On the structure of compact topological spaces

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Introduction

This thesis is about some properties of compact and related topological spaces.

In the first section we study discrete subspaces of topological spaces. For example in a T_2 space the points of an ordinary (ω type) convergent sequence are discrete. Some kind of spaces (metric, M_1 spaces, ...) are fully determined by the convergent sequences. But there are compact spaces, for example βN, which have no nontrivial convergent ω-sequences. It is well known that in a compact space every non-isolated point is the limit point of a convergent transfinite sequence. For a convergent ω-sequence it is also true that any initial segment doesn't accumulate to the limit point. If we generalize this property to transfinite convergence, we get the concept of a "free sequence". We prove that if a compact space has tightness greater or equal to a regular cardinal $\kappa > \omega$, then the space has a convergent κ-type free sequence.

A convergent (free) sequence in some sense has thin closure. Therefore it is an interesting problem how large can be the closure of a discrete subspace. We prove that if a weak version of GCH holds, in any countably tight compactum X there is a discrete subspace D with $|\overline{D}| = |X|$.

We also prove that if a countably compact space is the union of countably many D subspaces then it is compact, and if a compact Hausdorff space is the union of fewer than $N(\mathbb{R})$ = $\operatorname{cov}(\mathcal{M})$ left-separated subspaces then it is scattered.

Next we study the d-separable spaces, that is the spaces which have σ-discrete dense subspaces and we answer several problems raised by V. V. Tkachuk.

We prove the following result: If in a compact T_1 space X there is a λ-branching family of closed sets then X cannot be covered by fewer than λ many discrete subspaces. (A family of sets $\mathcal{F}$ is λ-branching iff $|\mathcal{F}| < \lambda$ but one can form λ many pairwise disjoint intersections of subfamilies of $\mathcal{F}$.) As a consequence, we obtain the following strengthening of the well-known Čech-Pospišil theorem: If X a is compact T_2 space such that all points $x \in X$ have character $\chi(x, X) \geq \kappa$ then X cannot be covered by fewer than 2^κ many discrete subspaces.

We call a topological space κ-compact if every subset of size κ has a complete accumulation point in it. Let $\Phi(\mu, \kappa, \lambda)$ denote the following statement: $\mu < \kappa < \lambda = \operatorname{cf}(\lambda)$ and there is $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ such that $|\{\xi : |S_\xi \cap A| = \mu\}| < \lambda$ whenever $A \in [\kappa]^{<\kappa}$. We show that if $\Phi(\mu, \kappa, \lambda)$ holds and the space X is both μ-compact and λ-compact then X is κ-compact as well. Moreover, from PCF theory we deduce $\Phi(\operatorname{cf}(\kappa), \kappa, \kappa^+)$ for every singular cardinal κ. As a corollary we get that a linearly Lindelöf and $\aleph_\omega$-compact space is uncountably compact, that is κ-compact for all uncountable cardinals κ.

The next section is about the relation of calibers and density. We prove, among others, the following theorems:

– If X is a T_3 space with no free sequences of length λ and $(\lambda, \lambda, \kappa)$ is a caliber of

X then $d(X) \leq \mu^{<\lambda}$ for some cardinal $\mu < \kappa$.

– If X is T_3 and $X = \bigcup \mathcal{C}$ with $|\mathcal{C}| < \kappa$ and the members of $\mathcal{C}$ are compact with no free sequences of length μ, moreover (μ, κ) is a caliber of X then $d(X) < \kappa$.

– If X is T_3 and $X = \bigcup \mathcal{C}$ with $|\mathcal{C}| \leq \kappa$ and C is compact with no free sequences of length κ for every $C \in \mathcal{C}$, moreover κ is a caliber of X then $d(X) < \kappa$.

These results provide strengthenings and generalizations of some results of Šapirovskii and of Arhangelskii, respectively.

We define the projective π-character $p\,\pi\chi(X)$ of a space X as the supremum of the values $\pi\chi(Y)$ where Y ranges over all (Tychonov) continuous images of X. Our main result says that every Tychonov space X has a π-base whose order is $\leq p\,\pi\chi(X)$, that is every point in X is contained in at most $p\,\pi\chi(X)$-many members of the π-base. Since $p\,\pi\chi(X) \leq t(X)$ for compact X, this is a significant generalization of a celebrated result of Šhapirovskii.

We answer several questions of V. Tkachuk from [*Point-countable π-bases in first countable and similar spaces*, Fund. Math. 186 (2005), pp. 55–69.] by showing the following results:

– There is a ZFC example of a first countable, 0-dimensional Hausdorff space with no point-countable π-base (in fact, the minimum order of a π-base of the space can be made arbitrarily large).

– If there is a κ-Suslin line then there is a first countable GO space of cardinality κ^+ in which the order of any π-base is at least κ.

– It is consistent to have a first countable, hereditarily Lindelöf regular space having uncountable π-weight and ω_1 as a caliber (of course, such a space cannot have a point-countable π-base).

The final section is about characterizing continuity by preserving compactness and connectedness. Let us call a function f from a space X into a space Y *preserving* if the image of every compact subspace of X is compact in Y and the image of every connected subspace of X is connected in Y. By elementary theorems a continuous function is always preserving. Evelyn R. McMillan [46] proved in 1970 that if X is Hausdorff, locally connected and Frèchet, Y is Hausdorff, then the converse is also true: any preserving function $f : X \to Y$ is continuous. The main result of this part is that if X is any product of connected linearly ordered spaces (e.g. if $X = \mathbb{R}^\kappa$) and $f : X \to Y$ is a preserving function into a regular space Y, then f is continuous.

Notations

We use all the standard notations of set theory.

- If $\mathcal{I}$ is an ideal on the set X such that $[X]^{<\omega} \subset \mathcal{I}$ then the *covering number of* $\mathcal{I}$ defined as $\operatorname{cov}(\mathcal{I}) = \min\{|\mathcal{J}| : \mathcal{J} \subset \mathcal{I} \text{ és } \bigcup \mathcal{J}\} = X$
- If $\mathcal{M}$ is the ideal on $\mathbb{R}$ generated by the nowhere dense subsets then $N(\mathbb{R}) = \operatorname{cov}(\mathcal{M})$ denotes the *Novak-number* of the real line.
- $s = \min\{|\mathcal{A}| : \mathcal{A}$ splitting family$\}$ the *splitting number*.
  $\mathcal{A} \subset [\omega]^\omega$ is a *splitting family* if $\forall\, Y \in [\omega]^\omega \ \exists\, A \in \mathcal{A}\ |Y \cap A| = |Y \setminus A| = \omega$
- $p = \min\{|\mathcal{F}| : \mathcal{F} \subset [\omega]^\omega$ strongly centered and $\forall A \in [\omega]^\omega \exists F \in \mathcal{F} |A \setminus F| = \omega\}$
  $\mathcal{F} \subset [\omega]^\omega$ *strongly centered* if $\forall f \in [\mathcal{F}]^{<\omega}\ |\cap f| = \omega$
- If $\mathcal{B} \subset P(X)$ then the *order of* $\mathcal{B}$
  $\operatorname{ord}(\mathcal{B}) = \sup\{\operatorname{ord}(x, \mathcal{B}) : x \in X\}$ where $\operatorname{ord}(x, \mathcal{B}) = |\{B \in \mathcal{B} : x \in B\}|$
  The family $\mathcal{B}$ is *point-countable* if $\operatorname{ord}(\mathcal{B}) \leq \omega$

Cardinal functions

In this dissertation we assume that all topological spaces are infinite T_1 spaces. τ denotes the open sets of X and $\tau^* = \tau \setminus \{\emptyset\}$ the family of the nonempty open sets. Now let X be a topological space and $x \in X$ an arbitrary point in the space.

- $w(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a base of $X\}$ is the *weight* of X.
  The space X is said to be an M_2 space if $w(X) = \omega$.
- $\pi(X) = \min\{|\mathcal{B}| : \mathcal{B}$ is a π-base $\}$ is the π-*weight* of X.
- $d(X) = \min\{|S| : \overline{S} = X\}$ is the *density* of X.
- $L(X) = \min\{\kappa : \forall \mathcal{G}$ open cover $\exists \mathcal{G}' \in [\mathcal{G}]^{\leq\kappa}$ subcover $\}$ is the *a Lindelöf-number* of X.
  The space X said to be *Lindelöf* if $L(X) = \omega$.
- $c(X) = \sup\{|\mathcal{C}| : \mathcal{C} \subset \tau$ is cellular $\}$ is the *cellularity* of X.
  Here "cellular" means "pairwise disjoint".
  The space X has the *Suslin-property (shortly Suslin)* if $c(X) = \omega$.
- $s(X) = \sup\{|D| : D \subset X$ is discrete $\}$ is the *spread* of X.

A space S is *left (right) separated* if there is a well-ordering of S such that every final (initial) segment is open.

- $z(X) = hd(X) = \sup\{|S| : S$ is left separeted $\}$.
- $h(X) = hL(X) = \sup\{|S| : S$ is right separeted $\}$.

$\{x_\alpha : \alpha \in \kappa\} \subset X$ is a *free sequence* if $\forall \alpha \in \kappa\ \overline{\{x_\beta : \beta < \alpha\}} \cap \overline{\{x_\beta : \beta \geq \alpha\}} = \emptyset$.

- $F(X) = \sup\{|S| : S$ is a free sequence $\}$.

Some of the (global) cardinal functions (like $s(X),\ z(X),\ h(X),\ F(X)$) are defined in the form:
$\varphi(X) = \sup\{|Y| : Y \subset X$ and $\psi(Y)\}$
where ψ is a property of spaces that is inherited by subspaces. In this case we can define the "$\widehat{\ }$version" of φ as follows:
$\widehat{\varphi}(X) = \min\{|Y| : Y \subset X$ and $\neg\psi(Y)\}$
In this way we get the cardinal functions $\widehat{s}(X),\ \widehat{z}(X),\ \widehat{h}(X),\ \widehat{F}(X)$.

- $\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \subset \tau$ is a neighbourhood base of $x\}$ is the *character of* x in X.
  $\chi(X) = \sup\{\chi(x, X) : x \in X\}$ is the *character* of X.
  The space X is said to be an M_1 space or *first-countable* space if $\chi(X) \leq \omega$.
- $\psi(x, X) = \min\{|\mathcal{B}| : \mathcal{B} \subset \tau, \bigcap \mathcal{B} = \{x\}\}$ is the *pseudo-character of* x in X
  $\psi(X) = \sup\{\psi(x, X) : x \in X\}$ is the *pseudo-character* of X.
- $t(x, X) = \min\{\kappa : \forall A \subset X, x \in \operatorname{cl} A \Longrightarrow \exists B \subset A, |B| \leq \kappa, x \in \overline{B}\}$ is the *tightness of* x in X.
  $t(X) = \sup\{t(x, X) : x \in X\}$ is the *tightness* of X.
- $\pi\chi(x, X) = \min\{|\mathcal{B}| : \mathcal{B}$ local π-base of x-ben$\}$ is the π-*character of* x in X.
  $\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$ is the π-*character* of X.

1 Discrete subspaces

1.1 Convergent free sequences in compact spaces

While there are compact T_2 spaces with no nontrivial convergent ω-sequences, βN being perhaps the best known such space, every infinite compact T_2 space contains nontrivial convergent transfinite sequences. Indeed, as it is easy to see, if in such a space $\{A_\alpha : \alpha \in \varphi\}$ is a (strictly) decreasing sequence of closed sets with $\bigcap\{A_\alpha : \alpha \in \varphi\} = p$, a singleton, then for every choice of points $q_\alpha \in A_\alpha \setminus A_{\alpha+1}$, the sequence $\{q_\alpha : \alpha \in \varphi\}$ convergences to p, provided that φ is a limit ordinal.

Moreover, it is also easy to see that if the character of a point p in such a space is equal to κ then a decreasing sequence of closed sets $\{A_\alpha : \alpha \in \kappa\}$ does exist with

$$\bigcap\{A_\alpha : \alpha \in \kappa\} = p,$$

hence p is the limit of a κ-sequence. Consequently, since every infinite compact T_2 space has a separable closed infinite subspace, in which then every point has character $\leq \mathfrak{c} = 2^\omega$, we get that every such space has convergent sequences of length $\leq \mathfrak{c}$.

On the other hand, it has also been known (see [8] or [71]) that for example, βN contains a convergent sequence of length ω_1 , without assuming CH, hence the natural question arises whether every infinite compact T_2 space contains a convergent ω or ω_1, sequence? This question was first formulated by Hušek in the late seventies, and a related stronger problem was independently raised by István Juhász around the same time: Does every nonfirst countable compact T_2 space contain a convergent ω_1 sequence?

In this section we show that every compact T_2 space of uncountable tightness contains a convergent ω_1 sequence, moreover assuming CH every nonfirst countable compact T_2 space does. Also we obtain that under CH every compact T_2 space with a small diagonal is metrizable, thus solving another problem of Husek from [24]. More generally, using some results of Dow from [12] we obtain that these consequences will hold in any generic extension obtained by adding Cohen reals to any ground model satisfying CH.

1.1.1 The Main theorem

The main result of this section says that if κ is any uncountable regular cardinal and X is a compact T_2 space containing a free sequence of length κ, then X also contains a convergent such sequence. Let us note that having a free sequence $\{p_\alpha : \alpha \in \kappa\}$ converging to p in X also yields a closed set $F \subset X$ with $\chi(p, F) = \kappa$. Indeed, $F = \overline{\{p_\alpha : \alpha \in \kappa\}}$ works since the sets $\overline{\{p_\alpha : \alpha \in \kappa \setminus \beta\}}$ are clopen in F for all $\beta \in \kappa$, for the sequence $\{p_\alpha : \alpha \in \kappa\}$ is free, i.e.,

$$\overline{\{p_\alpha : \alpha \in \beta\}} \cap \overline{\{p_\alpha : \alpha \in \kappa \setminus \beta\}} = \emptyset$$

for $\beta \in \kappa$.

The proof of the main result is split into two cases according to whether X contains a closed set that maps continuously onto 2^κ or not. In the first case we get somewhat stronger results and now we turn to their discussion.

Theorem 1.1. *Let κ be any uncountable cardinal and $f : X \to 2^\kappa$ be an irreducible continuous map of the compact T_2 space X onto 2^κ . Then for every point $x \in X$ we have*

(i) there is a (relatively) discrete subspace $D \subset X \setminus \{x\}$ with $|D| = \kappa$ that "converges" to x in the strong sense that every neighbourhood of x contains all but countably many elements of D;

(ii) moreover, if $\mathrm{cf}(\kappa) > \omega$, then there is a free sequence $\{x_\alpha : \alpha \in \kappa\} \subset X \setminus \{x\}$ that converges to x.

Proof. We need several simple facts concerning irreducible maps that are probably well known, although we have not found them in the literature. For the sake of completeness we present them here with their proofs. First, the image of a regular closed set (in particular, of a clopen set) A under an irreducible closed map $f : X \to Z$ is regular closed. Indeed, if we had $y \in f(A) \setminus \overline{\operatorname{int} f(A)} \neq \emptyset$ then $A \setminus f^{-1}\left(\overline{\operatorname{int} f(A)}\right)$, hence

$$G = \operatorname{int} A \setminus f^{-1}\left(\overline{\operatorname{int} f(A)}\right) \neq \emptyset$$

because A is regular closed. Since f is irreducible we have $\operatorname{int} f(G) \neq \emptyset$, contradicting that both $\operatorname{int} f(G) \setminus \operatorname{int} f(A)$ and $f(G) \cap \operatorname{int} f(A) = \emptyset$.

Next we show that if f is as above and $A,\ B$ are regular closed sets in X, then

$$\operatorname{int} f(A \cap B) = \operatorname{int} f(A) \cap \operatorname{int} f(B).$$

Indeed, assume that $\operatorname{int} f(A \cap B) \subsetneqq \operatorname{int} f(A) \cap \operatorname{int} f(B)$. Then $C = (\operatorname{int} f(A) \cap \operatorname{int} f(B)) \setminus f(A \cap B) \neq \emptyset$, hence $D = \operatorname{int} A \cap f^{-1}(C) \neq \emptyset$, as well. But then $D \cap B = \emptyset$ and $f(D) \setminus C \setminus f(B)$; consequently, $f(X \setminus D) = Y$, contradicting that f is irreducible.

Let us now return to the proof of the theorem, where, for technical reasons, we first assume that X is also 0-dimensional.

