion and Proposition 3.1.12 in Subsection 3.1.4), which further implies $\mathrm{coind}(B_0(M_{\boldsymbol{r}}^{(d)}(G))) = h + d + 1$. This gives the conclusion by Theorem 3.1.6.

The second statement follows by the homotopy equivalence of $H(K_n)$ with $\mathbb{S}^{n-2}$ and the chromatic number of $M_{\boldsymbol{r}}^{(d)}(K_n)$ being $n + d$. □

The above mentioned conjecture of Chang, Huang, and Zhu for $n + d$ even is a special case with $\boldsymbol{r} = (2, 2, \ldots, 2)$ and $n \geq d + 2$. Since $n + n$ is always even, the answer $\chi_c(M^n(K_n)) = 2n$ to their question also follows.

Corollary 3.1.25 also implies a recent result of Lam, Lin, Gu, and Song [102] who proved that for the generalized Mycielskian of odd order complete graphs $\chi_c(M_r(K_{2m-1})) = 2m$.

Lam, Lin, Gu, and Song [102] also determined the circular chromatic number of the generalized Mycielskian of even order complete graphs. They proved $\chi_c(M_r(K_{2m})) = 2m + 1/(\lfloor (r-1)/m \rfloor + 1)$. This result can be used to bound the circular chromatic number of the Borsuk graph $B(2s, \alpha)$ from above.

Theorem 3.1.26 *For the Borsuk graph $B(n, \alpha)$ we have*

(i) $\chi_c(B(n, \alpha)) = n + 1$ *if n is odd and α is large enough;*

(ii) $\chi_c(B(n, \alpha)) \to n$ *as $\alpha \to 2$ if n is even.*

Proof. The lower bound of part (i) immediately follows from Theorem 3.1.6 considering again the finite subgraph G_P of $B(n,\alpha)$ defined in [111] and already mentioned in the proof of Lemma 3.1.19. The matching upper bound is provided by $\chi(B(n,\alpha)) = n+1$ for large enough α, see [111] and Subsection 3.1.4.

For (ii) we have $\chi_c(B(n,\alpha)) > \chi(B(n,\alpha)) - 1 \geq n$. For an upper bound we use that $\chi_c(M_r(K_n)) \to n$ if r goes to infinity by the result of Lam, Lin, Gu, and Song [102] quoted above. By the result of Stiebitz [149] and Lemma 3.1.19 we have a graph homomorphism from $B(n,\alpha)$ to $M_r(K_n)$ for any r and large enough α. As (p,q)-colorings can be defined in terms of graph homomorphisms (see [25]), we have $\chi_c(G) \leq \chi_c(H)$ if there exists a graph homomorphism from G to H. This completes the proof of part (ii) of the theorem. □

Circular chromatic number of odd chromatic Schrijver graphs

In this subsection we show that the parity condition on $\chi(\mathrm{SG}(n,k))$ in Corollary 3.1.24 is relevant, for odd chromatic Schrijver graphs the circular chromatic number can be arbitrarily close to its lower bound.

Theorem 3.1.27 *For every $\varepsilon > 0$ and every odd $t \geq 3$ if $n \geq t^3/\varepsilon$ and $t = n - 2k + 2$, then*

$$1 - \varepsilon < \chi(\mathrm{SG}(n,k)) - \chi_c(\mathrm{SG}(n,k)) < 1.$$

The second inequality is well-known and holds for any graph. We included it only for completeness. To prove the first inequality we need some preparation. We remark that the bound on n in the theorem is not best possible. Our method proves $\chi(\mathrm{SG}(n,k)) - \chi_c(\mathrm{SG}(n,k)) \geq 1 - 1/i$ if i is a positive integer and $n \geq 6(i-1)\binom{t}{3} + t$.

First we extend our notion of wide coloring.

Definition 39 *For a positive integer s we call a vertex coloring of a graph s-wide if the two end vertices of any walk of length $2s-1$ receive different colors.*

Our original wide colorings are 3-wide, while 1-wide simply means proper. Gyárfás, Jensen, and Stiebitz [72] investigated s-wide colorings (in different terms) and mention (referring to a referee in the $s > 2$ case) the existence of homomorphism universal graphs for s-wide colorability with t colors. We give a somewhat different family of such universal graphs. (These graphs were later independently found by Baum and Stiebitz [13].) In the $s = 2$ case the color-criticality of the given universal graph is proven in [72] implying its minimality among graphs admitting 2-wide t-colorings. Later in Subsection 3.1.6 we generalize this result showing that the members of our family are color-critical for every s. (This result was also independently obtained later by Baum and Stiebitz [13].) Thus they must be minimal and therefore isomorphic to a retract of the corresponding graphs given in [72].

Definition 40 *Let H_s be the path on the vertices $0, 1, 2, \ldots, s$ (i and $i-1$ connected for $1 \leq i \leq s$) with a loop at s. We define $W(s,t)$ to be the graph with*

$$V(W(s,t)) = \{(x_1 \ldots x_t) : \forall i\ x_i \in \{0, 1, \ldots, s\}, \exists! i\ x_i = 0,\ \exists j\ x_j = 1\},$$

$$E(W(s,t)) = \{\{x_1 \ldots x_t, y_1 \ldots y_t\} : \forall i\ \{x_i, y_i\} \in E(H_s)\}.$$

Note that $W(s,t)$ is an induced subgraph of the direct power H_s^t (cf. the introduction of the generalized Mycielski construction in Subsection 3.1.4).

Proposition 3.1.28 *A graph G admits an s-wide coloring with t colors if and only if there is a homomorphism from G to $W(s,t)$.*

Proof. For the if part color vertex $\boldsymbol{x} = x_1 \ldots x_t$ of $W(s,t)$ with $c(\boldsymbol{x}) = i$ if $x_i = 0$. Any walk between two vertices colored i either has even length or contains two vertices $\boldsymbol{y}$ and $\boldsymbol{z}$ with $y_i = z_i = s$. These $\boldsymbol{y}$ and $\boldsymbol{z}$ are both at least at distance s apart from both ends of the walk, thus our coloring of $W(s,t)$ with t colors is s-wide. Any graph admitting a homomorphism φ to $W(s,t)$ is s-widely colored with t colors by $c_G(v) := c(\varphi(v))$.

For the only if part assume c is an s-wide t-coloring of G with colors $1, \ldots, t$. Let $\varphi(v)$ be an arbitrary vertex of $W(s,t)$ if v is an isolated vertex of G. For a non-isolated vertex v of G let $\varphi(v) = \boldsymbol{x} = x_1 \ldots x_t$ with $x_i = \min(s, d_i(v))$, where $d_i(v)$ is the distance of color class i from v. It is clear that $x_i = 0$ for $i = c(v)$ and for no other i, while $x_i = 1$ for the colors of the neighbors of v in G. Thus the image of φ is indeed in $V(W(s,t))$. It takes an easy checking that φ is a homomorphism. $\square$

The following lemma is a straightforward extension of the argument given in the proof of Theorem 3.1.3.

Lemma 3.1.29 *If $t = n - 2k + 2$ and $n \geq (2s-2)t^2 - (4s-5)t$ then $\mathrm{SG}(n,k)$ admits an s-wide t-coloring.*

Proof. We use the notation introduced in the proof of Theorem 3.1.3.