For $S \subset \kappa$ a subset H of 2^κ is called S-determined if $z \in H$ implies $z' \in H$ whenever $z' \in 2^\kappa$ and $z' \restriction S = z \restriction S$. It is well known (cf. [15]) that every regular closed (or open) subset of 2^κ is S-determined for some countable $S \subset \kappa$. Moreover, it is obvious that if H is S-determined then so are $\overline{H}$ and $\operatorname{int} H$ as well.

For $\alpha \in \kappa$ and $i \in 2$ we shall put

$$U_{\alpha,i} = \{z \in 2^\kappa : z(\alpha) = i\}$$

so the $U_{\alpha,i}$ form the canonical subbase for the product topology on 2^κ.

Now, for any subset $S \subset \kappa$ we let $\mathcal{A}_S$ be the collection of all those clopen subsets A of X that contain x and whose f-image $f(A)$ is S-determined. According to our preliminary remarks, for every clopen neighbourhood A of x there is some countable set $S \subset \kappa$ such that $A \in \mathcal{A}_S$. Moreover, each collection $\mathcal{A}_S$ is closed under finite intersections. Finally, we set $\bigcap \mathcal{A}_S = F_S$. Then F_S is a closed set containing x, and clearly $S \subset T$ implies $F_S \subset F_T$.

Let us put $f(x) = y$ and for any $S \subset \kappa$ we set

$$\Phi_S = \{z \in 2^\kappa : z \upharpoonright S = y \upharpoonright S\}.$$

We claim that $f(F_S) = \Phi_S$ holds for every $S \subset \kappa$. Indeed, first note that

$$f(F_S) = f(\bigcap \mathcal{A}_S) = \bigcap \{f(A) : A \in \mathcal{A}_S\}$$

because X is compact and $\mathcal{A}_S$ is closed under finite intersections. But for all $A \in \mathcal{A}_S$ we have $y \in f(A)$ and $f(A)$ is S-determined, hence $\Phi_S \subset f(A)$, i.e., $\Phi_S \subset f(F_S)$. On the other hand, for each $\alpha \in S$ we clearly have $f^{-1}(U_{\alpha,y(\alpha)}) \in \mathcal{A}_S$. Consequently,

$$f(F_S) \subset \bigcap \{U_{\alpha,y(\alpha)} : \alpha \in S\} = \Phi_S$$

hence $f(F_S) = \Phi_S$.

Now for every $\alpha \in \kappa$ we let y^α be the point of 2^κ that agrees with y for $\beta \in \kappa \setminus \{\alpha\}$ but disagrees with it at α, i.e.,

$$y^\alpha(\beta) = \begin{cases} y(\beta) & \text{if } \beta \neq \alpha \\ 1 - y(\alpha) & \text{if } \beta = \alpha \end{cases} .$$

Since $\Phi_{\kappa \setminus \{\alpha\}} = \{y, y_\alpha$, we may pick the points $x^\alpha \in F_{\kappa \setminus \{\alpha\}}$ such that $f(x^\alpha) = y^\alpha$ for all $\alpha \in \kappa$ and claim that the set $D = \{x^\alpha : \alpha \in \kappa\}$ satisfies (i).

That D is discrete follows immediately from the fact that the set $\{y^\alpha : \alpha \in \kappa\}$ is discrete. Next, if A is any clopen neighbourhood of x then there is a countable set $S \subset \kappa$ with $A \in \mathcal{A} : S$. But then for every $\alpha \in \kappa \setminus S$, we have

$$x^\alpha \in F_{\kappa \setminus \{\alpha\}} \subset F_S \subset A,$$

hence (i) holds because X is 0-dimensional.

As for (ii), let us first define the points $y_\alpha \in 2^\kappa$ for $\alpha \in \kappa$ by the stipulation

$$y_\alpha(\beta) = \begin{cases} y(\beta) & \text{if } \beta \leq \alpha \\ 1 - y(\beta) & \text{if } \alpha < \beta < \kappa \end{cases} .$$

Clearly $\{y_\alpha : \alpha \in \kappa\}$ is a free sequence in 2^κ. Moreover, $y_\alpha \in \Phi_{\alpha+1}$ for each $\alpha \in \kappa$. Since $f(F_{\alpha+1}) = \Phi_{\alpha+1}$, we can choose points $x_\alpha \in F_{\alpha+1}$ for $\alpha \in \kappa$ such that $f(x_\alpha) = y_\alpha$.

Now $\{x_\alpha : \alpha \in \kappa\}$ is a free sequence in X: if $z \in \overline{\{x_\alpha : \alpha \in \beta\}}$ then $f(z) \in \overline{\{y_\alpha : \alpha \in \beta\}}$ by the continuity of f; hence $f(z) \notin \overline{\{y_\alpha : \alpha \in \beta \setminus \beta\}}$ because $\{y_\alpha\}$ is free; thus $z \notin \overline{\{x_\alpha : \alpha \in \beta \setminus \beta\}}$.

Finally to see that $\{x_\alpha : \alpha \in \kappa\}$ converges to x it clearly suffices to show that $\bigcap\{F_{\alpha+1} : \alpha \in \kappa\} = \{x\}$ since the $F_{\alpha+1}$ are decreasing and $x_\alpha \in F_{\alpha+1} \setminus F_{\alpha+2}$. But for every clopen neighbourhood A of x there is a countable $S \subset \kappa$ with $A \in \mathcal{A}_S$, hence by $\operatorname{cf}(\kappa) > \omega$ there is an $\alpha \in \kappa$ such that $S \subset \alpha$. Consequently, $F_{\alpha+1} \subset F_\alpha \subset F_S \subset A$.

Thus, using again that X is 0-dimensional, (ii) has been established.

Now let us consider the general case with an arbitrary compact T_2 space X. By Alexandrov's well-known theorem (see, e.g., [15, 3.2.2]), there is a 0- dimensional compact T_2 space Z admitting an irreducible map $g : Z \to X$ onto X. Then the composition $h = f \circ g : Z \to 2^\kappa$ is also irreducible, hence the above considerations can be applied to h and a fixed point $z \in Z$ with $g(z) = x$. This gives us points $\{z^\alpha : \alpha \in \kappa\}$ with $h(z^\alpha) = y^\alpha$ and $\{z_\alpha : \alpha \in \kappa$ with $h(z_\alpha) = y_\alpha$. Now it is straightforward to check that the set $D = \{x^\alpha = g(z^\alpha) : \alpha \in \kappa\}$ and the free sequence $\{x_\alpha = g(z_\alpha) : \alpha \in \kappa\}$ are as required by (i) and (ii), respectively. $\square$

Now we turn to the formulation of our main result.

Theorem 1.2. *Let κ be an uncountable regular cardinal. If a compact T_2 space X contains a free sequence $S = \{x_\alpha : \alpha \in \kappa\}$ of length κ then it also contains one that is convergent.*

Proof. Let us first note that the closure $\overline{S}$ of S can be mapped continuously onto the space $\kappa + 1$ taken with its usual order topology. Indeed, if for any limit ordinal $\lambda < \kappa$ we set

$$A_\lambda = \bigcap \overline{\{x_\beta : \beta \in \lambda \setminus \alpha\}},$$

or equivalently, since S is free,

$$A_\lambda = \{x \in X : \lambda = \min\{\alpha : x \in \{x_\beta : \beta \in \alpha\}'\}\},$$

then the map $f : \overline{S} \to \kappa + 1$ (defined below) is clearly continuous:

$$f(x) = \begin{cases} n & x = x_n,\ n < \omega \\ \alpha + 1 & x = x_\alpha,\ \alpha \in \kappa \setminus \omega \\ \lambda & x \in A_\lambda,\ \lambda \leq \kappa \text{ limit} \end{cases}$$

Thus we may assume without any loss of generality that X admits an irreducible map f onto $\kappa+1$. Moreover, using 1.1 (ii), we may also assume that no closed subspace of X can be mapped onto 2^κ. This, however, by Sapirovskil's celebrated result (cf. [25, 3.18]) implies that every nonempty closed subset F of X has a point p with $\pi\chi(p, F) < \kappa$.

For every ordinal $\alpha < \kappa$ let us define $Z_\alpha = f^{-1}((\kappa + 1) \setminus \alpha)$. In particular, then the set $Z_\kappa = f^{-1}(\{\kappa\})$ is nowhere dense in X because f is irreducible.

Let $p \in Z_\kappa$ be any point having π-character $\pi\chi(p, Z_\kappa) = \pi$ less than κ in Z_κ. We are going to show that there is a free sequence of length κ converging to p. This will clearly establish our result.

First we claim that for every open subset G of X if $G \cap Z_\kappa \neq \emptyset$ then $f(G) \cap \kappa$ is cofinal in κ. Indeed, otherwise we had an $\alpha \in \kappa$ such that $G \setminus Z_\kappa \subset f^{-l}[\alpha]$, hence as Z_κ is nowhere dense we had

$$G \subset \overline{G \setminus Z_\kappa} \subset \overline{f^{-1}[\alpha]} \subset f^{-1}[\alpha + 1],$$

contradicting that $G \cap Z_\kappa \neq \emptyset$.

Using $\pi\chi(p, Z_\kappa) = \pi(< \kappa)$, we may choose open sets $\{G_\gamma : \gamma \in \pi\}$ in X such that $\{G_\gamma \cap Z_\kappa : \gamma \in \pi\}$ forms a local π-base at p in Z_κ. Let us pick a point $q_\gamma \in G_\gamma \cap Z_\kappa$ for each $\gamma \in \pi$ and then choose the open set B_γ in X such that $q_\gamma \in B_\gamma \subset \overline{B_\gamma} \subset G_\gamma$. By the above claim, then

$$C_\gamma = \kappa \cap \overline{f(B_\gamma)}$$

is a closed unbounded subset of κ, hence by $\pi < \kappa$ so is $C = \bigcap\{C_\gamma : \gamma \in \pi\}$. (Here we use the regularity of κ.)

Now, for every $\alpha \in \kappa$ we let $\mathcal{V}_\alpha$ be the collection of all those open neighbourhoods V of p for which $\overline{B_\gamma} \cap Z_\kappa \subset V$ implies $\overline{B_\gamma} \cap Z_\alpha$ for all $\gamma \in \pi$. Clearly $V_1, V_2 \in \mathcal{V}_\alpha$ implies $V_1 \cap V_2 \in \mathcal{V}_\kappa$. Moreover $\mathcal{V}_\alpha \subset \mathcal{V}_\beta$ if $\alpha < \beta$. Let us put

$$F_\alpha = \bigcap\{\overline{V} : V \in \mathcal{V}_\alpha\}$$

Then we have $F_\alpha \supset F_\beta$ for $\alpha < \beta$ and

$$f(F_\alpha) = \bigcap\{f(\overline{V}) : V \in \mathcal{V}_\alpha\}$$

by our above remarks.

But for every $V \in \mathcal{V}_\alpha$, there is a $\gamma \in \pi$ such that $\overline{B_\gamma} \cap Z_\alpha \subset G_\gamma \cap Z_\alpha \subset V$. Consequently,

$$C \setminus \alpha \subset C_\gamma \setminus \alpha \subset f(\overline{B_\gamma} \cap Z_\alpha) \subset f(V).$$

Thus we obtain $C \setminus \alpha \subset f(F_\alpha)$ and, in particular, $F_\alpha \neq \{p\}$.

Next we show that for every neighbourhood V of p there is some $\alpha \in \kappa$ with $V \in \mathcal{V}_\alpha$. Indeed, if $\overline{B_\gamma} \cap Z_\kappa \subset V$ then

$$Z_\kappa = \bigcap\{Z_\alpha : \alpha \in \kappa\},$$

and $Z_\alpha \subset Z_\beta$ for $\beta < \alpha$ imply that, as X is compact, there is some $\alpha_\gamma \in \kappa$ such that $\overline{B_\gamma} \cap Z_{\alpha_\gamma} \subset V$. Thus if a $\alpha \in \kappa$ is chosen with $\alpha_\gamma \le \alpha$ for each $\gamma \in \pi$, and this is possible since κ is regular and $\pi < \kappa$, then $V \in \mathcal{V}_\alpha$ indeed.

Putting all this together we obtain from the sets F_α a strictly decreasing sequence of closed sets whose intersection is $\{p\}$. By suitably thinning out, we may assume that $F_{\alpha+1} \subsetneqq F_\alpha$ for all $\alpha \in \kappa$; hence if $p_\alpha \in F_{\alpha+1} \setminus F_\alpha$ then the sequence $\{p_\alpha : \alpha \in \kappa\}$ converges to p. Finally, by passing to a subsequence we may assume that if $\alpha < \beta$ then $f(p_alpha) < f(p_\beta)$, and then the subsequence $\{p_{\alpha+1} : \alpha \in \kappa\}$ is also free because its f-image $\{f(p_{\alpha+1}) : \alpha \in \kappa\}$ is.

□

1.1.2 Applications

In this section we are going to present several results that are more or less immediate consequences of our main theorem and other known results. To start with, we consider a result which, for $\kappa = \omega$, was formulated in the introduction.

Corollary 1.3. *If $2^\kappa = \kappa^+$ and X is a compact T_2 space of character $\chi(X) > \kappa$, then X has a closed subset F with a point p such that $\chi(p, F) = \kappa^+$, hence a convergent κ^+-sequence as well.*

Proof. If $t(X) > \kappa$ then this is immediate from 1.2, even without assuming $2^\kappa = \kappa^+$. (Remember that a convergent free κ^+-sequence yields F and p as required.) Now if $t(X) \le \kappa$ then for any point $p \in X$ with $\chi(p, X) \ge \kappa^+$ we may apply 6.14(b) of [25] with $\lambda = \kappa^+$ to obtain a set $Y \subset X$ with $|Y| \le \kappa^+$ such that

$$\chi(p, Y) = \chi(p, \overline{Y}) \ge \kappa^+.$$

If $|Y| \le \kappa$ then we have $w(\overline{Y}) < 2^{d(\overline{Y})} \le 2^\kappa = \kappa^+$, hence we must have $\chi(p, \overline{Y}) = \kappa^+$.

If, on the other hand, $|Y| = \kappa^+$ then let us write $Y = \bigcup\{Y_\alpha : \alpha \in \kappa^+\}$ with $|Y_\alpha| = \kappa$ and $Y_\alpha \subset Y_\beta$ for $\alpha < \beta < \kappa^+$. Since $t(X) \le \kappa$, we also have $\overline{Y} = \bigcup\{\overline{Y}_\alpha : \alpha \in \kappa\}$. But then for each $\alpha \in \kappa^+$ we have $w(\overline{Y}_\alpha) < \kappa^+$ as above, hence (cf. [25]) we have

$$w(\overline{Y}) = nw(\overline{Y}) \le \sum\{nw(\overline{Y}_\alpha) : \alpha \in \kappa^+\} \le \kappa^+,$$

and thus again $\chi(p, \overline{Y}) = \kappa^+$. □

For the case $\kappa = \omega$ we can prove the following partial strengthening of 1.3.

Corollary 1.4. *If V satisfies CH and W is an extension of V obtained by adding some Cohen reals to V, then in W every nonfirst countable compact T_2 space X has a convergent ω_1-sequence. (Note that $W = V$ is permitted here.)*

Proof. As above, using our main theorem we can assume that $t(X) = \omega$, hence, by 5.8(b) of [12], every point $p \in X$ with $\chi(p, X) > \omega$ is the limit of an ω_1-sequence. $\square$

In [24] Hušek introduced the notion of small diagonal and asked whether CH implies that every compact T_2 space with a small diagonal is metrizable. (Recall that X is said to have small diagonal if for every uncountable set $H \subset X^2 \setminus \Delta$ there is a neighbourhood U of Δ with $H \setminus U$ also uncountable. In other words, this means that no ω_1-sequence from $X^2 \setminus \Delta$ can converge to Δ.) We can now establish the following stronger result.

Corollary 1.5. *If W is as in 1.4 then in W every compact T_2 space X with a small diagonal is metrizable.*

Proof. By 5.8(c) of [12] this is so if $t(X) \leq \omega$. If on the other hand $t(X) > \omega$, then by 1.2 there is a convergent ω_1-sequence $\{p_\alpha : \alpha \in \omega_1\}$ in X consisting of distinct points. But if $\langle p_\alpha \rangle$ converges to p then $\{(p_\alpha, p_{\alpha+1}) : \alpha \in \omega_1\} \subset X^2 \setminus \Delta$ converges to $(p, p) \in \Delta$, which shows that X cannot have small diagonal. $\square$

Let us now recall from the introduction Hu'sek's other conjecture saying that every compact T_2 space contains a convergent ω- or ω_1-sequence. Our next result shows that this is implied by a conjecture formulated by István Juhász in [28], namely, that every compact T_2 space of countable tightness has a point of character $\leq \omega_1$. In fact, a little more is true.

Corollary 1.6. *If every compact T_2 space of countable tightness has a point of character $\leq \omega_1$ then in every infinite compact T_2 space X there is a closed set F with a point $p \in F$ such that $\chi(p, F) = \omega$ or $\chi(p, F) = \omega_1$.*

Proof. If $t(X) > \omega$ then we can simply apply 1.2. Next if $t(X) = \omega$ and X is also scattered, then X contains the one-point compactification of an infinite set, hence a closed set F with a point p satisfying $\chi(p, F) = \omega$ or $\chi(p, F) = \omega_1$. Finally if $t(X) = \omega$ and X is not scattered, then X contains a closed set F that is dense in itself and, by our assumption, for some point $p \in F$ we have $\chi(p, F) = \omega$ or $\chi(p, F) = \omega_1$. $\square$

In [29] it was shown that the assumption of 1.6 is valid in a generic extension obtained by adding ω_1 Cohen reals to an arbitrary ground model. Moreover, Dow proved in [13] that this assumption, hence by 1.6 also Hušek's conjecture, is valid under PFA. Next we give a very simple and quick proof of a result form [30]. In order to formulate this we recall that a space X is said to omit a cardinal λ if $|X| > \lambda$ but X does not contain a closed subset of size λ.

Corollary 1.7. *If $2^{2^\kappa} = \kappa^{++}$ then no compact space X may omit both κ^+ and κ^{++}.*

Proof. Let $|X| > \kappa^{++}$. If $t(X) \leq \kappa$ then for every $Y \subset X$ with $|Y| = \kappa^{++}$ we have $|\overline{Y}| = \kappa^{++}$ because

$$\overline{Y} = \bigcup \{\overline{Z} : Z \subset Y \text{ and } |Z| \leq \kappa\}.$$

If, however, $t(X) > \kappa$, then by 1.2 there is a convergent κ^+-sequence $S = \{x_\alpha : \alpha \in \kappa^+\}$ in X. Then we have

$$\overline{S} = \bigcup \overline{\{x_\beta : \beta \in \alpha\}} : \alpha \in \kappa^+\} \cup \{x\},$$

where x is the limit of S, hence clearly either $|\overline{S}| = \kappa^+$ or $|\overline{S}| = \kappa^{++}$. □

Note that the same proof actually yields in ZFC that no compact T_2 space can omit every cardinal in the interval $[\kappa^+, 2^\kappa]$.

We finish with a result that is not connected with our main theorem, still we thought to add it here because it is directly related to Hušek's conjecture. This result uses the set theoretic principle $\clubsuit$ (club), introduced by Ostaszewski in [49], saying that with every limit ordinal $\lambda \in \omega_1$ we can associate an ω-type subset $S_\lambda \subset \lambda$ with $\bigcup S_\lambda = \lambda$ such that for every uncountable set $H \subset \omega_1$ there is a λ with $S_\lambda \subset H$. $\clubsuit$ is known to be consistent with the continuum being arbitrarily large.

Theorem 1.8. *Assume $\clubsuit$ and let X be a countably compact infinite T_2 space that contains no nontrivial convergent ω-sequence. Then there is an ω_1-size supspace $Y = \{x_\alpha : \alpha \in \omega_1\}$ of X such that in Y every (relatively) open set is either countable or co-countable*

Proof. Let $\langle S_\lambda : \lambda \in L_1 \rangle$ be a $\clubsuit$-sequence with L_1 denoting the set of limit ordinals in ω_1. By transfinite induction on a $\alpha \in \omega_1$, we are going to define infinite closed sets $F_\alpha \subset X$ and points $x_\alpha \in F_\alpha$ as follows.

Let us put $F_0 = X$, and if the infinite closed set F_α has been defined then we pick $x_\alpha \in F_\alpha$ and $F_{\alpha+1} \subset F_\alpha \setminus \{x_\alpha\}$ such that $F_{\alpha+1}$ be closed and infinite.

If $\lambda \in L_1$ and the F_α have been defined for all $\alpha \in \lambda$ then $\langle F_\alpha : \alpha \in \lambda \rangle$ is a decreasing ω-sequence of closed sets, hence the set $F_\lambda = \{x_\alpha : \alpha \in S_\lambda\}'$ of all accumulation points of the sequence $\{x_\alpha : \alpha \in S_\lambda\}$ is nonempty by countable compactness. Moreover, it is infinite because X contains no convergent ω-sequence. This completes the induction.

It is immediate from our construction that for every $\lambda \in L_1$ we have

$$\{X_\alpha : \alpha \in \omega_1 \setminus \lambda\} \subset \overline{\{x_\beta : \beta \in S_\lambda\}}.$$

Consequently, since for every uncountable set $Z \subset Y$ there is a $\lambda \in L_1$ with $\{x_beta : \beta \in S_\lambda\} \subset Z$, any uncountable closed (or open) subset of Y must be co-countable. The proof is thus completed. □

Let us first note that Y is an S-space if it is also T_3. Moreover it is obvious that Y can have at most one complete accumulation point. Consequently, we have

Corollary 1.9. *If $\clubsuit$ holds then every infinite initially ω_1-compact T_2-space has a nontrivial convergent ω or ω_1-sequence.*

1.2 Discrete subspaces of countably tight compacta

In this section *compactum* will mean an infinite compact Hausdorff space. The cardinal function $g(X)$ denotes the supremum of cardinalities of closures of discrete subspaces of a space X; this was introduced in [2]. It was also asked there if $|X| = g(X)$ holds for every compactum X. Much later, at a meeting in Budapest in 2003, Archangelskiĭ also asked the following (slightly) stronger question: Does every compactum X contain a discrete subspace D with $|\overline{D}| = |X|$? (If, following the notation of [25], we denote by $\widehat{g}(X)$ the smallest cardinal κ such that for every discrete $D \subset X$ we have $|\overline{D}| < \kappa$ then this latter question asks if $\widehat{g}(X) = |X|^+$ for every compactum X.)

It was noted by K. Kunen that the answer to the second question is "no" if there is an inaccessible cardinal, because for every non-weakly compact inaccessible cardinal λ there is even an ordered compactum X with $\widehat{g}(X) = |X| = \lambda$. Moreover A. Dow in [11] gave, with the help of a forcing argument, a consistent counterexample to the first question. It remains open if there are ZFC counterexamples to either question.

On the other hand, A. Dow proved in [11] the following positive ZFC results for *countably tight* compacta.

Proposition 1.10. *(A. Dow, see in [11]) Let X be a countably tight compactum. Then*

(i) $|X| \le g(X)^\omega$;

(ii) if $|X| \le \aleph_\omega$ then $\widehat{g}(X) = |X|^+$.

A. Dow also formulated the following conjecture in [11]: For every such X we do have $\widehat{g}(X) = |X|^+$. In what follows we shall confirm this conjecture under a slight weakening of GCH, namely the assumption that for any cardinal κ the power 2^κ is a finite successor of κ; or equivalently: every limit cardinal is strong limit.

It is trivial that for any space X we have $s(X) \le g(X)$ $\big($and $\widehat{s}(X) \le \widehat{g}(X)\big)$, however the following stronger inequality, which we shall use later, is also true: $h(X) \le g(X)$ $\big($and $\widehat{h}(X) \le \widehat{g}(X)\big)$. This is so because for any right separated (or equivalently: scattered) space Y the set $I(Y)$ of all isolated points of Y, that is clearly discrete, is dense in Y.

We start by giving a new, simpler proof of 1.10 (i).

Proof of 1.10 (i). Let X be a countably tight compactum and M be a countably closed elementary submodel of $H(\vartheta)$ for a large enough regular cardinal ϑ with $X \in M$, $g(X) \subset M$ and $|M| = g(X)^\omega$.

First we show that $X \cap M$ is compact. Indeed, if D is any countable discrete subspace of $X \cap M$ then $D \in M$ because M is countably closed, hence $\overline{D} \in M$ and so $g(X) \subset M$ implies $\overline{D} \subset M$. As X is countably tight, it follows that the X-closure of any discrete subspace of $X \cap M$ is contained in $X \cap M$, hence the subspace $X \cap M$ has the property that the closure of any discrete subspace of it is compact. But then it is well-known (see e.g. [63]) that $X \cap M$ is compact.

Next we show that $X = X \cap M$; this will clearly complete the proof. Assume, indirectly, that $x \in X \setminus M$. By 2.10 (a) of [25] we have $\psi(X) \le h(X) \le g(X)$, hence again by $g(X) \subset M$ there is for each point $y \in X \cap M$ a local ψ-base $\mathcal{U}_y \subset M$. Pick for each $y \in X \cap M$ an element $U_y \in \mathcal{U}_y \subset M$ with $x \notin U_y$. Since $X \cap M$ is compact, from the open cover $\{U_y : y \in X \cap M\}$ of $X \cap M$ we may select a finite subcover $\mathcal{U} = \{U_{y_1}, \dots, U_{y_n}\}$. But then $\mathcal{U} \in M$ and $x \notin \cup\,\mathcal{U}$, a contradiction.

□

Before giving our next result we shall prove two lemmas that may turn out to have independent interest. Both lemmas say something about T_3 spaces.

Lemma 1.11. *Let X be a countably compact T_3 space and λ be a strong limit cardinal of countable cofinality such that $\pi\chi(X) < \lambda$, moreover $|G| \ge \lambda$ for each (non-empty) open subset G of X. Then we actually have $\widehat{g}(X) > \lambda^\omega$, i. e. X has a discrete subspace D with $|\overline{D}| \ge \lambda^\omega$.*

Proof. By a well-known result of Šapirovskiĭ (see [25], 2.37), for every open G in X we have $c(G) \ge \lambda$, because otherwise we had $w(G) \le \pi\chi(G)^{c(G)} < \lambda$, hence also $|G| \le 2^{w(G)} < \lambda$ as λ is strong limit. But then by a result of Erdős and Tarski (see [25], 4.1) we also have $\widehat{c}(G) > \lambda$, i.e. G contains λ many pairwise disjoint open subsets, because λ is singular.

Now given an open set $G \subset X$ and a point $x \in G$ let us fix an open neighbourhood $V(G, x) = V$ of x such that $\overline{V} \subsetneqq G$. By the above we may also fix a family $\mathcal{U}(G, x)$ of open subsets of $G \setminus \overline{V}$ with pairwise disjoint closures and with $|\mathcal{U}(G, x)| = \lambda$. This is possible because X is T_3.

We next define for all finite sequences $s \in \lambda^{<\omega}$ open sets G_s and points $x_s \in G_s$ by recursion on $|s|$ as follows. To start with, we set $G_\emptyset = X$ and pick $x_\emptyset \in G_\emptyset$ arbitrarily. Once $x_s \in G_s$ are given, then the sets $\{G_{s\frown\alpha} : \alpha \in \lambda\}$ are chosen so as to enumerate $\mathcal{U}(G_s, x_s)$ in a one-to-one manner. Then the points $x_{s\frown\alpha} \in G_{s\frown\alpha}$ are chosen arbitrarily.

Let us set $D = \{x_s : s \in \lambda^{<\omega}\}$. Then D is discrete because, by the construction, we clearly have $D \cap V(G_s, x_s) = \{x_s\}$ for each $s \in \lambda^{<\omega}$. For every ω-sequence $f \in \lambda^\omega$, using the countable compactness of X, we may choose an accumulation point x_f of the set $\{x_{f\restriction n} : n \in \omega\}$. Clearly, we have $x_f \in \overline{G}_{f\restriction n}$ for all $n \in \omega$. For $f_1, f_2 \in \lambda^\omega$ with $f_1 \ne f_2$ then $x_{f_1} \ne x_{f_2}$ because if n is minimal with $f_1(n) \ne f_2(n)$ then

$$\overline{G}_{f_1\restriction n+1} \cap \overline{G}_{f_2\restriction n+1} = \emptyset\,.$$

This shows that $|\overline{D}| \ge \lambda^\omega$, hence $\widehat{g}(X) > \lambda^\omega$, and the proof is completed. □

As we shall see below, using 1.10 (i) and 1.11 one can already prove $|X| = g(X)$ for countably tight compacta, under the assumption that $2^\kappa < \kappa^{+\omega}$ for all κ. However, to get the stronger result $\widehat{g}(X) = |X|^+$ for the case when $|X|$ is an inaccessible cardinal, we shall need the following lemma.

Lemma 1.12. *Let κ, λ, μ be cardinals with $\kappa \to (\lambda, \mu)^2$ and X be a T_3 space in which there is a left-separated subspace of cardinality κ (i.e. $\widehat{z}(X) > \kappa$). Then either X has a discrete subspace of size λ (i.e. $\widehat{s}(X) > \lambda$) or there is in X a free sequence of length μ (i.e. $\widehat{F}(X) > \mu$).*

Proof. Let $Y \subset X$ be left-separated by the well-ordering $\prec$ in order-type κ. Then for each $y \in Y$ we can fix a *closed* neighbourhood N_y such that $\overline{\{z \in Y : z \prec y\}} \cap N_y = \emptyset$.

Let us then define the coloring $c\colon [Y]^2 \to 2$ by the following stipulation: for $y, z \in Y$ with $y \prec z$ we have $c(\{y, z\}) = 0$ if and only if $z \notin N_y$. By $\kappa \to (\lambda, \mu)^2$ then we either have a 0-homogeneous subset of Y of size λ or a 1-homogeneous subset of size μ. But clearly, if $S \subset Y$ is 0-homogeneous then S is discrete because then $S \cap N_y = \{y\}$ for all $y \in S$, and if S is 1-homogeneous then S is free in X because in this case for any $y \in S$ we have $\overline{\{z \in S : z \prec y\}} \cap N_y = \emptyset$ and $\overline{\{x \in S : y \preceq x\}} \subset N_y$. □

We are now ready to present our result.

Theorem 1.13. *Assume that $2^\kappa < \kappa^{+\omega}$ holds for all cardinals κ. Then for every countably tight compactum X we have $\widehat{g}(X) = |X|^+$, i.e. there is a discrete subspace $D \subset X$ with $|\overline{D}| = |X|$.*

Proof. Let us first consider the case in which $|X|$ is a limit cardinal, hence by our assumption a strong limit cardinal. If $|X|$ is also singular then by a result Hajnal and Juhász (see [25], 4.2) we have $\widehat{s}(X) = |X|^+$, i.e. there is even a discrete $D \subset X$ with $|D| = |X|$. If, on the other hand, $|X|$ is regular, hence inaccessible, then we have $d(X) = |X|$ (and so $\widehat{z}(X) = |X|^+$) because $|X|$ is strong limit, moreover by a well-known result of partition calculus (see e.g. [16]) we have $|X| \to (|X|, \omega_1)^2$. But obviously in a countably tight compactum there is no free sequence of uncountable length, hence applying lemma 1.12 we again obtain a discrete $D \subset X$ with $|D| = |X|$.

So next we may assume that $|X|$ is a successor cardinal: $|X| = \lambda^{+n}$ where λ is limit and $n \in \omega \setminus \{0\}$.

Now, if $|X|$ is ω-inaccessible, i.e. for any $\mu < |X|$ we have $\mu^\omega < |X|$, then by 1.1 (i) we must have $g(X) = |X|$. But then, $g(X)$ being a successor cardinal, we must also have $\widehat{g}(X) = |X|^+$.

If, on the other hand, $|X|$ is ω-accessible then our assumptions clearly imply $cf(\lambda) = \omega$ and $\lambda < |X| \le \lambda^\omega$. If $\widehat{h}(X) = |X|^+$ then by $\widehat{h}(X) \le \widehat{g}(X)$ we also have $\widehat{g}(X) = |X|^+$, hence we may assume that $\widehat{h}(X) \le |X|$. Note that, as we have seen above, X does have a discrete subspace of cardinality λ because λ is singular strong limit, so trivially we also have $\widehat{h}(X) > \lambda$, consequently $\widehat{h}(X)$ must be a successor cardinal, say μ^+ where $\lambda \le \mu < |X|$.