Let $n \geq t(2(s-1)(t-2)+1)$ as in the statement and let c_0 be the coloring defined in the mentioned proof. The lower bound on n now allows to assume that $|C_i| \geq (s-1)(t-2)+1$. We show that c_0 is s-wide.

Consider a walk $x_0 x_1 \ldots x_{2s-1}$ of length $(2s-1)$ in $\mathrm{SG}(n,k)$ and let $i = c_0(x_0)$. Then $C_i \subseteq x_0$. By Lemma 3.1.11 $|x_0 \setminus x_{2s-2}| \leq (s-1)(t-2) < |C_i|$. Thus x_{2s-2} is not disjoint from C_i. As x_{2s-1} is disjoint from x_{2s-2}, it does not contain C_i and thus its color is not i. $\square$

Lemma 3.1.30 *$W(s,t)$ admits a homomorphism to $M_s(K_{t-1})$.*

Proof. Recall our notation for the (iterated) generalized Mycielskians from Subsection 3.1.4.

We define the following mapping from $V(W(s,t))$ to $V(M_s(K_{t-1}))$.

$$\varphi(x_1 \dots x_t) := \begin{cases} (s - x_t, i) & \text{if } x_t \neq x_i = 0 \\ (s, *) & \text{if } x_t = 0. \end{cases}$$

One can easily check that φ is indeed a homomorphism. $\square$

Proof of Theorem 3.1.27. By Lemma 3.1.29, if $n \geq (2s-2)t^2 - (4s-5)t$, then $\mathrm{SG}(n,k)$ has an s-wide t-coloring, thus by Proposition 3.1.28 it admits a homomorphism to $W(s,t)$. Composing this with the homomorphism given by Lemma 3.1.30 we conclude that $\mathrm{SG}(n,k)$ admits a homomorphism to $M_s(K_{t-1})$, implying $\chi_c(\mathrm{SG}(n,k)) \leq \chi_c(M_s(K_{t-1}))$.

We continue by using Lam, Lin, Gu, and Song's result [102], who proved, as already quoted earlier, that $\chi_c(M_s(K_{t-1})) = t - 1 + \frac{1}{\lfloor \frac{2s-2}{t-1} \rfloor + 1}$ if t is odd. Thus, for odd t and $i > 0$ integer we choose $s = (t-1)(i-1)/2 + 1$ and $\chi(\mathrm{SG}(n,k)) - \chi_c(\mathrm{SG}(n,k)) = t - \chi_c(\mathrm{SG}(n,k)) \geq 1 - 1/i$ follows from the $n \geq 6(i-1)\binom{t}{3} + t$ bound.

To get the form of the statement claimed in the theorem we choose $i = \lfloor 1/\varepsilon \rfloor + 1$. $\square$

Remark 3.7. It is not hard to see that the graphs $M_s(K_{t-1})$ can also be interpreted as homomorphism universal graphs for a property related to wide colorings. Namely, a graph admits a homomorphism into $M_s(K_{t-1})$ if and only if it can be colored with t colors so that there is no walk of length $2s-1$ connecting two (not necessarily different) points of one particular color class, say, color class t. Realizing this, the statement of Lemma 3.1.30 is immediate. $\diamondsuit$

3.1.6 Further remarks

Color-criticality of $W(s,t)$

In this subsection we prove the edge color-criticality of the graphs $W(s,t)$ introduced in the previous subsection. This generalizes Theorem 2.3 in [72], see Remark 3.8 after the proof. (As we already mentioned, this result was later independently obtained also by Baum and Stiebitz [13].)

Theorem 3.1.31 *For every integer $s \geq 1$ and $t \geq 2$ the graph $W(s,t)$ has chromatic number t, but deleting any of its edges the resulting graph is $(t-1)$-chromatic.*

Proof. $\chi(W(s,t)) \geq t$ follows from the fact that some t-chromatic Schrijver graphs admit a homomorphism to $W(s,t)$ which is implied by Lemma 3.1.29 and Proposition 3.1.28. The coloring giving vertex $\boldsymbol{x} = x_1 \dots x_t$ of $W(s,t)$ color i iff $x_i = 0$ is proper proving $\chi(W(s,t)) \leq t$.

We prove edge-criticality by induction on t. For $t = 2$ the statement is trivial as $W(s,t)$ is isomorphic to K_2. Assume that $t \geq 3$ and edge-criticality holds for $t-1$. Let

$\{x_1 \dots x_t, y_1 \dots y_t\}$ be an edge of $W(s,t)$ and W' be the graph remaining after removal of this edge. We need to give a proper $(t-1)$-coloring c of W'.

Let i and j be the coordinates for which $x_i = y_j = 0$. We have $x_j = y_i = 1$, in particular, $i \neq j$. Let r be a coordinate different from both i and j. We may assume without loss of generality that $r = 1$, and also that $y_1 \geq x_1$. Coordinates i and j make sure that $x_2 x_3 \dots x_t$ and $y_2 y_3 \dots y_t$ are vertices of $W(s, t-1)$, and in fact, they are connected by an edge e.

A proper $(t-2)$-coloring of the graph $W(s,t-1) \setminus e$ exists by the induction hypothesis. Let c_0 be such a coloring. Let α be a color of c_0 and β a color that does not appear in c_0. We define the coloring c of W' as follows:

$$c(z_1 z_2 \dots z_t) = \begin{cases} \alpha & \text{if } z_1 < x_1,\ x_1 - z_1 \text{ is even} \\ \beta & \text{if } z_1 < x_1,\ x_1 - z_1 \text{ is odd} \\ \alpha & \text{if } z_1 = x_1 = 1,\ z_i \neq 1 \text{ for } i > 1 \\ \beta & \text{if } z_1 > x_1,\ z_i = x_i \text{ for } i > 1 \\ c_0(z_2 z_3 \dots z_t) & \text{otherwise.} \end{cases}$$

It takes a straightforward case analysis to check that c is a proper $(t-1)$-coloring of W'. □

Remark 3.8. Gyárfás, Jensen, and Stiebitz [72] proved the $s = 2$ version of the previous theorem using a homomorphism from their universal graph with parameter t to a generalized Mycielskian of the same type of graph with parameter $t-1$. In fact, our proof is a direct generalization of theirs using very similar ideas. Behind the coloring we gave is the recognition of a homomorphism from $W(s,t)$ to $M_{3s-2}(W(s,t-1))$. ♢

Hadwiger's conjecture and the Zig-zag theorem

Hadwiger's conjecture, one of the most famous open problems in graph theory, states that if a graph G contains no K_{r+1} minor, then $\chi(G) \leq r$. For detailed information on the history and status of this conjecture we refer to Toft's survey [154]. We only mention that even $\chi(G) = O(r)$ is not known to be implied by the hypothesis for general r.

As a fractional and linear approximation version, Reed and Seymour [129] proved that if G has no K_{r+1} minor then $\chi^*(G) \leq 2r$. This means that graphs with $\chi^*(G)$ and $\chi(G)$ appropriately close and not containing a K_{r+1} minor satisfy $\chi(G) = O(r)$.