Now let G be the union of all open subsets of X of size at most μ. Then we have $|G| \le \mu$ as well since otherwise we could easily produce in X a right separated subset of cardinality μ^+ contradicting $\widehat{h}(X) = \mu^+$. But then, in the closed subspace $X \setminus G$

of X, clearly each non-empty relatively open set has size $> \mu \geq \lambda$. By a result of Šapirovskiĭ(see [25], 3.14) every countably tight compactum has countable π-character, hence lemma 1.11 can be applied to the space $X \setminus G$ and the cardinal λ. Consequently, we have $\widehat{g}(X) \geq \widehat{g}(X \setminus G) > \lambda^\omega \geq |X|$, and so we may again conclude that $\widehat{g}(X) = |X|^+$. □

1.3 Two improvements on Tkaèenko's addition theorem

We start this section by recalling a few well-known definitions and introducing some related notation.A space X is said to be a D-space if for any neighbourhood assignment ϕ defined on X there is a closed discrete set $D \subset X$ such that $\cup\{\phi(x) : x \in D\} = X$. For any space X we set

- $D(X) = \min\{|\mathcal{A}| : X = \cup\mathcal{A}$ and A is a D-space for each $A \in \mathcal{A}\}$.
- $\mathrm{ls}(X) = \min\{|\mathcal{A}| : X = \cup\mathcal{A}$ and A is left-separated for each $A \in \mathcal{A}\}$.

(Note that both $D(X)$ and $\mathrm{ls}(X)$ can be finite.)
It was shown in [70] that left-separated spaces are D-spaces, hence we have $D(X) \leq \mathrm{ls}(X)$ for any X.
In [62], M. Tkačenko proved the following remarkable result: *If X is a countably compact T_3-space with $\mathrm{ls}(X) \leq \omega$ then*

(i) X is compact,

(ii) X is scattered,

(iii) X is sequential.

It is easy to see that if in a scattered compact T_2-space any countably compact subspace is compact then it is sequential, hence (iii) immediately follows from (i) and (ii), although this is not how (iii) was proved in [62].

The aim of this section is to improve (i) and (ii) as follows:

(A) *Any countably compact space X with $D(X) \leq \omega$ is compact.*

(B) *If X is compact T_2 with $\mathrm{ls}(X) < N(\mathbb{R})$ then X is scattered.*

Here $N(\mathbb{R})$ denotes the Novák number of the real line $\mathbb{R}$, i.e. the covering number $\mathrm{cov}(\mathcal{M})$ of the ideal $\mathcal{M}$ of all meager subsets of $\mathbb{R}$.

If X is any crowded (i.e. dense-in-itself) space and $Y \subset X$ then we denote by $N(Y, X)$ the relative Novák number of Y in X, that is the smallest number of nowhere dense subsets of X needed to cover Y. In particular, $N(X) = N(X, X)$ is the Novák number of X.

We should also mention that a weaker version of statement (A), in which $D(X) < \omega$ is assumed instead of $D(X) \le \omega$, has been established in [20].

Similarly as in [62], we can actually prove the following higher-cardinal generalization of statement (A).

Theorem 1.14. *Let κ be any infinite cardinal and X be initially κ-compact with $D(X) \le \kappa$. Then X is actually compact.*

The proof of Theorem 1.14 is based on the following lemma that may have some independent interest in itself.

Lemma 1.15. *Let X be any space and $Y \subset X$ its D subspace. If ρ is a regular cardinal such that X has no closed discrete subset of size ρ (i.e. $\hat{e}(X) \le \rho$), moreover $\mathcal{U} = \{U_\alpha : \alpha \in \rho\}$ is a strictly increasing open cover of X then there is a closed set $Z \subset X$ such that $Z \cap Y = \emptyset$ and $Z \not\subset U_\alpha$ for all $\alpha \in \rho$.*

Proof. If there is an $\alpha \in \rho$ with $Y \subset U_\alpha$ then $Z = X - U_\alpha$ is clearly as required. So assume from here on that $Y \not\subset U_\alpha$ for all $\alpha \in \rho$.

For every point $y \in Y$ let $\alpha(y)$ be the *minimal* ordinal α such that $y \in U_\alpha$ and then consider the neighbourhood assignment ϕ on Y defined by

$$\phi(y) = U_{\alpha(y)}.$$

Since Y is a D-space there is a set $E \subset Y$, closed and discrete in Y, such that $Y \subset \phi[E]$. We claim that $Z = E'$, the derived set of E, is now as required.

Indeed, Z is closed in X and $Z \cap Y = \emptyset$ as E has no limit point within Y. It remains to show that $Z \not\subset U_\alpha$ for all $\alpha \in \rho$. Assume, indirectly, that $Z \subset U_\alpha$ for some $\alpha \in \rho$. Note first that for any point $y \in Y \cap U_\alpha$ we have $\alpha(y) \le \alpha$, consequently $\phi[E \cap U_\alpha] \subset U_\alpha$. On the other hand, $Z = E' \subset U_\alpha$ implies that $E - U_\alpha$ is closed discrete in X, hence $|E - U_\alpha| < \rho$ by our assumption. But then

$$\beta = \sup\{\alpha(y) : y \in E - U_\alpha\} < \rho$$

because ρ is regular, consequently we have

$$Y \subset \phi[E] = \phi[E \cap U_\alpha] \cup \phi[E - U_\alpha] \subset U_\alpha \cup U_\beta = U_{\max\{\alpha,\beta\}},$$

contradicting that no member of $\mathcal{U}$ covers Y. □

Now, we can turn to the proof of our theorem.

Proof. It suffices to prove that for no regular cardinal ρ is there a strictly increasing open cover of X of the form $\mathcal{U} = \{U_\alpha : \alpha \in \rho\}$. For $\rho \le \kappa$ this is clear, for X is initially

κ-compact. So assume now that $\rho > \kappa$, and assume indirectly that $\mathcal{U} = \{U_\alpha : \alpha \in \rho\}$ is a strictly increasing open cover of X. Note also that X has no closed discrete subset of size $\rho > \kappa$ because X is initially κ-compact.

By $D(X) \leq \kappa$ we have $X = \cup\{Y_\nu : \nu \in \kappa\}$, where Y_ν is a D subspace of X for each $\nu \in \kappa$. Using lemma 1.15 then we may define by a straightforward transfinite recursion on $\nu \in \kappa$ closed sets $Z_\nu \subset X$ such that for each $\nu \in \kappa$ we have $Z_\nu \cap Y_\nu = \emptyset$, $Z_\nu \not\subset U_\alpha$ for all $\alpha \in \rho$, moreover $\nu_1 < \nu_2$ implies $Z_{\nu_1} \supset Z_{\nu_2}$. In this we make use of the fact that if $\nu < \kappa$ and $\{Z_\eta : \eta \in \nu\}$ is a decreasing sequence of closed sets in X such that $\cap\{Z_\eta : \eta \in \nu\} \subset U$ for some open $U \subset X$ then there is an $\eta \in \nu$ with $Z_\eta \subset U$ as well, using again the initial κ-compactness of X.

But then, applying once more that X is initially κ-compact, we conclude that

$$\cap\{Z_\nu : \nu \in \kappa\} \neq \emptyset,$$

contradicting that $X = \cup\{Y_\nu : \nu \in \kappa\}$. $\square$

It should be noted that in the above result no separation axiom is needed. This is in contrast with Tkačenko's result from [62].

Let us now turn to our second statement (B). Again, we need to first give a preparatory result. For this we recall the cardinal function $\delta(X)$ that was introduced in [72]:

$$\delta(X) = \sup\{d(S) : \ S \text{ is dense in } X\}.$$

Let us note here that if X is a compact T_2-space then $\delta(X) = \pi(X)$, as was shown in [33].

Lemma 1.16. *Assume that X is an arbitrary crowded topological space and $Y \subset X$ is its left-separated subspace. Then we have*

$$N(Y, X) \leq \delta(X),$$

consequently

$$N(X) \leq \mathrm{ls}(X) \cdot \delta(X).$$

Proof. We shall prove $N(Y, X) \leq \delta(X)$ by transfinite induction on the order type of the well-ordering that left-separates Y. So assume that $\prec$ is a left-separating well-ordering of Y such that if Z is any proper initial segment of Y, w.r.t. $\prec$, then $N(Z, X) \leq \delta(X)$.

Let G be the union of all those open sets U in X for which Y (or more precisely: $U \cap Y$) is dense in U. Clearly, then $Y \setminus G$ is nowhere dense in X and $Y \cap G$ is dense in G. The latter then implies

$$d(Y \cap G) \leq \delta(G) \leq \delta(X).$$

On the other hand, since $\prec$ left-separates $Y \cap G$, any dense subset of $Y \cap G$ must be cofinal in $Y \cap G$ w.r.t. $\prec$, hence we clearly have

$$\mathrm{cf}(Y \cap G, \prec) \leq d(Y \cap G) \leq \delta(X).$$

But any proper $\prec$-initial segment of $Y \cap G$ may be covered by $\delta(X)$ many nowhere dense sets, by the inductive hypothesis, hence we have

$$N(Y, X) \leq 1 + \delta(X) \cdot \delta(X) = \delta(X),$$

because $d(X)$ and so $\delta(X)$ is always infinite by definition. The second part now follows immediately. □

Note that again absolutely no separation axiom was needed in the above result. However, in the proof of the following theorem the assumption of Hausdorffness is essential.

Theorem 1.17. *Let X be a compact T_2-space satisfying $\mathrm{ls}(X) < N(\mathbb{R})$. Then X must be scattered.*

Proof. We actually prove the contrapositive form of this statement. So assume that X is not scattered, then it is well-known that some closed subspace $F \subset X$ admits an irreducible continuous closed map $f : F \to \mathbb{C}$ onto the Cantor set $\mathbb{C}$.

It is also well-known and easy to check that then we have $\delta(F) = \delta(\mathbb{C}) = \omega$, moreover $N(F) = N(\mathbb{C}) = N(\mathbb{R}) > \omega$. But then from lemma 1.16 we conclude that

$$\mathrm{ls}(X) \geq \mathrm{ls}(F) = \mathrm{ls}(F) \cdot \omega \geq N(F) = N(\mathbb{R}).$$ □

We would like to mention that 1.16 and 1.17 were motivated by the treatment of Tkačenko's results given in [60]. We also point out that theorems 1.14 and 1.17 yield a slight strengthening of Tkačenko's theorem in that the T_3 separation axiom may be replaced by T_2 in it. This is new even in the case of left-separated spaces (i. e. the assumption $\mathrm{ls}(X) = 1$) that preceded Tkačenko's result in [17].

Corollary 1.18. *Let X be a countably compact T_2 space that satisfies $\mathrm{ls}(X) \leq \omega$. Then X is compact, scattered, and sequential.*

We finish by formulating a couple of natural problems concerning our results.

Problem 1.19. *Is the upper bound $N(\mathbb{R})$ in theorem 1.17 sharp? Can it actually be replaced by the cardinality of the continuum (in ZFC, of course)?*

Note that as metric or compact spaces are all D-spaces, in theorem 1.17 one clearly cannot replace $\mathrm{ls}(X)$ with $D(X)$. Also, a compact (D-)space may fail to be sequential. Being left-separated, however, is clearly a hereditary property, hence left-separated spaces are actually hereditary D-spaces. Thus the following problems may be raised.

Problem 1.20. *Is a compact T_2 hereditary D-space sequential? Does it contain a point of countable character?*

Concerning this problem we note that it follows easily from theorem 1.14 that a compact T_2-space X satisfying $D(Y) \leq \omega$ for all $Y \subset X$ has countable tightness.

1.4 On d-separability of powers and $C_p(X)$

A space is called d-separable if it has a dense subset representable as the union of countably many discrete subsets. Thus d-separable spaces form a common generalization of separable and metrizable spaces. A. V. Arhangelskii was the first to study d-separable spaces in [3], where he proved for instance that any product of d-separable spaces is again d-separable. In [64], V. V. Tkachuk considered conditions under which a function space of the form $C_p(X)$ is d-separable and also raised a number of problems concerning the d-separability of both finite and infinite powers of certain spaces. He again raised some of these problems in his lecture presented at the 2006 Prague Topology Conference. In this note we give solutions to basically all his problems concerning infinite powers and to one concerning $C_p(X)$.

Theorem 1.21. *Let κ be an infinite cardinal and let X be a T_1 space satisfying $\widehat{s}(X^\kappa) > d(X)$. Then the power X^κ is d-separable.*

Proof. If X itself is discrete then all powers of X are obviously d-separable, hence in what follows we assume that X is not discrete. Consequently, we may pick an accumulation point of X that we fix from now on and denote it by 0. By definition, we may then find a dense subset S of X with $0 \notin S$ and $|S| = d(X) = \delta$. For any non-empty finite set of indices $a \in [\kappa]^{<\omega}$ we have then $|S^a| = \delta$ as well, hence we may fix a one-one indexing $S^a = \{s^a_\xi : \xi < \delta\}$.

Let us next fix an *increasing* sequence $\langle I_n : n < \omega\rangle$ of subsets of κ such that $\bigcup_{n<\omega} I_n = \kappa$ and $|\kappa \setminus I_n| = \kappa$ for each $n < \omega$. It follows from our assumptions then that for every $n < \omega$ there is a *discrete* subspace D_n of the "partial" power $X^{\kappa\setminus I_n}$ such that $|D_n| = \delta$. Thus we may also fix a one-one indexing of D_n of the form

$$D_n = \{y^n_\xi : \xi < \delta\}.$$

The discreteness of D_n means that for each $\xi < \delta$ there is an open set U^n_ξ in $X^{\kappa\setminus I_n}$ such that $U^n_\xi \cap D_n = \{y^n_\xi\}$.

Now fix $n \in \omega$ and pick a non-empty finite subset a of I_n. For each ordinal $\xi < \delta$ we define a point $x^{n,a}_\xi \in X^\kappa$ as follows:

$$x^{n,a}_\xi(\alpha) = \begin{cases} s^a_\xi(\alpha) & \text{if } \alpha \in a, \\ 0 & \text{if } \alpha \in I_n \setminus a, \\ y^n_\xi(\alpha) & \text{if } \alpha \in \kappa \setminus I_n. \end{cases}$$

Having done this, for any $n < \omega$ and $1 \le k < \omega$ we define a subset $E^{n,k} \subset X^\kappa$ by putting

$$E^{n,k} = \{x^{n,a}_\xi : \ a \in [I_n]^k \text{ and } \xi < \delta\}.$$

Now, for n and a as above and for $\xi < \delta$, let $W^{n,a}_\xi$ be the (obviously open) subset of X^κ consisting of those points $x \in X^\kappa$ that satisfy both $x(\alpha) \neq 0$ for all $\alpha \in a$ and $x \restriction (\kappa \setminus I_n) \in U^n_\xi$. Clearly, we have $x^{n,a}_\xi \in W^{n,a}_\xi$ and we claim that

$$W^{n,a}_\xi \cap E^{n,k} = \{x^{n,a}_\xi\}$$

whenever $a \in [I_n]^k$. Indeed, if $b \in [I_n]^k$ and $a \neq b$ then $|a| = |b| = k$ implies that $a \setminus b \neq \emptyset$, hence for any $\alpha \in a \setminus b$ and for any $\eta < \delta$ we have $x^{n,b}_\eta(\alpha) = 0$ showing that $x^{n,b}_\eta \notin W^{n,a}_\xi$. Moreover, for any ordinal $\eta < \delta$ with $\eta \neq \xi$ we have

$$x^{n,a}_\eta \restriction (\kappa \setminus I_n) = y^n_\eta \notin U^n_\xi,$$

hence again $x^{n,a}_\eta \notin W^{n,a}_\xi$. Thus we have shown that each set $E^{n,k}$ is discrete, while their union is trivially dense in X^κ. Consequently, X^κ is indeed d-separable. □

Let us note now that if X is any T_1 space containing at least two points then the power X^κ includes the Cantor cube 2^κ that is known to contain a discrete subspace of size κ. So if we apply this trivial observation to $\kappa = d(X)$, then we obtain immediately from theorem 1.21 the following corollary which answers problem 4.10 of [64]. This was asking if for every (Tychonov) space X there is a cardinal κ such that X^κ is d-separable.

Corollary 1.22. *For every T_1 space X the power $X^{d(X)}$ is d-separable.*

Next we show that if X is compact Hausdorff then even X^ω is d-separable, answering the second half of problem 4.2 from [64]. This will follow from the following result that we think is of independent interest.

Theorem 1.23. *If X is any compact T_2 space then X^2 contains a discrete subspace of size $d(X)$, that is $\widehat{s}(X^2) > d(X)$.*

Proof. Let us assume first that for every non-empty open subspace $G \subset X$ we also have $w(G) \geq d(X) = \delta$. We then define by transfinite induction on $\alpha < \delta$ distinct points $x_\alpha,\ y_\alpha \in X$ together with their *disjoint* open neighbourhoods $U_\alpha,\ V_\alpha$ as follows.

Suppose that $\alpha < \delta$, moreover $x_\beta \in U_\beta$ and $y_\beta \in V_\beta$ have already been defined for all $\beta < \alpha$. Then $\alpha < \delta = d(X)$ implies that there exists a non-empty open set $G_\alpha \subset X$ such that neither x_β nor y_β belongs to G_α for $\beta < \alpha$. Let us choose then a non-empty open set H_α such that $\overline{H_\alpha} \subset G_\alpha$ and consider the topology τ_α on $\overline{H_\alpha}$ generated by the traces of the open sets $U_\beta,\ V_\beta$ for all $\beta < \alpha$. Since

$$w(\overline{H_\alpha},\, \tau_\alpha) < \delta \leq w(H_\alpha) \leq w(\overline{H_\alpha}),$$

the topology τ_α is strictly coarser than the compact Hausdorff subspace topology of $\overline{H_\alpha}$ inherited from X, hence τ_α is not Hausdorff. We pick the two points $x_\alpha,\ y_\alpha \in \overline{H_\alpha}$ so that they witness the failure of the Hausdorffness of τ_α. Note that, in particular, this will imply

$$\langle x_\alpha,\, y_\alpha\rangle \notin U_\beta \times V_\beta$$

for all $\beta < \alpha$. We may then choose their disjoint open (in X) neighbourhoods $U_\alpha,\ V_\alpha$ inside G_α. This will clearly imply that we shall also have $\langle x_\alpha,\, y_\alpha\rangle \notin U_\gamma \times V_\gamma$ whenever $\alpha < \gamma < \delta$. Thus, indeed, $\{\langle x_\alpha,\, y_\alpha\rangle : \alpha < \delta\}$ is a discrete subspace of X^2.

Now, assume that X is an arbitrary compact Hausdorff space and call an open set $G \subset X$ *good* if we have $d(H) = d(G)$ for every non-empty open $H \subset G$. Clearly, every non-empty open set has a non-empty good open subset, hence if $\mathcal{G}$ is a maximal disjoint family of good open sets in X then $\bigcup \mathcal{G}$ is dense in X. Consequently we have

$$\sum\{d(G) : G \in \mathcal{G}\} \geq d(X).$$

But for every $G \in \mathcal{G}$ its square G^2 has a discrete subspace D_G with $|D_G| = d(G)$. Indeed, if H is open with $\emptyset \neq \overline{H} \subset G$ then for every non-empty open $U \subset \overline{H}$ we have $w(U) \geq d(U) = d(H) = d(\overline{H})$, so the first part of our proof applies to $\overline{H}$, that is $\overline{H}^2$ (and therefore G^2) has a discrete subspace of size $d(\overline{H}) = d(U)$. It immediately follows that $D = \bigcup\{D_G : G \in \mathcal{G}\}$ is discrete in X^2, moreover

$$|D| = \sum\{d(G) : G \in \mathcal{G}\} \geq d(X),$$

completing our proof. □

Any compact L-space, more precisely: a non-separable hereditarily Lindelof compact space (e. g. a Suslin line), demonstrates, alas only consistently, that in theorem 1.23 the square X^2 cannot be replaced by X itself. On the other hand, we should recall here Shapirovskii's celebrated result, see 3.13 of [25], which states that $d(X) \leq s(X)^+$ holds for any compact T_2 space X. This leads us to the following natural question.

Problem 1.24. *Is there a ZFC example of a compact T_2 space X that does not contain a discrete subspace of cardinality $d(X)$?*

Since X^2 embeds as a subspace into X^ω, theorems 1.21 and 1.23 immediately imply the following.

Corollary 1.25. *If X is any compact T_2 space then X^ω is d-separable.*

Of course, to get corollary 1.25 it would suffice to know $\widehat{s}(X^\omega) > d(X)$. Our next result shows, however, that if we know that some finite power of X has a discrete subspace of size $d(X)$ then we may actually obtain a stronger conclusion. To formulate this result we again fix a point $0 \in X$ and introduce the notation

$$\sigma(X^\omega) = \big\{x \in X^\omega \ :\ \{i < \omega : x(i) \neq 0\} \text{ is finite}\big\}.$$

Clearly, $\sigma(X^\omega)$ is dense in X^ω, hence the d-separability of the former implies that of the latter.

Theorem 1.26. *Let X be a space such that, for some $k < \omega$, the power X^k has a discrete subspace of cardinality $d(X)$. Then $\sigma(X^\omega)$ (and hence X^ω) is d-separable.*

Proof. Let us put again $d(X) = \delta$ and fix a dense set $S \subset X$ with $|S| = \delta$. By assumption, there is a discrete subspace $D \subset X^k$ with a one-one indexing $D = \{d_\xi : \xi < \delta\}$. Also, for each natural number $n \geq 1$ we have $|S^n| = \delta$, so we may fix a one-one indexing $S^n = \{s^n_\xi : \xi < \delta\}$.

Now, for any $1 \leq n < \omega$ and $\xi < \delta$ we define a point $x^n_\xi \in \sigma(X^\omega)$ with the following stipulations:

$$x^n_\xi(i) = \begin{cases} s^n_\xi(i) & \text{if } i < n, \\ d_\xi(i-n) & \text{if } n \leq i < n+k, \\ 0 & \text{if } n+k \leq i < \omega. \end{cases}$$

It is straight-forward to check that each $D_n = \{x^n_\xi : \xi < \delta\} \subset \sigma(X^\omega)$ is discrete, moreover $\bigcup_{n<\omega} D_n$ is dense in $\sigma(X^\omega)$. □

Actually, before we get too excited, let us point out that the d-separability of X^ω implies that some finite power of X has a discrete subspace of cardinality $d(X)$, in "most" cases, namely if $\mathrm{cf}(d(X)) > \omega$. Indeed, first of all, in this case there is a discrete $D \subset X^\omega$ with $|D| = d(X^\omega) = d(X)$. Secondly, for each point $x \in D$ there is a finite set of co-ordinates $a_x \in [\omega]^{<\omega}$ that supports a neighbourhood U_x of x such that $D \cap U_x = \{x\}$. But by $\mathrm{cf}(|D|) > \omega$ then there is some $a \in [\omega]^{<\omega}$ with $|\{x \in D : a_x = a\}| = |D| = d(X)$, and we are clearly done.

Let us mention though that the d-separability of the power X^ω does not imply that of some finite power of X. In fact, the Čech–Stone remainder ω^* demonstrates this because its ω^{th} power is d-separable by theorem 1.26 but no finite power of ω^* is d-separable, as it was pointed out in [64, 3.16 (b)].

Next we give a negative solution to one more problem of Tkachuk concerning the d-separability of powers. Problem 4.9 from [64] asks if the d-separability of some infinite power X^κ implies the d-separability of the countable power X^ω. We recall that a strong L-space is a non-separable regular space all finite powers of which are hereditarily Lindelöf.

Theorem 1.27. *Let X be a strong L-space with $d(X) = \omega_1$. Then X^{ω_1} is d-separable but X^ω is not. Moreover, there is a ZFC example of a 0-dimensional T_2 space Y such that Y^{ω_2} is d-separable but Y^{ω_1} (and hence Y^ω) is not.*

Proof. It is immediate from corollary 1.22 that X^{ω_1} is d-separable. Also, since all finite powers of X are hereditarily Lindelöf so is X^ω, hence

$$s(X^\omega) = \omega < \omega_1 = d(X^\omega)$$

implies that X^ω cannot be d-separable.

To see the second statement, we use Shelah's celebrated coloring theorem from [59], which says that $Col(\lambda^+, 2)$ holds for every uncountable regular cardinal λ, together with theorem [26, 1.11 (i)] saying that $Col(\lambda^+, 2)$ implies the existence of a 0-dimensional T_2 space Y that is a strong L_λ space. The latter means that $hL(Y^n) \leq \lambda$ for all finite n but $d(Y) > \lambda$. Without loss of generality, we may assume that $d(Y) = \lambda^+$. Thus from from corollary 1.22 we conclude that the power Y^{λ^+} is d-separable.

On the other hand, a simple counting argument as above yields that

$$s(Y^\lambda) \leq hL(Y^\lambda) \leq \lambda < \lambda^+ = d(Y) = d(Y^\lambda),$$

hence Y^λ obviously cannot be d-separable. In particular, if $\lambda = \omega_1$ then we obtain our claim. □

Finally, our next result answers the first part of problem 4.1 from [64] that asks for a ZFC example of a (Tychonov) space X such that $C_p(X)$ is not d-separable. (The second part asks the same for compact spaces.)

Theorem 1.28. *If $Col(\kappa, 2)$ holds for some successor cardinal $\kappa = \lambda^+$ then the Cantor cube of weight κ, $D(2)^\kappa$, has a dense subspace X such that $C_p(X)$ is not d-separable. Moreover, if X is a compact strong S_λ space of weight λ^+ then $C_p(X)$ is not d-separable.*

Proof. It was shown in [27, 6.4] (and mentioned in [26, 1.11]) that $Col(\kappa, 2)$ implies the existence of a strong κ-HFD$_w$ subspace $Y = \{y_\alpha : \alpha < \kappa\}$ of $D(2)^\kappa$ with the additional property that $y_\alpha(\beta) = 0$ for $\beta < \alpha < \kappa$.
It is also well-known (see e. g. [25, 5.4]) that $D(2)^\kappa$ has a dense subspace Z of cardinality λ. Let us now set $X = Y \cup Z$.

As Y is a strong κ-HFD$_w$, we have $s(Y^n) \leq hd(Y^n) \leq \lambda$ for each finite n and it is easy to see that then we also have $s(X^n) \leq hd(X^n) \leq \lambda$ whenever $n < \omega$. It was also pointed out in [27, 6.5] that every (relatively) open subset G of Y (and hence of X) satisfies either $|G| \leq \lambda$ or $|Y \setminus G| \leq \lambda$ (resp. $|X \setminus G| \leq \lambda$). This in turn obviously implies that no family $\mathcal{U}$ of open subsets of Y (resp. X) with $|\mathcal{U}| < \kappa$ can separate its points, hence we have

$$iw(X) = iw(Y) = \kappa > \lambda.$$

But then by [64, 3.6] neither $C_p(X)$ nor $C_p(Y)$ is d-separable. As we have noted above, $Col(\omega_2, 2)$ is provable in ZFC, so in particular we may conclude that the Cantor cube of weight ω_2 has a dense subspace X such that $C_p(X)$ is not d-separable.

To see the second statement of our theorem, consider a compact strong S_λ space X. This means that for each natural number n we have $s(X^n) \leq hd(X^n) \leq \lambda$ but $hL(X) > \lambda$. It is well-known that we may assume without any loss of generality that $w(X) = \lambda^+$ holds as well. But now the compactness of X immediately implies $iw(X) = w(X)$, hence again by [64, 3.6] the function space $C_p(X)$ is not d-separable. □

It is an intriguing open question if the existence of a cardinal λ for which there is a compact strong S_λ space is provable in ZFC. Note that by theorem 1.23 there is no compact strong L_λ space for any cardinal λ. On the other hand, the existence of compact strong S (i. e. S_ω) spaces was shown to follow from CH by K. Kunen, see e. g. [14, 2.4] and [47, 7.1].

1.5 A strengthening of the Čech-Pospišil theorem

In [44] the following problem was formulated and raised: If X is a crowded (i. e. dense-in-itself) compact T_2 space, is then $\mathrm{dis}(X) \geq \mathfrak{c}$? The cardinal function $\mathrm{dis}(X)$ was defined there as the smallest infinite cardinal κ such that X can be covered by κ many discrete subspaces. This problem was answered affirmatively by G. Gruenhage: He showed in [21] that if $f : X \twoheadrightarrow Y$ is a perfect onto map, with X and Y arbitrary topological spaces, then $\mathrm{dis}(X) \geq \mathrm{dis}(Y)$. Since any crowded compact T_2 space maps continuously (and hence perfectly) onto the closed interval $[0, 1]$, this clearly suffices.

Our aim here is to present another solution to the above problem which also may be considered as a significant strengthening of the, by now classical, Čech–Pospišil theorem, see [25, 3.16]. Although our solution goes in a completely different direction from that of Gruenhage, it makes use of a lemma of his that was playing a crucial role in his solution as well. Since [21] is still unpublished and because we would like to make our paper self-contained, we shall start by presenting a proof of Gruenhage's lemma. Our proof, we think, is also somewhat simpler than the one given in [21].

First we recall that a point x in a space X is a limit point of a set X iff for every neighbourhood U of x in X we have $U \cap A \setminus \{x\} \neq \emptyset$. (Of course, if X is T_1 then this is equivalent to $U \cap A$ being infinite.) Also, for any subset A of X we use A' to denote the derived set of all limit points of A. Note that $D \subset X$ is a discrete subspace iff $D \cap D' = \emptyset$.

Lemma 1.29. *(G. Gruenhage) Let X be any topological space and $K \subset X$ be its non-empty compact subset with $K \subset \cup\mathcal{D}$, where each $D \in \mathcal{D}$ is a discrete subspace of X. Then there exist $D \in \mathcal{D}$ and $\mathcal{E} \in [\mathcal{D}]^{<\omega}$ such that*

$$\emptyset \neq K \cap \overline{D} \cap \bigcap\{E' : E \in \mathcal{E}\} \subset D.$$

Proof. By Zorn's lemma we may choose a *maximal* subfamily $\mathcal{C} \subset \mathcal{D}$ such that the family $\{K\} \cup \{C' : C \in \mathcal{C}\}$ is centered (i. e. has the finite intersection property). As K is compact and each C' is closed, then we have

$$K \cap \bigcap\{C' : C \in \mathcal{C}\} \neq \emptyset,$$

hence there is some $D \in \mathcal{D}$ with

$$K \cap D \bigcap\{C' : C \in \mathcal{C}\} \neq \emptyset.$$

But then $D \notin \mathcal{C}$, for $D \cap D' = \emptyset$, so by the maximality of $\mathcal{C}$ there is a finite subfamily $\mathcal{E} \subset \mathcal{C}$ with

$$K \cap D' \cap \bigcap\{E' : E \in \mathcal{E}\} = \emptyset.$$

Since $\overline{D} = D \cup D'$, then D and $\mathcal{E}$ are as required. $\square$

Note that the sets of the form E' are closed, hence so is $\overline{D} \cap \bigcap\{E' : E \in \mathcal{E}\}$. Consequently, $K \cap \overline{D} \cap \bigcap\{E' : E \in \mathcal{E}\}$ is compact and discrete, and hence finite (and non-empty).

Now, to present the main result of this section we need a simple definition.

Definition 1.30. Let λ be an infinite cardinal. A family of sets $\mathcal{F}$ is said to be λ-*branching* if $|\mathcal{F}| < \lambda$ but one can form λ many pairwise disjoint intersections of subfamilies of $\mathcal{F}$.

Theorem 1.31. *If in a compact space X there is a λ-branching family of* closed *sets then* $\operatorname{dis}(X) \geq \lambda$.

Proof. Let $\mathcal{F}$ be a λ-branching family of closed subsets of X. So $|\mathcal{F}| < \lambda$ and we may fix a family $\mathcal{K}$ of pairwise disjoint non-empty sets with $|\mathcal{K}| = \lambda$ such that each $K \in \mathcal{K}$ is obtainable as the intersection of some subfamily $\mathcal{F}_K \subset \mathcal{F}$. Each $K \in \mathcal{K}$ is closed in X and therefore is also compact.