We know that the main examples of graphs in [133] for $\chi^*(G) << \chi(G)$ (Kneser graphs, Mycielski graphs), as well as many other graphs studied in this section, satisfy the hypothesis of the Zig-zag theorem, therefore their t-chromatic versions must contain $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ subgraphs. (We mention that for strongly topologically t-chromatic graphs this consequence, in fact, the containment of $K_{a,b}$ for every a, b satisfying $a + b = t$, was proven by Csorba, Lange, Schurr, and Waßmer [42].) However, a $K_{\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor}$ subgraph contains a $K_{\lfloor \frac{t}{2} \rfloor + 1}$ minor (just take a matching of size $\lfloor \frac{t-2}{2} \rfloor$ plus one point from each side of the bipartite graph) proving the following statement which shows that the same kind of approximation is valid for these graphs, too.

Corollary 3.1.32 *If a topologically t-chromatic graph contains no K_{r+1} minor, then $t < 2r$.*

□

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3.2 Colorful subgraphs in Kneser-like graphs

In this section we combine Ky Fan's theorem with ideas of Greene and Matoušek to prove a generalization of Dol'nikov's theorem. Using another variant of the Borsuk-Ulam theorem due to Tucker and Bacon, we also prove the presence of all possible completely multicolored t-vertex complete bipartite graphs in t-colored t-chromatic Kneser graphs and in several of their relatives. In particular, this implies a generalization of a recent result of G. Spencer and F. E. Su.

This section is based on the joint paper [145] with Gábor Tardos.

3.2.1 Introduction

The solution of Kneser's conjecture in 1978 by László Lovász [108] opened up a new area of combinatorics that is usually referred to as topological combinatorics [43], [21]. Many results of this area, including the first one by Lovász, belong to one of its by now most developed branches that applies the celebrated Borsuk-Ulam theorem to graph coloring problems. An account of such results and other applications of the Borsuk-Ulam theorem in combinatorics is given in the excellent book of Matoušek [116].

Recently it turned out that a generalization of the Borsuk-Ulam theorem found by Ky Fan [52] in 1952 can give useful generalizations and variants of the Lovász-Kneser theorem. Examples of such results can be found in [121, 144, 147, 148], cf. also the previous section.

In this section our aim is twofold. In Subsection 3.2.2 we show a further application of Ky Fan's theorem. More precisely, we give a generalization of Dol'nikov's theorem, which is itself a generalization of the Lovász-Kneser theorem. The proof will be a simple combination of Ky Fan's result with the simple proof of Dol'nikov's theorem given by Matoušek in [116] that was inspired by Greene's recent proof [69] of the Lovász-Kneser theorem.

In Section 3.2.3 we use another variant of the Borsuk-Ulam theorem due to Tucker [153] and Bacon [10] to show a property of optimal colorings of certain t-chromatic graphs, including Kneser graphs, Schrijver graphs, Mycielski graphs, Borsuk graphs, odd chromatic rational complete graphs, and two other types of graphs we have already seen in the previous section. The claimed property (in a somewhat weakened form) is that all the complete bipartite graphs $K_{l,m}$ with $l + m = t$ will have totally multicolored copies in proper t-colorings of the above graphs.

When applied to rational complete graphs this implies a somewhat different proof of Theorem 3.1.6 of the previous section.

When applied to Kneser graphs the above property implies a generalization of a recent result due to G. Spencer and F. E. Su [147, 148] which we will present in the last subsection.

3.2.2 A generalization of Dol'nikov's theorem

We recall some concepts and notations from [116]. For any family $\mathcal{F}$ of subsets of a fixed finite set we define the general Kneser graph $\mathrm{KG}(\mathcal{F})$ by

$$\begin{aligned} V(\mathrm{KG}(\mathcal{F})) &= \mathcal{F}, \\ E(\mathrm{KG}(\mathcal{F})) &= \{\{F, F'\} : F, F' \in \mathcal{F}, F \cap F' = \emptyset\}. \end{aligned}$$

When we refer to a Kneser graph (without the adjective "general") we mean the "usual" Kneser graph $\mathrm{KG}(n, k)$ that is identical to the general Kneser graph of a set system consisting of all k-subsets of an n-set.

A hypergraph H is m-colorable, if its vertices can be colored by (at most) m colors so that no hyperedge becomes monochromatic. As isolated vertices (not contained in any hyperedge) play no role we may identify hypergraphs with the set of their edges as done in the following definition. The m-colorability defect of a set system $\emptyset \notin \mathcal{F}$ is

$$\mathrm{cd}_m(\mathcal{F}) := \min \{|Y| : \{F \in \mathcal{F} : F \cap Y = \emptyset\} \text{ is } m\text{-colorable}\}.$$

Dol'nikov's theorem ([46]) *For any finite set system $\emptyset \notin \mathcal{F}$, the inequality*

$$\mathrm{cd}_2(\mathcal{F}) \leq \chi(\mathrm{KG}(\mathcal{F}))$$

holds.

This theorem generalizes the Lovász-Kneser theorem, as it is easy to check that if $\mathcal{F}$ consists of all the k-subsets of an n-set with $k \leq n/2$, then $\mathrm{cd}_2(\mathcal{F}) = n - 2k + 2$ which is the true value of $\chi(\mathrm{KG}(\mathcal{F}))$ in this case. On the other hand, as also noted in [116], equality between $\mathrm{cd}_2(\mathcal{F})$ and $\chi(\mathrm{KG}(\mathcal{F}))$ does not hold in general.

Recently Greene [69] found a very simple new proof of the Lovász-Kneser theorem. In [116] Matoušek observed that one can generalize Greene's proof so that it also gives Dol'nikov's theorem. Here we combine this proof with Ky Fan's theorem to obtain the following generalization.

Theorem 3.2.1 *Let $\mathcal{F}$ be a finite family of sets, $\emptyset \notin \mathcal{F}$ and $\mathrm{KG}(\mathcal{F})$ its general Kneser graph. Let $r = \mathrm{cd}_2(\mathcal{F})$. Then any proper coloring of $\mathrm{KG}(\mathcal{F})$ with colors $1, \ldots, m$ (m arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil r/2 \rceil, \lfloor r/2 \rfloor}$ such that the r different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.*

This theorem generalizes Dol'nikov's theorem, because it implies that any proper coloring must use at least $\mathrm{cd}_2(\mathcal{F})$ different colors.

Remark 3.9. Theorem 3.2.1 is clearly in the spirit of the Zig-zag theorem of [144] (see in the previous subsection) the special case of which for Kneser graphs was already established by Ky Fan in [53]. Recall that this theorem claims that if $\mathrm{coind}(B_0(G)) + 1 \geq t$, then the graph must contain a completely multicolored $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$ where the colors also alternate on the two sides with respect to their natural order. As we will show in Remark 3.10 the proof below can be modified to show that $\mathrm{coind}(B_0(G))+1$ is at least $\mathrm{cd}_2(\mathcal{F})$ for any $G = \mathrm{KG}(\mathcal{F})$. We mention that the slightly weaker inequality $\mathrm{ind}(B_0(\mathrm{KG}(\mathcal{F}))) + 1 \geq \mathrm{cd}_2(\mathcal{F})$ is shown in [118]. $\Diamond$

To prove the theorem we first recall (form (ii) of) Ky Fan's theorem [52] already stated in the previous section. We repeat its statement for easier reference (with slight differences in the notation). As we already mentioned, it is formulated a little differently in [52], while this form can be found in [10]. Recall that for a set A on the unit sphere $\mathbb{S}^h$ we denote by $-A$ its antipodal set, i.e., $-A = \{-\boldsymbol{x} : \boldsymbol{x} \in A\}$.

Ky Fan's theorem ([52]) *Let $A_1, \ldots, A_m$ be open subsets of the h-dimensional sphere $\mathbb{S}^h$ satisfying that none of them contains antipodal points (i.e., $\forall i$ $(-A_i) \cap A_i = \emptyset$) and that at least one of $\boldsymbol{x}$ and $-\boldsymbol{x}$ is contained in $\cup_{i=1}^m A_i$ for all $\boldsymbol{x} \in \mathbb{S}^h$.*

Then there exists an $\boldsymbol{x} \in \mathbb{S}^h$ and $h+1$ distinct indices $i_1 < \ldots < i_{h+1}$ such that $\boldsymbol{x} \in A_{i_1} \cap (-A_{i_2}) \cap \ldots \cap ((-1)^h A_{i_{h+1}})$.

Proof of Theorem 3.2.1. Let $h = \mathrm{cd}_2(\mathcal{F}) - 1$ and consider the sphere $\mathbb{S}^h$. We assume without loss of generality that the base set $X := \cup\mathcal{F}$ is finite and identify its elements with points of $\mathbb{S}^h$ in general position, i.e., so that at most h of them can be on a common hyperplane through the origin. Consider an arbitrary fixed proper coloring of $\mathrm{KG}(\mathcal{F})$ with colors $1, \ldots, m$. For every $\boldsymbol{x} \in \mathbb{S}^h$ let $H(\boldsymbol{x})$ denote the open hemisphere centered at $\boldsymbol{x}$. Define the sets $A_1, \ldots, A_m$ as follows. Set A_i will contain exactly those points $\boldsymbol{x} \in \mathbb{S}^h$ that have the property that $H(\boldsymbol{x})$ contains the points of some $F \in \mathcal{F}$ which is colored by color i in the coloring of $\mathrm{KG}(\mathcal{F})$ considered. The sets A_i are all open. None of them contains an antipodal pair of points, otherwise there would be two disjoint open hemispheres both of which contain some element of $\mathcal{F}$ that is colored i. But these two elements of $\mathcal{F}$ would be disjoint contradicting the assumption that the coloring was proper. Now we show that there is no $\boldsymbol{x} \in \mathbb{S}^h$ for which neither $\boldsymbol{x}$ nor $-\boldsymbol{x}$ is in $\cup_{i=1}^m A_i$. Color the points in $X \cap H(\boldsymbol{x})$ red, the points in $X \cap H(-\boldsymbol{x})$ blue and delete the points of X not colored, i.e., those on the "equator" between $H(\boldsymbol{x})$ and $H(-\boldsymbol{x})$. Since at most $h < \mathrm{cd}_2(\mathcal{F})$ points are deleted there exists some $F \in \mathcal{F}$ which became completely red or completely blue. All points of F are either in $H(\boldsymbol{x})$ or in $H(-\boldsymbol{x})$. This implies that $\boldsymbol{x}$ or $-\boldsymbol{x}$ should belong to A_i where i is the color of F in our fixed coloring of $\mathrm{KG}(\mathcal{F})$.

Thus our sets $A_1, \ldots, A_m$ satisfy the conditions and therefore also the conclusion of Ky Fan's theorem. Let $F_{i_j} \in \mathcal{F}$ be the set responsible for $\boldsymbol{x} \in A_{i_j}$ for j odd and for $-\boldsymbol{x} \in A_{i_j}$ for j even in the conclusion of Ky Fan's theorem. Then all the F_{i_j}'s with odd j must be disjoint from all the F_{i_j}'s with even j. Thus they form the complete bipartite graph claimed. $\Box$

Remark 3.10. Our claim in Remark 3.9 was that the proof of Theorem 3.2.1 implies $\mathrm{coind}(B_0(\mathrm{KG}(\mathcal{F}))) \geq \mathrm{cd}_2(\mathcal{F}) - 1$ for any $\mathcal{F}$ not containing the empty set. Here we sketch the proof of this claim which is similar to the proof of Proposition 3.1.9 in the previous section. Assume again without loss of generality that $X = \cup\mathcal{F}$ is finite and identify its elements with points of $\mathbb{S}^h$ in general position as in the proof of Theorem 3.2.1 with $h = \mathrm{cd}_2(\mathcal{F}) - 1$. For each vertex v of $\mathrm{KG}(\mathcal{F})$ and $\boldsymbol{x} \in \mathbb{S}^h$ let $D_v(\boldsymbol{x})$ be the smallest distance of a point in v (this point is an element of X) from the set $\mathbb{S}^h \setminus H(\boldsymbol{x})$. Notice that $D_v(\boldsymbol{x}) > 0$ iff $H(\boldsymbol{x})$ contains all points of v. Set $D(\boldsymbol{x}) := \sum_{v\in\mathcal{F}}(D_v(\boldsymbol{x}) + D_v(-\boldsymbol{x}))$. The argument in the proof of Theorem 3.2.1 implies $D(\boldsymbol{x}) > 0$. Therefore the map $f(\boldsymbol{x}) = (1/D(\boldsymbol{x}))(\sum_v D_v(\boldsymbol{x})||(v,1)|| + \sum_v D_v(-\boldsymbol{x})||(v,2)||)$ is a $\mathbb{Z}_2$-map from $\mathbb{S}^h$ to $||B_0(\mathrm{KG}(\mathcal{F}))||$, thus $\mathrm{coind}(B_0(\mathrm{KG}(\mathcal{F}))) \geq h$ as claimed. (With this formulation the proof of Theorem 3.2.1 can be completed by applying the the Zig-zag theorem.) $\Diamond$

An example

We would like to point out that Theorem 3.2.1 is really stronger than Dol'nikov's theorem, especially if looked at as an upper bound for $\mathrm{cd}_2(\mathcal{F})$. It is shown in [118] that every graph G is isomorphic to the general Kneser graph $\mathrm{KG}(\mathcal{F})$ of some set system $\mathcal{F}$.

Consider the graphs $U(m,r)$ of Definition 29 we have alreday dealt with in Subsection 2.3.6 and again in Remark 3.3. Let $\mathcal{F}(m,r)$ denote a set system for which $U(m,r) \cong \mathrm{KG}(\mathcal{F}(m,r))$. It follows from results in [47] that if $r \geq 3$ is fixed and m goes to infinity then $\chi(U(m,r))$ also grows above all limits. Thus Dol'nikov's theorem would not give any finite upper bound for $\mathrm{cd}_2(\mathcal{F}(m,r))$ if $r \geq 3$ is fixed and m goes to infinity. The same is true if we consider only the size (and not the coloring) of largest complete bipartite subgraphs in $U(m,r)$ which, due to combination of results in [42] and [118] can also provide an upper bound for $\mathrm{cd}_2(\mathcal{F}(m,r))$. (Indeed, it is easy to check that $U(m,r)$ contains $K_{\binom{m-2}{r-2},\binom{m-2}{r-2}}$ subgraphs.) In contrast, consider the proper coloring given as $(x,A) \mapsto x$. One easily checks that the largest balanced completely multicolored complete bipartite subgraph in this coloring is $K_{r-1,r-1}$, from which Theorem 3.2.1 gives the upper bound $2r-2$ on $\mathrm{cd}_2(\mathcal{F}(m,r))$, which is independent of m.

3.2.3 Applying a theorem of Tucker and Bacon

Preliminaries

We will use again the topological concepts already introduced in Subsection 3.1.3.