Arguing indirectly, assume that $\operatorname{dis}(X) < \lambda$ and fix a family $\mathcal{D}$ of discrete subspaces of X such that $|\mathcal{D}| < \lambda$ and $X = \cup\mathcal{D}$. Applying Lemma 1.29, for each set $K \in \mathcal{K}$ there are a member $D_K \in \mathcal{D}$ and a finite subfamily $\mathcal{E}_K \in [\mathcal{D}]^{<\omega}$ such that

$$\emptyset \neq S_K = K \cap \overline{D_K} \cap \bigcap\{E' : E \in \mathcal{E}_K\} \subset D_K.$$

Since $K = \cap\mathcal{F}_K$ and D'_K is also compact, we may find a *finite* subfamily $\mathcal{G}_K \subset \mathcal{F}_K$ such that $\bigcap\mathcal{G}_K \cap \bigcap\{E' : E \in \mathcal{E}_K\} \cap D'_K = \emptyset$ and hence

$$S_K \subset T_K = \bigcap\mathcal{G}_K \cap \overline{D_K} \cap \bigcap\{E' : E \in \mathcal{E}_K\} \subset D_K.$$

But then T_K is a compact and discrete set as well, and so it is finite. Consequently, we may extend $\mathcal{G}_K$ with finitely many further members of $\mathcal{F}_K$ to obtain a finite family $\mathcal{H}_K$ with $\mathcal{G}_K \subset \mathcal{H}_K \subset \mathcal{F}_K \subset \mathcal{F}$ in such a way that

$$S_K = \bigcap\mathcal{H}_K \cap D_K \cap \bigcap\{E' : E \in \mathcal{E}_K\}.$$

Now, we have both $|\mathcal{F}| < \lambda$ and $|\mathcal{D}| < \lambda$, so there are only fewer than λ many choices for $D_K,\ \mathcal{E}_K$ and $\mathcal{H}_K$, while $|\mathcal{K}| = \lambda$, hence there must be distinct sets $K, L \in \mathcal{K}$ such that $D_K = D_L,\ \mathcal{E}_K = \mathcal{E}_L$ and $\mathcal{H}_K = \mathcal{H}_L$, consequently $S_K = S_L$. But this is a contradiction because S_K and S_L are disjoint non-empty sets. $\square$

Let us emphasize that in Theorem 1.31 no separation axiom had to be assumed about our compact space X. In contrast to this, the following result, the promised strengthening of the Čech–Pospišil theorem, seems to require that our compact space be also Hausdorff.

Corollary 1.32. *If X is a compact T_2 space such that all points $x \in X$ have character $\chi(x, X) \geq \kappa$ then $\mathrm{dis}(X) \geq 2^\kappa$.*

Proof. The proof starts out exactly as in the proof of the original Čech–Pospišil theorem, that is one builds a Cantor-tree $\mathcal{T} = \{F_s : s \in 2^{<\kappa}\}$ of non-empty closed sets as e. g. in [25, 3.16]. For $\kappa = \omega$, each F_s is regular closed. For $\kappa > \omega$ we have $\chi(F_s, X) \leq |s| \cdot \omega < \kappa$ whenever $s \in 2^{<\kappa}$.

Now let $\mu = \log(2^\kappa) \leq \kappa$ (that is, $\mu = \min\{\lambda : 2^\lambda = 2^\kappa\}$) and set $\mathcal{F} = \{F_s : s \in 2^{<\mu}\}$. Then, by $\mathrm{cf}(2^\mu) > \mu$, we have

$$|\mathcal{F}| = 2^{<\mu} = \sum\{2^\lambda : \lambda < \mu\} < 2^\mu = 2^\kappa,$$

moreover the sets $F_t = \bigcap\{F_{t\restriction\alpha} : \alpha < \mu\}$ for $t \in 2^\mu$ are non-empty and pairwise disjoint, hence $\mathcal{F}$ is 2^κ-branching. Thus it follows from Theorem 1.31 that $\mathrm{dis}(X) \geq 2^\kappa$. □

In [44, Theorem 3] the following related result was proved: If X is a compact T_2 space such that all points $x \in X$ have character $\chi(x, X) \geq \kappa$ then $\mathrm{rs}(X) > \kappa$, where

$$\mathrm{rs}(X) = \min\{|\mathcal{R}| : X = \cup\mathcal{R} \text{ and each } R \in \mathcal{R} \text{ is right separated}\}.$$

Since every discrete (sub)space is right separated, this result is stronger than corollary 1.32 provided that $2^\kappa = \kappa^+$. On the other hand, now the following interesting open question can be raised.

Problem 1.33. *Can one replace in Corollary 1.32 $\mathrm{dis}(X)$ with $\mathrm{rs}(X)$, even if $2^\kappa > \kappa^+$?*

Finally, we would like to formulate one more open problem. Note that it follows immediately even from the original Čech–Pospišil theorem that if X is a compact T_2 space in which all points have character $\geq \kappa$ then

$$\Delta(X) = \min\{|G| : G \text{ is non-empty open in } X\} \geq 2^\kappa.$$

Consequently, an affirmative answer to the following question would yield another strengthening of Corollary 1.32.

Problem 1.34. *For X compact T_2, is $\mathrm{dis}(X) \geq \Delta(X)$?*

1.6 Interpolation of κ-compactness and PCF

We start by recalling that a point x in a topological space X is said to be a *complete accumulation point* of a set $A \subset X$ iff for every neighbourhood U of x we have $|U \cap A| = |A|$. We denote the set of all complete accumulation points of A by A°.

It is well-known that a space is compact iff every infinite subset has a complete accumulation point. This justifies to call a space κ-compact if every subset of cardinality κ in it has a complete accumulation point. Now, let κ be a singular cardinal and $\kappa = \sum\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ with $\kappa_\alpha < \kappa$ for each $\alpha < \mathrm{cf}(\kappa)$. Clearly, if a space X is both κ_α-compact for all $\alpha < \mathrm{cf}(\kappa)$ and $\mathrm{cf}(\kappa)$-compact then X is κ-compact as well. This trivial "extrapolation" property of κ-compactness (for singular κ) implies that in the above characterization of compactness one may restrict to subsets of regular cardinality.

The aim of this note is to present a new "interpolation" result on κ-compactness, i.e. one in which $\mu < \kappa < \lambda$ and we deduce κ-compactness of a space from its μ- and λ-compactness. Again, this works for singular cardinals κ and the proof uses non-trivial results from Shelah's PCF theory.

Definition 1.35. Let κ, λ, μ be cardinals, then $\Phi(\mu, \kappa, \lambda)$ denotes the following statement: $\mu < \kappa < \lambda = \mathrm{cf}(\lambda)$ and there is $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ such that $|\{\xi : |S_\xi \cap A| = \mu\}| < \lambda$ whenever $A \in [\kappa]^{<\kappa}$.

As we can see from our next theorem, this property Φ yields the promised interpolation result for κ-compactness.

Theorem 1.36. *Assume that $\Phi(\mu, \kappa, \lambda)$ holds and the space X is both μ-compact and λ-compact. Then X is κ-compact as well.*

Proof. Let Y be any subset of X with $|Y| = \kappa$ and, using $\Phi(\mu, \kappa, \lambda)$, fix a family $\{S_\xi : \xi < \lambda\} \subset [Y]^\mu$ such that $|\{\xi : |S_\xi \cap A| = \mu\}| < \lambda$ whenever $A \in [Y]^{<\kappa}$. Since X is μ-compact we may then pick a complete accumulation point $p_\xi \in S_\xi{}^\circ$ for each $\xi < \lambda$.

Now we distinguish two cases. If $|\{p_\xi : \xi < \lambda\}| < \lambda$ then the regularity of λ implies that there is $p \in X$ with $|\{\xi < \lambda : p_\xi = p\}| = \lambda$. If, on the other hand, $|\{p_\xi : \xi < \lambda\}| = \lambda$ then we can use the λ-compactness of X to pick a complete accumulation point p of this set. In both cases the point $p \in X$ has the property that for every neighbourhood U of p we have $|\{\xi : |S_\xi \cap U| = \mu\}| = \lambda$.

Since $S_\xi \cap U \subset Y \cap U$, this implies using $\Phi(\mu, \kappa, \lambda)$ that $|Y \cap U| = \kappa$, hence p is a complete accumulation point of Y, hence X is indeed κ-compact. □

Our following result implies that if $\Phi(\mu, \kappa, \lambda)$ holds then κ must be singular.

Theorem 1.37. *If $\Phi(\mu, \kappa, \lambda)$ holds then we have $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$.*

Proof. Assume that $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\mu$ witnesses $\Phi(\mu, \kappa, \lambda)$ and fix a strictly increasing sequence of ordinals $\eta_\alpha < \kappa$ for $\alpha < \mathrm{cf}(\kappa)$ that is cofinal in κ. By the regularity of

$\lambda > \kappa$ there is an ordinal $\xi < \lambda$ such that $|S_\xi \cap \eta_\alpha| < \mu$ holds for each $\alpha < \mathrm{cf}(\kappa)$. But this S_ξ must be cofinal in κ, hence from $|S_\xi| = \mu$ we get $\mathrm{cf}(\mu) \le \mathrm{cf}(\kappa) \le \mu$.

Now assume that we had $\mathrm{cf}(\mu) < \mathrm{cf}(\kappa)$ and set $|S_\xi \cap \eta_\alpha| = \mu_\alpha$ for each $\alpha < \mathrm{cf}(\kappa)$. Our assumptions then imply $\mu^* = \sup\{\mu_\alpha : \alpha < \mathrm{cf}(\kappa)\} < \mu$ as well as $\mathrm{cf}(\kappa) < \mu$, contradicting that $S_\xi = \cup\{S_\xi \cap \eta_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ and $|S_\xi| = \mu$. This completes our proof. $\square$

According to theorem 1.37 the smallest cardinal μ for which $\Phi(\mu, \kappa, \lambda)$ may hold for a given singular cardinal κ is $\mathrm{cf}(\kappa)$. Our main result says that this actually does happen with the natural choice $\lambda = \kappa^+$.

Theorem 1.38. *For every singular cardinal κ we have* $\Phi(\mathrm{cf}(\kappa), \kappa, \kappa^+)$.

Proof. We shall make use of the following fundamental result of Shelah from his PCF theory: There is a strictly increasing sequence of length $\mathrm{cf}(\kappa)$ of regular cardinals $\kappa_\alpha < \kappa$ cofinal in κ and such that in the product

$$\mathbb{P} = \prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$$

there is a scale $\{f_\xi : \xi < \kappa^+\}$ of length κ^+. (This is Main Claim 1.3 on p. 46 of [59].)

Spelling it out, this means that the κ^+-sequence $\{f_\xi : \xi < \kappa^+\} \subset \mathbb{P}$ is increasing and cofinal with respect to the partial ordering $<^*$ of eventual dominance on $\mathbb{P}$. Here for $f, g \in \mathbb{P}$ we have $f <^* g$ iff there is $\alpha < \mathrm{cf}(\kappa)$ such that $f(\beta) < g(\beta)$ whenever $\alpha \le \beta < \mathrm{cf}(\kappa)$.

Now, to show that this implies $\Phi(\mathrm{cf}(\kappa), \kappa, \kappa^+)$, we take the set

$$H = \cup\{\{\alpha\} \times \kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$$

as our underlying set. Note that then $|H| = \kappa$ and every function $f \in \mathbb{P}$, construed as a set of ordered pairs (or in other words: identified with its graph) is a subset of H of cardinality $\mathrm{cf}(\kappa)$.

We claim that the scale sequence

$$\{f_\xi : \xi < \kappa^+\} \subset [H]^{\mathrm{cf}(\kappa)}$$

witnesses $\Phi(\mathrm{cf}(\kappa), \kappa, \kappa^+)$. Indeed, let A be any subset of H with $|A| < \kappa$. We may then choose $\alpha < \mathrm{cf}(\kappa)$ in such a way that $|A| < \kappa_\alpha$. Clearly, then there is a function $g \in \mathbb{P}$ such that we have

$$A \cap (\{\beta\} \times \kappa_\beta) \subset \{\beta\} \times g(\beta)$$

whenever $\alpha \le \beta < \mathrm{cf}(\kappa)$. Since $\{f_\xi : \xi < \kappa^+\}$ is cofinal in $\mathbb{P}$ w.r.t. $<^*$, there is a $\xi < \kappa^+$ with $g <^* f_\xi$ and obviously we have $|A \cap f_\eta| < \mathrm{cf}(\kappa)$ whenever $\xi \le \eta < \kappa^+$. $\square$

Note that the above proof actually establishes the following more general result: If for some increasing sequence of regular cardinals $\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ that is cofinal in κ there is a scale of length $\lambda = \mathrm{cf}(\lambda)$ in the product $\prod\{\kappa_\alpha : \alpha < \mathrm{cf}(\kappa)\}$ then $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$ holds.

Before giving some further interesting application of the property $\Phi(\mu, \kappa, \lambda)$, we present a result that enables us to "lift" the first parameter $\mathrm{cf}(\kappa)$ in theorem 1.38 to higher cardinals.

Theorem 1.39. *If $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$ holds for some singular cardinal κ then we also have $\Phi(\mu, \kappa, \lambda)$ whenever $\mathrm{cf}(\kappa) < \mu < \kappa$ with $\mathrm{cf}(\mu) = \mathrm{cf}(\kappa)$.*

Proof. Let us put $\mathrm{cf}(\kappa) = \varrho$ and fix a strictly increasing and cofinal sequence $\{\kappa_\alpha : \alpha < \varrho\}$ of cardinals below κ. We also fix a partition of κ into disjoint sets $\{H_\alpha : \alpha < \varrho\}$ with $|H_\alpha| = \kappa_\alpha$ for each $\alpha < \varrho$.

Let us now choose a family $\{S_\xi : \xi < \lambda\} \subset [\kappa]^\varrho$ that witnesses $\Phi(\mathrm{cf}(\kappa), \kappa, \lambda)$. Since λ is regular, we may assume without any loss of generality that $|H_\alpha \cap S_\xi| < \varrho$ holds for every $\alpha < \varrho$ and $\xi < \lambda$. Note that this implies $|\{\alpha : H_\alpha \cap S_\xi \neq \emptyset\}| = \varrho$ for each $\xi < \lambda$.

Now take a cardinal μ with $\mathrm{cf}(\mu) = \varrho < \mu < \kappa$ and fix a strictly increasing and cofinal sequence $\{\mu_\alpha : \alpha < \varrho\}$ of cardinals below μ. To show that $\Phi(\mu, \kappa, \lambda)$ is valid, we may use as our underlying set $S = \cup\{H_\alpha \times \mu_\alpha : \alpha < \varrho\}$, since clearly $|S| = \kappa$.

For each $\xi < \lambda$ let us now define the set $T_\xi \subset S$ as follows:

$$T_\xi = \cup\{(S_\xi \cap H_\alpha) \times \mu_\alpha : \alpha < \varrho\}.$$

Then we have $|T_\xi| = \mu$ because $|\{\alpha : H_\alpha \cap S_\xi \neq \emptyset\}| = \varrho$. We claim that $\{T_\xi : \xi < \lambda\}$ witnesses $\Phi(\mu, \kappa, \lambda)$.

Indeed, let $A \subset S$ with $|A| < \kappa$. For each $\alpha < \rho$ let B_α denote the set of all first co-ordinates of the pairs that occur in $A \cap (H_\alpha \times \mu_\alpha)$ and set $B = \cup\{B_\alpha : \beta < \varrho\}$. Clearly, we have $B \subset \kappa$ and $|B| \le |A| < \kappa$, hence $|\{\xi : |S_\xi \cap B| = \varrho\}| < \lambda$

Now, consider any ordinal $\xi < \lambda$ with $|S_\xi \cap B| < \varrho$. If $\langle \gamma, \delta \rangle \in (T_\xi \cap A) \cap (H_\alpha \times \mu_\alpha)$ for some $\alpha < \varrho$ then we have $\gamma \in S_\xi \cap B_\alpha$, consequently $H_\alpha \cap S_\xi \cap B \neq \emptyset$. This implies that

$$W = \{\alpha : (T_\xi \cap A) \cap (H_\alpha \times \mu_\alpha) \neq \emptyset\}$$

has cardinality $\le |S_\xi \cap B| < \varrho$. But for each $\alpha \in W$ we have

$$|T_\xi \cap (H_\alpha \times \mu_\alpha)| \le \varrho \cdot \mu_\alpha < \mu,$$

hence

$$T_\xi \cap A = \cup\{(T_\xi \cap A) \cap (H_\alpha \times \mu_\alpha) : \alpha \in W\}$$

implies $|T_\xi \cap A| < \mu$ as well. But this shows that $\{T_\xi : \xi < \lambda\}$ indeed witnesses $\Phi(\mu, \kappa, \lambda)$. $\square$

Arhangel'skii has recently introduced and studied in [4] the class of spaces that are κ-compact for all uncountable cardinals κ and, quite appropriately, called them *uncountably compact*. In particular, he showed that these spaces are Lindelöf.

We recall that the spaces that are κ-compact for all uncountable *regular* cardinals κ have been around for a long time and are called linearly Lindelöf. Moreover, the question under what conditions is a linearly Lindelöf space Lindelöf is important and well-studied. Note, however, that a linearly Lindelöf space is obviously comapct iff it is countably compact, i.e. ω-compact. This should be compared with our next result that, we think, is far from being obvious.

Theorem 1.40. *Every linearly Lindelöf and $\aleph_\omega$-compact space is uncountably compact hence, in particular, Lindelöf.*

Proof. Let X be a linearly Lindelöf and $\aleph_\omega$-compact space. According to the (trivial) extrapolation property of κ-compactness that we mentioned in the introduction, X is κ-compact for all cardinals κ of uncountable cofinality. Consequently, it only remains to show that X is κ-compact whenever κ is a singular cardinal of countable cofinality with $\aleph_\omega < \kappa$.

But, according to theorems 1.38 and 1.39, we have $\Phi(\aleph_\omega, \kappa, \kappa^+)$ and X is both $\aleph_\omega$-compact and κ^+-compact, hence theorem 1.36 implies that X is κ-compact as well. □

Arhangel'skii gave in [4] the following surprising result which shows that the class of uncountably compact T_3-spaces is rather restricted: Every uncountably compact T_3-space X has a (possibly empty) compact subset C such that for every open set $U \supset C$ we have $|X \setminus U| < \aleph_\omega$. Below we show that in this result the T_3 separation axiom can be replaced by T_1 plus van Douwen's property wD, see e.g. 3.12 in [69]. Since uncountably compact T_3-spaces are normal, being also Lindelöf, and the wD property is a very weak form of normality, this indeed is an improvement.

Definition 1.41. A topological space X is said to be κ-concentrated on its subset Y if for every open set $U \supset Y$ we have $|X \setminus U| < \kappa$.

So what we claim can be formulated as follows.

Theorem 1.42. *Every uncountably compact T_1 space X with the wD property is $\aleph_\omega$-concentrated on some (possibly empty) compact subset C.*

Proof. Let C be the set of those points $x \in X$ for which every neighbourhood has cardinality at least $\aleph_\omega$. First we show that C, as a subspace, is compact. Indeed, C is clearly closed in X, hence Lindelöf, so it suffices to show for this that C is countably compact.

Assume, on the contrary, that C is not countably compact. Then, as X is T_1, there is an infinite closed discrete $A \in [C]^\omega$. But then by the wD property there is an infinite

$B \subset A$ that expands to a discrete (in X) collection of open sets $\{U_x : x \in B\}$. By the definition of C we have $|U_x| \geq \aleph_\omega$ for each $x \in B$.

Let $B = \{x_n : n < \omega\}$ be any one-to-one enumeration of B. Then for each $n < \omega$ we may pick a subset $A_n \subset U_{x_n}$ with $|A_n| = \aleph_n$ and set $A = \cup\{A_n : n < \omega\}$. But then $|A| = \aleph_\omega$ and A has no complete accumulation point, a contradiction.

Next we show that X is $\aleph_\omega$ concentrated on C. Indeed, let $U \supset C$ be open. If we had $|X \setminus U| \geq \aleph_\omega$ then any complete accumulation point $X \setminus U$ is not in U but is in C, again a contradiction. $\square$

The following easy result, that we add or the sake of completeness, yields a partial converse to theorem 1.42.

Theorem 1.43. *If a space X is κ-concentrated on a compact subset C then X is λ-compact for all cardinals $\lambda \geq \kappa$.*

Proof. Let $A \subset X$ be any subset with $|A| = \lambda \geq \kappa$. We claim that we even have $A^\circ \cap C \neq \emptyset$. Assume, on the contrary, that every point $x \in C$ has an open neighbourhood U_x with $|A \cap U_x| < \lambda$. Then the compactness of C implies $C \subset U = \cup\{U_x : x \in F\}$ for some finite subset F of C. But then we have $|A \cap U| < \lambda$, hence $|A \setminus U| = \lambda \geq \kappa$, contradicting that X is κ-concentrated on C. $\square$

Putting all these theorems together we immediately obtain the following result.

Corollary 1.44. *Let X be a T_1 space with property wD that is $\aleph_n$-compact for each $0 < n < \omega$. Then X is uncountably compact if and only if it is $\aleph_\omega$-concentrated on some compact subset.*

2 Calibers, free sequences and density

All spaces considered in this section are assumed to be T_3.

Let us start by recalling that a cardinal κ is said to be a caliber of a space X (in symbols: $\kappa \in \mathrm{Cal}(X)$) if among any κ open subsets of X there are always κ many with non-empty intersection. Obviously, if $\mathrm{cf}(\kappa) > d(X)$ holds then $\kappa \in \mathrm{Cal}(X)$, however, as was shown by Šanin in [53], the converse of this statement is false, e.g., because the property "$\kappa \in \mathrm{Cal}(X)$" is fully productive. Therefore, it is of some interest to find additional conditions on X such that they ensure the truth of the converse or at least provide some upper bound for the density $d(X)$ of X.

As an example we may mention Šapirovskii's theorem that for a compact space X and a regular cardinal κ we have $d(X) < \kappa$ (in fact even $\pi(X) < \kappa$) provided that $\kappa \in \mathrm{Cal}(X)$ and $t(X) < \kappa$ (see [25, 3.25]), or the recent result of Arhangelskii from [7] saying that if X is Lindelof with $T(X) = \omega$ and $\kappa \in \mathrm{Cal}(X)$ and $t(X) < \kappa$ then $d(X) \leq 2^\omega$. (Recall from [31] that $T(X)$ is defined as the smallest cardinal κ such that whenever $\{F_\alpha : \alpha \in \varrho\}$ is an increasing sequence of closed subsets of X with $\varrho = \mathrm{cf}(\varrho) > \kappa$ then $\bigcup\{F_\alpha : \alpha \in \varrho\}$ is closed as well.)

Now, it is well known that $t(X) = T(X) = F(X)$ for a compact space X, where $F(X)$ denotes the supremum of the sizes of all free sequences in X, moreover it is easy to show that $F(X) = \omega$ if X is Lindelof and $T(X) = \omega$. Hence in both results mentioned above we consider spaces in which limitations for the sizes of their free sequences are given. Our aim in this section is to show that this is the crucial assumption together with the caliber assumption.

In addition to the notation $F(X)$ (following [25, 1.22]) we shall also make use of the notation $\widehat{F}(X)$ that is defined as the smallest cardinal such that X has no free sequence of that size. Thus $\widehat{F}(X) \leq \varrho$ means that X contains no free sequence of size (or length) ϱ.

We shall also consider modifications of the notion of caliber to pairs and triples of cardinals. If $\lambda \leq \kappa$ then the pair (λ, κ) is said to be a (pair) caliber of the space X (and this will be denoted by $(\lambda, \kappa) \in \mathrm{Cal}_2(X)$) if among any κ open subsets of X there are always λ many with non-empty intersection. Also, if $\mu \leq \lambda \leq \kappa$ then the triple (μ, λ, κ) is said to be a (triple) caliber of X (in symbols: $(\mu, \lambda, \kappa) \in \mathrm{Cal}_3(X)$ provided that among any κ open sets in X we can always find a collection of size λ such that any subcollection of this of size $< \mu$ has non-empty intersection.

Clearly, if $\lambda \in \mathrm{Cal}(X)$ then $(\lambda, \kappa) \in \mathrm{Cal}_2(X)$ for all $\kappa \geq \lambda$, moreover $(\lambda, \kappa) \in \mathrm{Cal}_2(X)$ implies $(\lambda, \lambda, \kappa) \in \mathrm{Cal}_3(X)$. We conclude the introduction of this section with the following simple result connecting the density with calibers.

Lemma 2.1. *Let X be a space with $T(X) < d(X) = \varrho = \mathrm{cf}(\varrho)$. Then $\varrho \notin \mathrm{Cal}(X)$.*

Proof. Clearly, the assumptions imply that we can write X in the form $X = \bigcup\{K_\alpha :$

$\alpha \in \varrho\}$, where each K_α is closed and $K_\beta \subsetneq K_\alpha$ if $\beta < \alpha < \varrho$. Then the family $\{X \setminus K_\alpha : \alpha \in \varrho$ of open subsets of X witnesses that $\varrho \notin \mathrm{Cal}(X)$. $\square$

2.1 X has no "long" free sequences

The main result of this section was directly motivated by [7, Theorem 5.1].

Theorem 2.2. *Assume that for the space X and the infinite cardinals $\lambda \le \kappa$ we have both $\widehat{F}(X) \le \lambda$ and $(\lambda, \lambda, \kappa) \in \mathrm{Cal}_3(X)$. Then there is a cardinal $\mu < \kappa$ such that $d(X) \le \mu^{<\lambda}$.*

Proof. Assume, indirectly, that $d(X) > \mu^{<\lambda}$ for all cardinals $\mu < \kappa$.

Let us fix a choice function φ on $\mathcal{P}(X) \setminus \{\emptyset\}$ and then define for $\alpha < \kappa$ subsets $Y_\alpha \subset X$ with $|Y_\alpha| \le |\alpha|^{<\lambda}$ and open sets U_α in X as follows:

Set $Y_0 = \emptyset$. If Y_α satisfying $|Y_\alpha| \le |\alpha|^{<\lambda}$ has been chosen then, by the indirect assumption, Y_α is not dense in X hence, as X is T_3, we can choose a non-empty open set U_α in X such that $\overline{Y}_\alpha \cap \overline{U}_\alpha = \emptyset$.

Next, let $\mathcal{H}_\alpha$ denote the family of all those subcollections $\mathcal{U} \subset \{U_\beta : \beta \le \alpha\}$ for which we have both $|\mathcal{U}| < \lambda$ and $\bigcap \mathcal{U} \ne \emptyset$, and then put

$$Y_{\alpha+1} = Y_\alpha \cup \left\{\varphi\left(\bigcap \mathcal{U}\right) : \mathcal{U} \in \mathcal{H}_\alpha\right\}.$$

Clearly, we have $|Y_{\alpha+1}| \le |Y_\alpha| + |\mathcal{H}_\alpha| \le |\alpha|^{<\lambda} + |\alpha+1|^{<\lambda} = |\alpha+1|^{<\lambda}$.

If α is limit and Y_β has been defined for all $\beta < \alpha$ such that $|Y_\beta| \le \beta|^{<\lambda}$, we simply set $Y_\alpha = \bigcup\{Y_\beta : \beta \in \alpha\}$; then we clearly have $|Y_\alpha| \le |\alpha|^{<\lambda}$ as well.

Applying $(\lambda, \lambda, \kappa) \in \mathrm{Cal}_3(X)$ to the family $\{U_\alpha : \alpha \in \kappa\}$ we can find a set $I \in [\kappa]^\lambda$ of order type $\mathrm{tp}(I) = \lambda$ such that for every $\alpha \in I$ we have

$$\bigcap \{U_\beta : \beta \in (\alpha + 1) \cap I\}.$$

Then the point $\varphi(\bigcap \{U_\beta : \beta \in (\alpha+1) \cap I\})$ is well-defined and $y_\alpha \in Y_{\alpha+1}$ by our construction. Clearly, this implies that $\{y_\beta : \beta \in \alpha \cap I\} \subset Y_\alpha$ whenever $qalpha \in I$, moreover $y_\gamma \in U_\alpha$ if $\gamma \in I$ and $\gamma \ge \alpha$, by definition of y_γ. Consequently we have

$$\bigcap \{U_\beta : \beta \in (\alpha + 1) \cap I\} \ne \emptyset$$

because $\overline{Y}_\alpha \cap \overline{U}_\alpha = \emptyset$, and thus $\{y_\alpha : \alpha \in I\}$ is a free sequence in X of size λ, contradicting that $\widehat{F}(X) \le \lambda$. $\square$

It is instructive to isolate the following particular instance of our theorem: $\lambda = \varrho^+$ and $\kappa = (2^\varrho)^+$ for some fixed cardinal ϱ. In this case for every cardinal $\mu < \kappa$ we have $\mu^{<\lambda} \le (2^\varrho)^+ = 2^\varrho$ and thus we obtain the following result.

Corollary 2.3. *If $F(X) \le \varrho$ and $(\varrho^+, \varrho^+, (2^\varrho)^+) \in \mathrm{Cal}_3(X)$ then $d(X) \le 2^\varrho$.*

For $\varrho = \omega$ this is clearly a strengthening of [7, Theorem 5.1]. As we have pointed out in the introduction, the assumption there, namely $T(X) = \omega$ together with the Lindelöfness of X, is really only needed to obtain $F(X) = \omega$. Ironically, as it turns out, the assumption $T(X) = \omega$ may be used to get further improvements on the bound for the density, at least under some extra assumptions on calibers and cardinal exponentiation.

Theorem 2.4. *Assume $2^\varrho = \varrho^{(+n)}$ for some $n < \omega$, moreover $F(X) \le \varrho, T(X) \le \varrho$ and $\{\varrho^+, \dots, \varrho^{(+n)}\} \subset \mathrm{Cal}(X)$. Then $d(X) \le \varrho$.*

Proof. First, as $\varrho 3+ \in \mathrm{Cal}(X)$ implies $(\varrho^+, \varrho^+, (2^\varrho)^+) \in \mathrm{Cal}_3(X)$, we conclude from Corollary 2.3 that $d(X) \le 2^\varrho = \varrho^{(+n)}$. But then $d(X) > \varrho$ would mean that $d(X) = \varrho^{(+i)}$ where $0 < i \le n$, hence by $T(X) \le \varrho$ and Lemma 2.1 we would have $\varrho^{(+i)} \notin \mathrm{Cal}(X)$, a contradiction. □

Note that if we have $n = 1$ in Theorem 2.4, i.e., if GCH holds at ϱ, then what we get is the following.

Corollary 2.5. *If $2^\varrho = \varrho^+$, moreover $F(X) \le \varrho$, $T(X) \le \varrho$ and $\varrho^+ \in \mathrm{Cal}(X)$ then $d(X) \le \varrho$.*

Again, for $\varrho = \omega$, this yields the following interesting partial strengthening of [7, Theorem 5.1]: If CH holds and X is Lindelof with $T(X) = \omega$ and $\omega_1 \in \mathrm{Cal}(X)$ then X is separable. It is an interesting open question whether or not this last statement remains valid without CH?

Now we formulate one more interesting Corollary of Theorem 2.4.

Corollary 2.6. *Assume that $\aleph_\omega$ is strong limit cardinal, moreover X is a space such that $T(X) = \omega$, $\widehat{F}(X) \le \aleph_\omega$ and $\aleph_n \in \mathrm{Cal}(X)$ for each n with $0 < n < \omega$. Then X is separable.*

Proof. Let us first deal with the case in which $\widehat{F}(X) < \aleph_\omega$, say $F(X) = \aleph_k$. Then we may apply Theorem 2.4 with $\varrho = \aleph_k$ and first conclude that $d(X) \le \aleph_k$. But now we also have $\aleph_i \in \mathrm{Cal}(X)$ for $1 \le i \le k$ that, by Lemma 2.1, implies $d(X) = \omega$ because we now have $T(X) = \omega$ as well.

Now let us assume that $\widehat{F}(X) = \aleph_\omega$. Let $\mathcal{H}$ be the family of all those non-empty open subsets H of X for which $\widehat{F}(\overline{H}) = \widehat{F}(\overline{U})$ is satisfied for every non-empty open subset U of H. Clearly, every non-empty open subset G of X has a subset $H \subset G$ with $\widehat{F}(\overline{H})$ minimal, hence $H qin \mathcal{H}$ (i.e., $\mathcal{H}$ is a π-base of X).

Next, we show that $\widehat{F}(\overline{H}) < \aleph_\omega$ whenever $H \in \mathcal{H}$. Assume, indirectly, that $\widehat{F}(\overline{H}) = \aleph_\omega$. There is a sequence $\langle U_n : n \in \omega \rangle$ of non-empty open subsets of H with $U_n \cap U_m = \emptyset$ if $n \ne m$. But then $H \in \mathcal{H}$ implies that, for every $n \in \omega$, we have $\widehat{F}(\overline{U}) = \aleph_\omega > \aleph_n$, whenever $U \subset U_n$, hence there is a free sequence S_n with $|S_n| = \aleph_n$ and $\overline{S}_n \subset U_n$. But then $S = \bigcup\{S_n : n \in \omega\}$ is clearly also a free sequence in X with $|S| = \aleph_\omega$, contradicting $\widehat{F}(X) = \aleph_\omega$.