Recall that in applications of the topological method one often associates box complexes to graphs. These give rise to topological spaces the index and coindex of which can serve to obtain lower bounds for the chromatic number of the graph. Following ideas in earlier works by Alon, Frankl, Lovász [5] and others, the paper [118] introduces several box complexes. We extensively used the box complex $B_0(G)$ in the previous section and referred to it again in the previous subsection, too. We briefly mentioned once (see the proof of Proposition 3.1.8) another box complex $B(G)$ the definition of which we make explicit below. As also mentioned after the proof of Proposition 3.1.8 $||B(G)||$ is known to be $\mathbb{Z}_2$-homotopy equivalent to $H(G)$ (see its definition right before Proposition 3.1.8) by a result of Csorba [39], cf. also [161] and [118].

Definition 41 *The box complex $B(G)$ is a simplicial complex on the vertices $V(G) \times \{1,2\}$. For subsets $S, T \subseteq V(G)$ the set $S \uplus T := S \times \{1\} \cup T \times \{2\}$ forms a simplex if and only if $S \cap T = \emptyset$, the vertices in S have at least one common neighbor, the same is true for T, and the complete bipartite graph with sides S and T is a subgraph of G. The $\mathbb{Z}_2$-map $S \uplus T \mapsto T \uplus S$ acts simplicially on $B(G)$ making the body $||B(G)||$ of the complex a free $\mathbb{Z}_2$-space.*

Thus the difference between $B(G)$ and $B_0(G)$ is that the former does not contain those simplices $\emptyset \uplus T$ and $S \uplus \emptyset$ where the vertices in T, respectively in S, do not have a common neighbor.

Note that $B(G)$ is also a functor like $B_0(G)$ and using the $\mathbb{Z}_2$-homotopy equivalence of $||B(G)||$ and $H(G)$ the inequality (3.1) can also be written in the following form.

$$\chi(G) \geq \operatorname{ind}(B(G)) + 2 \geq \operatorname{ind}(B_0(G)) + 1 \geq \operatorname{coind}(B_0(G)) + 1 \geq \operatorname{coind}(B(G)) + 2 \quad (3.2)$$

We have already seen in the previous section that there are several interesting graph families the members of which satisfy the inequalities in (3.2) with equality. These include, for example, Kneser graphs, and a longer list is given in Corollary 3.2.4 below. (We note that some of the graphs in Corollary 3.2.4 give equality only in the first three of the above inequalities, cf. what is said below about the homomorphism universal graphs for local colorings.)

A colorful $K_{l.m}$-theorem

In their recent paper [42] Csorba, Lange, Schurr, and Waßmer proved that if $\operatorname{ind}(B(G)) = l+m-2$ then G must contain the complete bipartite graph $K_{l,m}$ as a subgraph. They called this "the $K_{l,m}$-theorem". In case of those graphs that satisfy $\operatorname{coind}(B_0(G)) + 1 = \chi(G)$ (see Corollary 3.2.4), the following statement generalizes their result. We use again the notation $[t] := \{1, \ldots, t\}$.

Theorem 3.2.2 *Let G be a graph for which $\chi(G) = \operatorname{coind}(B_0(G)) + 1 = t$. Let $c : V(G) \to [t]$ be a proper coloring of G and let $A, B \subseteq [t]$ form a bipartition of the color set, i.e., $A \cup B = [t]$ and $A \cap B = \emptyset$.*

Then there exists a complete bipartite subgraph $K_{l,m}$ of G with sides L, M such that $|L| = l = |A|$, $|M| = m = |B|$, $\{c(v) : v \in L\} = A$, and $\{c(v) : v \in M\} = B$. In particular, all vertices of this $K_{l,m}$ receive different colors at c.

For the proof we will use a modified version of the following theorem.

Tucker-Bacon theorem ([153, 10]) *If $C_1, \ldots, C_{h+2}$ are closed subsets of $\mathbb{S}^h$,*

$$\cup_{i=1}^{h+2} C_i = \mathbb{S}^h, \quad \forall i: \ C_i \cap (-C_i) = \emptyset,$$

and $j \in \{1, \ldots, h+1\}$, then there is an $\boldsymbol{x} \in \mathbb{S}^h$ such that

$$\boldsymbol{x} \in \cap_{i=1}^{j} C_i, \text{ and } -\boldsymbol{x} \in \cap_{i=j+1}^{h+2} C_i.$$

Tucker proved the above theorem in the 2-dimensional case and remarked that it is also true in the higher dimensional cases. Bacon [10] shows that this theorem is equivalent to 14 other statements that include standard forms of the Borsuk-Ulam theorem, and also Ky Fan's theorem.

It is a routine matter to see that the Tucker-Bacon theorem also holds for open sets C_i. (One can simply use the fact that for a collection of open sets C_i covering $\mathbb{S}^h$ one can define closed sets C_i' so that $C_i' \subseteq C_i$ for all i and $\cup_{i=1}^{h+2} C_i' = \cup_{i=1}^{h+2} C_i = \mathbb{S}^h$. Cf. [1], Satz VII, p. 73, quoted also in [10].)

The modified version we need is the following.

Tucker-Bacon theorem, second form. *If $C_1, \ldots, C_{h+1}$ are open subsets of $\mathbb{S}^h$,*

$$\cup_{i=1}^{h+1} (C_i \cup (-C_i)) = \mathbb{S}^h, \quad \forall i: \; C_i \cap (-C_i) = \emptyset,$$

and $j \in \{0, \ldots, h+1\}$, then there is an $\boldsymbol{x} \in \mathbb{S}^h$ such that

$$\boldsymbol{x} \in C_i \text{ for } i \leq j \text{ and } -\boldsymbol{x} \in C_i \text{ for } i > j.$$

Proof. Let $D_{h+2} = \mathbb{S}^h \setminus \left(\cup_{i=1}^{h+1} C_i\right)$. Then $D_{h+2} \cap (-D_{h+2}) = \emptyset$ by the first condition on the sets C_i. Since D_{h+2} is closed there is some $\varepsilon > 0$ bounding the distance of any pair of points $\boldsymbol{x} \in D_{h+2}$ and $\boldsymbol{y} \in -D_{h+2}$ from below. Let C_{h+2} be the open $\frac{\varepsilon}{2}$-neighborhood of D_{h+2}. Then the open sets $C_1, \ldots, C_{h+2}$ satisfy the conditions (of the open set version) of the Tucker-Bacon theorem. Therefore its conclusion holds. Neglecting the set C_{h+2} in this conclusion the proof is completed for $j > 0$. To see the statement for $j = 0$ one can take the negative of the value $\boldsymbol{x}$ guaranteed for $j = h+1$. $\square$

Remark 3.11. Frédéric Meunier [122] noted that the second form of the Tucker-Bacon theorem can also be deduced directly from the above given form of Ky Fan's theorem (applied for $m = h+1$) by exchanging some of the given open sets with their antipodal set (separately for each possible value of j) and indexing appropriately. $\Diamond$

Proof of Theorem 3.2.2. Let G be a graph with $\chi(G) = \text{coind}(B_0(G)) + 1 = t$ and fix an arbitrary proper t-coloring $c: V(G) \to [t]$. Let $g: \mathbb{S}^{t-1} \to ||B_0(G)||$ be a $\mathbb{Z}_2$-map that exists by $\text{coind}(B_0(G)) = t-1$.

We define for each color $i \in [t]$ an open set C_i on $\mathbb{S}^{t-1}$. For $\boldsymbol{x} \in \mathbb{S}^{t-1}$ we let $\boldsymbol{x}$ be an element of C_i iff the minimal simplex $S_{\boldsymbol{x}} \uplus T_{\boldsymbol{x}} \in B_0(G)$ whose body contains $g(\boldsymbol{x})$ has a vertex $v \in S_{\boldsymbol{x}}$ for which $c(v) = i$. These C_i's are open. If an $\boldsymbol{x} \in \mathbb{S}^{t-1}$ is not covered by any C_i then $S_{\boldsymbol{x}}$ must be empty, which also implies $T_{\boldsymbol{x}} \neq \emptyset$. Since $S_{-\boldsymbol{x}} = T_{\boldsymbol{x}}$ this further implies $-\boldsymbol{x} \in \cup_{i=1}^t C_i$, thus $\cup_{i=1}^t (C_i \cup (-C_i)) = \mathbb{S}^{t-1}$ follows.

If a set C_i contained an antipodal pair $\boldsymbol{x}$ and $-\boldsymbol{x}$ then $S_{-\boldsymbol{x}} = T_{\boldsymbol{x}}$ would contain an i-colored vertex u, while $S_{\boldsymbol{x}}$ would contain an i-colored vertex v. Since $S_{\boldsymbol{x}} \uplus T_{\boldsymbol{x}}$ is a simplex of $B_0(G)$ u and v must be adjacent, contradicting that c is a proper coloring. Thus the C_i's satisfy the conditions of (the second form of) the Tucker-Bacon theorem.

Let $j = |A| = l$ and relabel the colors so that colors $1, \dots, j$ be in A, and the others be in B. The indices of the C_i's are relabelled accordingly. We apply the second form of the Tucker-Bacon theorem with $h = t-1$. It guarantees the existence of an $\boldsymbol{x} \in \mathbb{S}^{t-1}$ with the property $\boldsymbol{x} \in C_i$ for $i \in A$ and $-\boldsymbol{x} \in C_i$ for $i \in B$. Then $S_{\boldsymbol{x}}$ contains vertices $u_1, \dots, u_l$ with $c(u_i) = i$ for all $i \in A$ and $S_{-\boldsymbol{x}} = T_{\boldsymbol{x}}$ contains vertices $v_1, \dots, v_m$ with $c(v_i) = l+i$ for all $(l+i) \in B$. Since all vertices of $S_{\boldsymbol{x}}$ are connected to all vertices in $T_{\boldsymbol{x}}$ by the definition of $B_0(G)$, they give the required completely multicolored $K_{l,m}$ subgraph. □

Graphs that are subject of the colorful $K_{l,m}$ theorem

Let us put the statement of Theorem 3.2.2 into the perspective of our earlier work in [144] presented in the previous section. There we investigated the local chromatic number of graphs that is defined in [47], see Definition 25 in Subsection 2.3.1 (page 55).

With similar techniques to those applied in this section we have shown using Ky Fan's theorem that $\mathrm{coind}(B_0(G)) \geq t-1$ implies that G must contain a completely multicolored $K_{\lceil t/2\rceil, \lfloor t/2\rfloor}$ subgraph in *any* proper coloring with the colors alternating with respect to their natural order on the two sides of this complete bipartite graph. This is the Zig-zag theorem in [144] we already referred to in Remark 3.9 and we have seen in the previous section. The Zig-zag theorem implies that any graph satisfying its condition must have $\psi(G) \geq \lceil t/2\rceil + 1$. In [144], i.e., in the previous section, we have shown for several graphs G for which $\chi(G) = \mathrm{coind}(B_0(G))+1 = t$ that it can be colored with $t+1$ colors so that no vertex has more than $\lfloor t/2\rfloor + 1$ colors on its neighbors. When t is odd, this established the exact value $\psi(G) = \lceil t/2\rceil + 1$ for these graphs. For odd t this also means that the only type of $K_{l,m}$ subgraph with $l+m = t$ that must appear completely multicolored when using $t+1$ colors is the $K_{\lceil t/2\rceil, \lfloor t/2\rfloor}$ subgraph guaranteed by the Zig-zag theorem (apart from the empty graph $K_{t,0}$). (In principle completely multicolored $K_{\frac{t+1}{2}, \frac{t+1}{2}}$ subgraphs could still appear without increasing $\psi(G)$, but it is explained after the proof of Lemma 3.1.10 why they do actually not occur.) If we use only t colors, however, then the situation is quite different. It is true for any graph F that if it is properly colored with $\chi(F)$ colors then each color class must contain a vertex that sees all other colors on the vertices adjacent to it. If it were not so, we could completely eliminate a color class by recoloring each of its vertices to a color which is not present on any of its neighbors. In the context of local chromaticity this means (as already mentioned in Subsection 3.1.1) that if $\psi(F) < \chi(F)$ then it can only be attained by a coloring that uses strictly more than $\chi(F)$ colors. Now Theorem 3.2.2 says that if G satisfies $\chi(G) = \mathrm{coind}(B_0(G))+1 = t$ then all t-colorings give rise not only to completely multicolored $K_{\lceil t/2\rceil,\lfloor t/2\rfloor}$'s that are guaranteed by the Zig-zag theorem, and $K_{1,t-1}$'s that must appear in any optimal coloring (with all possible choices of the color being on the single vertex side), but to all possible completely multicolored complete bipartite graphs on t vertices.

To conclude this subsection we list explicitly some classes of graphs G that satisfy the $\chi(G) = \mathrm{coind}(B_0(G)) + 1$ condition of Theorem 3.2.2. We recall that in the previous section we used the term *topologically t-chromatic* for graphs G with $\mathrm{coind}(B_0(G)) \geq t-1$. With this notation we are listing graphs G that are topologically $\chi(G)$-chromatic, i.e., for which this specific lower bound on their chromatic number is tight.

Standard examples

For the following graph families it is well known that they (or their appropriate members) satisfy the condition in Theorem 3.2.2. For a detailed discussion we refer to [116] and the previous section.

Kneser graphs. The argument presented in Remark 3.10 proves that the general Kneser graph $G = \mathrm{KG}(\mathcal{F})$ satisfies $\chi(G) = \mathrm{coind}(B_0(G)) + 1$ as long as $\chi(\mathrm{KG}(\mathcal{F})) = \mathrm{cd}_2(\mathcal{F})$. This family includes the graphs $\mathrm{KG}(n, k)$.

Schrijver graphs. The graph denoted by $\mathrm{SG}(n, k)$ is just the general Kneser graph $\mathrm{KG}(\mathcal{F})$ for the set system $\mathcal{F}$ consisting of exactly those k-subsets of $[n]$ that contain neither a pair $\{i, i+1\}$, nor $\{1, n\}$ (cf. Definition 33 on page 72). Though $\chi(\mathrm{KG}(\mathcal{F})) \neq \mathrm{cd}_2(\mathcal{F})$ for this $\mathcal{F}$ if $k > 1$, all graphs $\mathrm{SG}(n, k)$ satisfy the condition in Theorem 3.2.2.

Borsuk graphs. The graph $B(n, \alpha)$ has $\mathbb{S}^{n-1}$ as its vertex set and two vertices are adjacent if their distance is at least $\alpha < 2$, see Definition 36 on page 92 and cf. [48, 111]. If α is close enough to 2, then the chromatic number of these graphs is $n + 1$. The paper [111] shows that some finite subgraphs of $B(n, \alpha)$ also have the required properties.

Mycielski and generalized Mycielski graphs. We discussed these graphs (and the above ones as well) in more detail in Subsection 3.1.4. Below we recall some of their relevant properties.

When applying the generalized Mycielski construction to an arbitrary graph, the clique number does not increase (except in the trivial case when $r = 1$) while the chromatic number may or may not increase. If it does it increases by 1. Generalizing Stiebitz's result [149] (see also in [72, 116]) Csorba [40, 41] proved that $B(M_r(G))$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of $B(G)$ for every graph G. (Csorba's result is in terms of the so-called homomorphism complex $\mathrm{Hom}(K_2, G)$ but this is known to be $\mathbb{Z}_2$-homotopy equivalent to $B(G)$ by results in [39, 118, 161].) Together with Csorba's already mentioned other result in [39] stating the $\mathbb{Z}_2$-homotopy equivalence of $B_0(G)$ and the suspension of $B(G)$, the foregoing implies that if a graph G satisfies $\chi(G) = \mathrm{coind}(B_0(G)) + 1$, then the analogous equality will also hold for the graph $M_r(G)$. In this case the chromatic number does increase by 1. Thus iterating the construction d times (perhaps with varying parameters r) we arrive to a graph which has chromatic number $\chi(G)+d$ and still satisfies that its chromatic number is equal to its third (in fact, for $d > 0$ also the fourth) lower bound given in (3.2).

In the following paragraphs we list three more families of graphs that satisfy the condition in Theorem 3.2.2. Members of the third family are well-known graphs but probably not in the present context. The other two families are less known, though we have already met them in the previous section where it was proven that they belong here.

Homomorphism universal graphs for local colorings

Recall Definition 29 of the graphs $U(m,r)$ (we used the letter k in place of r there) the vertices and edges of which are given by

$$\begin{aligned} V(U(m,r)) &= \{(i,A) : i \in [m], A \subseteq [m], |A| = r-1, i \notin A\} \\ E(U(m,r)) &= \{\{(i,A),(j,B)\} : i \in B, j \in A\} \end{aligned}$$

Recall that it is shown in [47] that these graphs characterize local colorability in the following sense: a graph G has an m-coloring attaining $\psi(G) \le r$ if and only if G admits a homomorphism to $U(m,r)$. As already mentioned above, for all odd $t \ge 3$ we showed in the previous section for several t-chromatic graphs satisfying the conditions of Theorem 3.2.2 that their local chromatic number is $\lceil t/2 \rceil + 1$ and it is attained with a coloring using $t+1$ colors. It follows that for odd t the t-chromatic graph $U(t+1, \frac{t+3}{2})$ also satisfies the conditions of Theorem 3.2.2. (Indeed, by the functoriality of $B_0(G)$, $\mathrm{coind}(B_0(F)) \ge t-1$ and the existence of a homomorphism $F \to G$ implies $\mathbb{S}^{t-1} \to ||B_0(F)|| \to ||B_0(G)||$ and thus $\mathrm{coind}(B_0(G)) \ge t-1$, cf. also Remark 3.3 in Section 3.1.)

The graphs $U(t+1, \frac{t+2}{2})$ with t even also belong here. It is proven in [146] that $\mathrm{coind}(B_0(U(t+1, \frac{t+2}{2})) = t-1$, but the proof is rather different than the previous argument above. The t-colorability of these graphs is also easy to check. We also mention the result from [146] according to which the fourth lower bound on the chromatic number in (3.2) is not tight for these graphs (while it is for the graphs of the previous paragraphs). This shows that Theorem 3.2.2 in its present form is somewhat stronger than it would be with the stronger requirement $\chi(G) = \mathrm{coind}(B(G)) + 2$ in place of $\chi(G) = \mathrm{coind}(B_0(G)) + 1$. We needed the second form of the Tucker-Bacon theorem for obtaining this stronger form.

We mention that $\chi(U(t+1, \lfloor \frac{t+3}{2} \rfloor)) = t$ is a special case of Theorem 2.6 in [47].

Homomorphism universal graphs for wide colorings

Recall the definition of graphs $W(s,t)$ in Definition 40.

The graphs $W(2,t)$ are defined in [72] in somewhat different terms. It is shown there that a graph can be colored properly with t colors so that the neighborhood of each color class is an independent set if and only if it admits a homomorphism into $W(2,t)$. The described property is equivalent to having a t coloring where no walk of length 3 can connect vertices of the same color. Similarly, a graph F admits a homomorphism into $W(s,t)$ if and only if it can be colored with t colors so that no walk of length $2s-1$ can connect vertices of the same color. Such colorings were called s-wide in the previous section. Other graphs having the mentioned property of $W(s,t)$ are also defined in [72]. The graphs $W(s,t)$ are defined and shown to be minimal with respect to the above property in [144] (see Theorem 3.1.31) and independently also in [13]. The t-colorability of $W(s,t)$ is obvious: $c(x_1 \dots x_t) = i$ if $x_i = 0$ gives a proper coloring. It was also shown in the previous section, that several of the above mentioned t-chromatic graphs (e.g., $B(t-1,\alpha)$ for α close enough to 2 and $\mathrm{SG}(n,k)$ with $n-2k+2=t$ and n,k large enough with respect to s and t) admit a homomorphism to $W(s,t)$. This implies $\mathrm{coind}(B_0(W(s,t))) \geq t-1$ (with equality, since $\chi(W(s,t)) = t$, cf. also Remark 3.6 in Subsection 3.1.4).

Rational complete graphs

Our last example of a graph family satisfying the conditions of Theorem 3.2.2 consists of certain rational (or circular) complete graphs $K_{p/q}$, as they are called, for example, in [77]. The graph $K_{p/q}$ is defined for positive integers $p \geq 2q$ on vertex set $\{0,\dots,p-1\}$ with $\{i,j\}$ being an edge if and only if $q \leq |i-j| \leq p-q$. The widely investigated chromatic parameter $\chi_c(G)$, the circular chromatic number of graph G (cf. [159], or Section 6.1 in [77]) can be defined as the infimum of those values p/q for which G admits a homomorphism to $K_{p/q}$ (cf. Definition 38 on page 95). Recall that $\chi(G)-1 < \chi_c(G) \leq \chi(G)$ for every graph G. In [102] it is shown that certain odd-chromatic generalized Mycielski graphs can have their circular chromatic number arbitrarily close to the above lower bound. Building on this we showed similar results also for odd chromatic Schrijver graphs and Borsuk graphs in the previous section. As it is also known that $K_{p/q}$ admits a homomorphism into $K_{r/s}$ whenever $r/s \geq p/q$ (see, e.g., as Theorem 6.3 in [77]), the above and the functoriality of $B_0(G)$ together imply that $\mathrm{coind}(B_0(K_{p/q}))+1 = \chi(K_{p/q}) = \lceil p/q \rceil$ whenever $\lceil p/q \rceil$ is odd.

We remark that the oddness condition is crucial here. It also follows from results in the previous section (see Theorem 3.1.6 and cf. also [121] for some special cases) that the graphs $K_{p/q}$ with $\lceil p/q \rceil$ even and p/q not integral do not satisfy the conditions of Theorem 3.2.2. Here we state more: the conclusion of Theorem 3.2.2 does not hold for these graphs. Indeed, let us color vertex i with the color $\lfloor i/q \rfloor + 1$. This is a proper coloring with the minimal number $\lceil p/q \rceil$ of colors, but it does not contain a complete bipartite graph with all the even colors on one side and all the odd colors on the other. (The remaining case of p/q even and integral is not especially interesting as $K_{p/q}$ with p/q integral is homomorphic equivalent to the complete graph on p/q vertices and therefore trivially satisfies the $\chi(G) = \mathrm{coind}(B_0(G)) + 1$ condition.)

Taking the contrapositive in the above observation we obtain another proof of Theorem 3.1.6 the special case of which for Kneser graphs and Schrijver graphs was independently obtained by Meunier [121].

Corollary 3.2.3 (See as Theorem 3.1.6 in the previous section, cf. also [121]) *If* $\mathrm{coind}(B_0(G))$ *is odd for a graph* G*, then* $\chi_c(G) \geq \mathrm{coind}(B_0(G)) + 1$*. In particular, if* G *satisfies* $\chi(G) = \mathrm{coind}(B_0(G)) + 1$ *and this number is even, then* $\chi_c(G) = \chi(G)$*.* □

Proof. If a graph G has $\chi_c(G) = p/q$, then G admits a homomorphism to $K_{p/q}$, thus by the functoriality of $B_0(G)$ we have $\mathrm{coind}(B_0(K_{p/q})) \geq \mathrm{coind}(B_0(G))$. Then $\lceil p/q \rceil = \chi(K_{p/q}) \geq \mathrm{coind}(B_0(K_{p/q}))+1 \geq \mathrm{coind}(B_0(G))+1$. If in addition $\mathrm{coind}(B_0(G))+1 > p/q$, then all the previous inequalities hold with equality by the integrality of the coindex. Then $K_{p/q}$ satisfies the conditions of Theorem 3.2.2, thus it must satisfy its conclusion. But we just have seen that this is not so if $\lceil p/q \rceil$ is even and p/q is not integral. Thus $\mathrm{coind}(B_0(G)) + 1$ cannot be even in this case. □

The proof of Corollary 3.2.3 in the previous section and also the proof in [121] relies on Ky Fan's theorem. The above argument shows that Ky Fan's theorem can be substituted by (the second form of) the Tucker-Bacon theorem in obtaining this result. Nevertheless, it may be worth noting, that the missing bipartite graph in the above optimal coloring of an even-chromatic $K_{p/q}$ is one the presence of which would also be required by the Zig-zag theorem. In this sense the above proof is not that much different from the earlier one.

The entire collection

Our examples are collected in the following corollary.

Corollary 3.2.4 *For any proper* t*-coloring of any member of the following* t*-chromatic families of graphs the property described as the conclusion of Theorem 3.2.2 holds.*

(i) *Kneser graphs* $\mathrm{KG}(n,k)$ *with* $t = n - 2k + 2$*,*

(ii) *Schrijver graphs* $\mathrm{SG}(n,k)$ *with* $t = n - 2k + 2$*,*

(iii) *Borsuk graphs* $B(t-1,\alpha)$ *with large enough* $\alpha < 2$ *and some of their finite subgraphs,*

(iv) $U(t+1, \lfloor \frac{t+3}{2} \rfloor)$*, for any* $t \geq 2$*,*

(v) $W(s,t)$ *for every* $s \geq 1, t \geq 2$*,*

(vi) *rational complete graphs* $K_{p/q}$ *for* $t = \lceil p/q \rceil$ *odd,*

(vii) *the* t*-chromatic graphs obtained by* $1 \leq d \leq t-2$ *iterations of the generalized Mycielski construction starting with a* $(t-d)$*-chromatic version of any graph appearing on the list above.*

□

Generalization of Spencer and Su's result

Recently Gwen Spencer and Francis Edward Su [147, 148] found an interesting consequence of Ky Fan's theorem. They prove that if the Kneser graph $\mathrm{KG}(n,k)$ is colored optimally, that is, with $t = n-2k+2$ colors, but otherwise arbitrarily, then the following holds. Given any bipartition of the color set $[t]$ into partition classes B_1 and $B_2 = [t] \setminus B_1$ with $|B_1| = \lfloor t/2 \rfloor$, there exists a bipartition of the ground set $[n]$ into E_1 and E_2, such that, the k-subsets of E_i as vertices of $\mathrm{KG}(n,k)$ are all colored with colors from B_i and every color in B_i does occur (i=1,2).

Theorem 3.2.2 implies an analogous statement where no special requirement is needed about the sizes of B_1 and B_2. It can also be obtained by simply replacing Ky Fan's theorem by the Tucker-Bacon theorem in Spencer and Su's argument.

Corollary 3.2.5 *Let $t = n-2k+2$ and fix an arbitrary proper t-coloring c of the Kneser graph $\mathrm{KG}(n,k)$ with colors from the color set $[t]$. Let B_1 and B_2 form a bipartition of $[t]$, i.e., $B_1 \cup B_2 = [t]$ and $B_1 \cap B_2 = \emptyset$. Then there exists a bipartition (E_1, E_2) of $[n]$ such that for $i = 1,2$ we have $\{c(v) : v \in V(\mathrm{KG}(n,k)), v \subseteq E_i\} = B_i$.*

Proof. Set $A = B_1$ and $B = B_2$ and consider the complete bipartite graph Theorem 3.2.2 returns for this bipartition of the color set. Let the vertices on the two sides of this bipartite graph be $u_1, \dots, u_{|A|}$ and $v_1, \dots, v_{|B|}$. All vertices u_i and v_j are subsets of $[n]$. Since u_i is adjacent to v_j for every i,j we have that $E_1' := \cup_{i=1}^{|A|} u_i$ and $E_2' := \cup_{j=1}^{|B|} v_j$ are disjoint. If there are elements of $[n]$ that are neither in E_1' nor in E_2' then put each such element into either one of the sets E_i' thus forming the sets E_1 and E_2. We show that these sets E_i satisfy our requirements. It follows from the construction that $E_1 \cap E_2 = \emptyset$ and $E_1 \cup E_2 = [n]$. It is also clear that all colors from B_1 appear as the color of some k-subset of E_1, namely, the k-subsets $u_1, \dots, u_{|A|}$. Since E_2 is disjoint from E_1 no k-subset of E_2 can be colored by any of the colors from B_1. Thus each k-subset of E_2 is colored by a color from B_2, and all these colors appear on some k-subset of E_2 by the presence of $v_1, \dots, v_{|B|}$. Exchanging the role of E_1 and E_2 we get that all k-subsets of E_1 are colored with some color of B_1 and the proof is complete. $\square$

Remark 3.12. The same argument proves a similar statement for the general Kneser graph $\mathrm{KG}(\mathcal{F})$ in place of $\mathrm{KG}(n,k)$ as long as we have $t = \chi(\mathrm{KG}(\mathcal{F})) = \mathrm{coind}(B_0(\mathrm{KG}(\mathcal{F}))) + 1$. Such graphs include the Schrijver graphs $\mathrm{SG}(n,k)$ with $t = n-2k+2$ and (by the argument presented in Remark 3.10) the graphs $\mathrm{KG}(\mathcal{F})$ with a family $\emptyset \notin \mathcal{F}$ satisfying $t = \chi(\mathrm{KG}(\mathcal{F})) = \mathrm{cd}_2(\mathcal{F})$. $\Diamond$

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