To conclude the proof, let $\mathcal{C}$ be a maximal disjoint family of members of $\mathcal{H}$. Since $\omega_1 \in \mathrm{Cal}(X)$, $\mathcal{C}$ is countable, and by the first part of the proof we have $d(\overline{H}) = d(H) = \omega$ whenever $H \in \mathcal{H}$, because both $T(\overline{H}) = \omega$ and $\{\omega_n : n \in \omega \setminus \{0\}\} \subset \mathrm{Cal}(H)$ are clearly "inherited" by $\overline{H}$ from X. So X is separable because, as $\mathcal{H}$ is a π-base, $\bigcup \mathcal{C}$ is dense in X. □

Actually, the second part of the above proof can be avoided because, as it turns out, $\widehat{F}(X) = \aleph_\omega$ cannot occur under the assumptions of Corollary 2.6, namely if $\aleph_\omega$, is strong limit. Although this fact does not properly belong to the theme of this section, we provide a proof just for completeness. Before formulating the result, let us agree on the following: for a space X we denote by $\widehat{L}(X)$ the smallest cardinal κ such that every open cover of X has a subcover of size less than κ.

Theorem 2.7. *Assume $\lambda > cf(\lambda) = \omega$ and X is a (T_3!) space such that if $S \subset X$ is a free sequence with $|S| < \lambda$ then $\widehat{L}(\overline{S}) \le \lambda$. Then $\widehat{F}(X) \neq \lambda$.*

Proof. Assume that $\widehat{F}(X) \le \lambda$, we shall show that then $\widehat{F}(X) < \lambda$. To see this, first note that if $\langle K_n : n \in \omega\rangle$ is any sequence of closed sets in X such that $K_n \cap \overline{\bigcup\{K_m : m > n\}} = \emptyset$ for every $n \in \omega$ then there is a $\mu < \lambda$ such that $\widehat{F}(K_n) \le \mu$ for all large enough $n \in \omega$, since otherwise we could easily "put together" a free sequence of size λ in X.

Next we show that for every point $p \in X$ there is an open set U with $p \in U$ such that $\widehat{F}(\overline{U}) < \lambda$. Indeed, assume that $\widehat{F}(\overline{U}) = \lambda$ whenever $p \in U$.

We claim that for every $\mu < \lambda$ and for every open U with $p \in U$ there is a free sequence S in $\overline{U}$ with $|S| = \mu$, and $p \notin \overline{S}$. Indeed, let $S_0 \dotplus S_1$ be any free sequence in $\overline{U}$ of order type $\mu \dotplus \mu$, where $\dotplus$ denotes addition of order types. Then either $p \notin \overline{S}_0$ or $p \notin \overline{S}_1$.

Using the claim above we define a sequence $\langle S_n : n \in \omega\rangle$ of free sequences with $|S_n| = \lambda_n \nearrow \lambda$ and a sequence of open sets $\langle U_n : n \in \omega\rangle$ such that $\overline{S}_n \subset U_n,\ p \notin U_n$ and $U_n \cap S_m = \emptyset$ for $n < m$.

If $S_k,\ U_k$ have been de?ned for all $k < n$ with these properties then $p \notin \bigcup\{\overline{U}_k : k < n\}$, and so we have an open neighbourhood W_n of p with $\overline{W}_n \cap \bigcup\{\overline{U}_k : k < n\} = \emptyset$. Now we can choose a free sequence S_n with $p \notin \overline{S}_n \subset W_n$ and $|S_n| = \lambda_n$ by the claim and then the open set $U_n \subset W_n$ with $\overline{S}_n \subset \underline{U_n \text{ and } p \notin \overline{U}_n}$ since X is T_3. Now the sequence $\langle S_n : n \in \omega\rangle$ clearly satisfies $\overline{S}n \cap \overline{\bigcup\{\overline{S}_m : m > n\}} = \emptyset$ for all $n \in \omega$, and this is impossible by our introductory remark.

Now, it follows immediately that actually there is a cardinal $\mu < \lambda$ such that for every $p \in X$ there is an open neighbourhood U with $\widehat{F}(\overline{U}) \le \mu$. Indeed, otherwise we could choose distinct points $p_n \in X$ with $\widehat{F}(\overline{U}) > \lambda_n$ if $p_n \in U$ with U open, for all

$n \in \omega$. As X is T_3, we may assume that $\{p_n : n \in \omega\}$ forms a discrete subspace in X, and so we may also fix for each p_n a neighbourhood U_n so that $\{U_n : n \in \omega\}$ is pairwise disjoint. Let us now pick for every $n \in \omega$ a free sequence S_n in X with $|S_n| = \lambda_n$ and $\overline{S}_n \subset U_n$. Then we clearly have $\overline{S}_n \cap \overline{\bigcup\{\overline{S}_n : m > n\}} = \emptyset$ for each n, contradicting again our introductory remark.

Let H be any closed set in X with $\widehat{F}(H) = \lambda$ and let $S \subset H$ be a free sequence, then $|S| < \lambda$. Thus by our assumption we have $\widehat{L}(\overline{S}) \le \lambda$, hence S can be covered by a family $\mathcal{U}$ of less than λ many open sets U with $\widehat{F}(\overline{U}) \le \mu < \lambda$ for $U \in \mathcal{U}$. Clearly this implies that every free sequence in X that is contained in $\bigcup\mathcal{U}$ has size at most $\mu \cdot |\mathcal{U}| < \lambda$, consequently by $\widehat{F}(H) = \lambda$ we have $\widehat{F}(H \setminus \bigcup\mathcal{U}) = \lambda$ as well.

This fact allows us again to define inductively a sequence $\langle S_n : n \in \omega\rangle$, where S_n is free in X with $|S_n| = \lambda_n$ for all n and also satisfying

$$\overline{S}_n \cap \overline{\bigcup\{\overline{S}_m : m > n\}} = \emptyset$$

for all $n \in \omega$, arriving at a contradiction as above. □

Let us note that if λ is strong limit then we have

$$\widehat{L}(\overline{S}) \le w(\overline{S})^+ \le \left(2^{|S|}\right)^+ < \lambda$$

for any subset S of X with $|S| < \lambda$, hence the second condition of Theorem 2.7 holds trivially.

Also, by an old result of Hajnal and Juhász (see [25, 4.3]), we can never have $\widehat{s}(X) = \lambda$ for a singular cardinal λ of cofinality ω if X is T_3, where of course $\widehat{s}(X)$ denotes the smallest cardinal κ such that X has no discrete subspace of size κ. The question if this also holds for the cardinal function F instead of s remains open.

2.2 X is the union of "few" compact subspaces with no "long" free sequences

In this section we are going to prove two main theorems whose proofs, while similar to each other, are quite different from that of Theorem 2.2. Both will use a caliber assumption on a space X and an assumption that X is union of a "small" number of compact subspaces, all without "long" free sequences, and conclude that the density of X is "small". Of course, in view of the equality of F and t for compact spaces, we could also formulate this by saying that X is the union of a "small" number of compact subspaces of "small" tightness. Now, the precise statements of the results read as follows.

Theorem 2.8. *Assume that* $X = \bigcup\{C_\alpha : \alpha \in \kappa\}$ *where* C_κ *is compact and* $\widehat{F}(C_\alpha) \le \kappa$ *for each* $\alpha \in \kappa$*, moreover* $\kappa \in \mathrm{Cal}(X)$*. Then* $d(X) < \kappa$*.*

Theorem 2.9. *Assume that $\omega < \mu \le \kappa$ and $X = \bigcup \mathcal{C}$ where $|\mathcal{C}| < \kappa$ and every $C \in \mathcal{C}$ is compact with $\widehat{F}(C) \le \mu$, moreover $(\mu, \kappa) \in \mathrm{Cal}_2(X)$. Then $d(X) < \kappa$.*

Before we can prove these results, we need to introduce a new concept that will turn out to play an important role in producing free sequences.

Definition 2.10. Let X be a space and $\mathcal{H} \subset \mathcal{P}(X)$ be any family of subsets of X. We say that the sequence of pairs

$$\vec{s} = \langle \langle U_\alpha, V_\alpha \rangle : \alpha \in \eta \rangle$$

is *loose* over $\mathcal{H}$ if the following two conditions are satisfied:

(i) $U_\alpha,\ V_\alpha$ are non-empty open sets in X with $\overline{U}_\alpha \cap \overline{V}_\alpha = \emptyset$ for every $\alpha < \eta$;

(ii) if $a, b \in [\eta]^{<\omega}$ with $a < b$ and $H \in \mathcal{H}$ are such that

$$\bigcap \{U_\alpha : \alpha \in a\} \cap \bigcap \{V_\beta : \beta \in b\} \cap H \neq \emptyset$$

then for every γ with $b < \gamma < \eta$ we have

$$\bigcap \{U_\alpha : \alpha \in a\} \cap \bigcap \{V_\beta : \beta \in b\} \cap V_\gamma \cap H \neq \emptyset$$

as well.

To shorten exposition, let us introduce here the following piece of notation: if $a, b \in [\eta]^{<\omega}$ then

$$W(a, b) = \bigcap \{U_\alpha : \alpha \in a\} \cap \bigcap \{V_\beta : \beta \in b\}$$

Thus (ii) says that $W(a, b) \cap H \neq \emptyset$ implies $W(a, b \cup \{\gamma\}) \cap H \neq \emptyset$ whenever $a, b \in [\eta]^{<\omega},\ a < b < \gamma < \eta$ and $H \in \mathcal{H}$.

Let us note that if $\vec{s}$ is loose over $\mathcal{H}$ then so is every subsequence of $\vec{s}$, moreover $\vec{s}$ is loose over any subfamily of $\mathcal{H}$.

The following result tells us how we can obtain "long" loose sequences over appropriate families of sets in a given space.

Lemma 2.11. *Let X be a space and $\mathcal{H} \subset \mathcal{P}(X)$ be a family of subsets of X.*

(A) if $|\mathcal{H}| < d(X)$ then there is a loose sequence over $\mathcal{H}$ of length $d(X)$;

(B) if, in addition, $\mathcal{H} \subset \tau(X)$, i.e., all elements of $\mathcal{H}$ are open in X then $|\mathcal{H}| < \pi(X)$ implies that there is a loose sequence over $\mathcal{H}$ of length $\pi(X)$.

Proof.

(A) We are going to define by transfinite recursion on $\alpha < d(X)$ non-empty open sets $U_\alpha,\ V_\alpha$ as follows. Assume that $\alpha < d(X)$ and $\vec{s}_\alpha = \langle\langle U_\beta, V_\beta\rangle : \beta \in \alpha\rangle$ has been defined already in such a way that $\vec{s}_\alpha$ is loose over $\mathcal{H}$. Consider the family

$$\mathcal{W}_\alpha = \{W(a,b) \cap H : a,b \in [\alpha]^{<\omega} \text{ and } H \in \mathcal{H}\} \setminus \{\emptyset\}$$

then clearly $|\mathcal{W}| < d(X)$. Therefore, we can find a non-empty open set G_α such that $W \setminus G_\alpha \neq \emptyset$ for every $W \in \mathcal{W}_\alpha$. Since X is T_3, we can choose two (non-empty) open sets $U_\alpha,\ W_\alpha$ such that $U_\alpha \subset \overline{U}_\alpha \subset \overline{W}_\alpha \subset G_\alpha$. Set $V_\alpha = X \setminus \overline{W}_\alpha$, it is obvious then that $\vec{s}_{\alpha+1} = \vec{s}_\alpha \frown \langle U_\alpha, V_\alpha\rangle$ will also be loose over $\mathcal{H}$, hence $\vec{s} = \langle\langle U_\alpha, V_\alpha\rangle : \alpha \in d(X)\rangle$ is as required.

(B) The proof in this case is the same as in case (A), the only difference being in noticing that for each $\alpha < \pi(X)$ the family $\mathcal{W}_\alpha$ will consist of open sets and so $|\mathcal{W}_\alpha| < \pi(X)$ will suffice to imply the existence of an open set G_α as above. □

The following easy result, that actually gives an alternative characterization of free sequences, will be used to produce free sequences from appropriate loose ones.

Lemma 2.12. *Let X be any space (T_3 is not used here!) and $\langle p_\alpha : \alpha \in \eta\rangle$ be a sequence of points of X. Then the following statements (1) and (2) are equivalent:*

(1) $\langle p_\alpha : \alpha \in \eta\rangle$ is a free sequence in X;

(2) there is a sequence $\langle\langle K_\alpha, L_\alpha\rangle : \alpha \in \eta\rangle$ of pairs of disjoint closed sets X such that

$$p_\alpha \in \bigcap\{K_\xi : \xi \le \alpha\} \cap \bigcap\{L_\xi : \alpha < \xi < \eta\}$$

for all $\alpha \in \eta$.

Proof.

If $\langle p_\alpha : \alpha \in \eta\rangle$ is free then clearly (2) will be satisfied with $K_\alpha = \overline{\{p_\xi : \xi \ge \alpha\}}$ and $L_\alpha = \overline{\{p_\xi : \xi < \alpha\}}$.

In the other direction, if $\langle p_\alpha : \alpha \in \eta\rangle$ and $\langle\langle K_\alpha, L_\alpha\rangle : \alpha \in \eta\rangle$ satisfy (2) then for every $\alpha \in \eta$ we have $\{p_\xi : \xi < \alpha\} \subset L_\alpha$ and $\{p_\xi : \xi \ge \alpha\} \subset K_\alpha$, hence $\langle p_\alpha : \alpha \in \eta\rangle$ is free in X. □

Next we first give the proof of Theorem 2.9.

Proof of Theorem 2.9. Assume, indirectly, that $\kappa \le d(X)$, then Lemma 2.11 (A) implies the existence of a sequence $\vec{s} = \langle\langle U_\alpha, V_\alpha\rangle : \alpha \in \kappa\rangle$ that is loose over $\mathcal{C}$. Since $(\mu, \kappa) \in \mathrm{Cal}_2(X)$, there is a set $I \in [\kappa]^\mu$ with $\mathrm{tp}(I) = \mu$ such that $\bigcap\{U_\alpha : \alpha \in I\} \neq \emptyset$, hence as $X = \bigcup \mathcal{C}$, there is some $C \in \mathcal{C}$ with

$$\bigcap\{U_\alpha : \alpha \in I\} \cap C \neq \emptyset$$

as well.

Now $\vec{s}$ restricted to I is loose over $\{C\}$ and clearly for every $a \in [I]^{<\omega}$ we have

$$\bigcap\{U_\alpha : \alpha \in a\} \cap C \neq \emptyset.$$

Consequently, an easy induction yields that if $a, b \in [I]^{<\omega}$ with $a < b$ then

$$W(a,b) \cap C \neq \emptyset.$$

Since C is compact, this clearly implies that for every $\alpha \in I$ we have

$$\bigcap\{\overline{U}_\xi : \xi \in (\alpha+1) \cap I\} \cap \bigcap\{\overline{V}_\xi : \xi \in I \setminus (\alpha+1)\} \cap C \neq \emptyset,$$

hence applying Lemma 2.12 to the sequence

$$\big\langle\big\langle \overline{U}_\alpha \cap C, \overline{V}_\alpha \cap C\big\rangle : \alpha \in I\big\rangle$$

we get a free sequence of length $\operatorname{tp}(I) = \mu$ in C, contradicting $\widehat{F}(C) \leq \mu$. □

Note that if X is compact with $\widehat{F}(X) \leq \mu$, i.e., we can have $\mathcal{C} = \{X\}$ in Theorem 2.9, then Lemma 2.11 (B) can be applied in the above proof, hence we get the following result that strengthens Shapirovskiĭ's result mentioned in the introduction:

Corollary 2.13. *If X is compact, $\widehat{F}(X) \leq \mu$, and $(\mu, \kappa) \in \mathrm{Cal}_2(X)$ then $\pi(X) < \kappa$.*

Now, to prove Theorem 2.8, we actually need a slight technical variation of the notion of a loose sequence.

Definition 2.14. Assume that $\mathcal{H} = \{H_\alpha : \alpha \in \eta\} \subset \mathcal{P}(X)$. We say that the sequence $\langle\langle U_\alpha, V_\alpha\rangle : \alpha \in \eta\rangle$ is *weakly loose* over $\mathcal{H}$ if the following two conditions are satisfied:

(i) U_α, V_α are non-empty open sets in X and $\overline{U}_\alpha \cap \overline{V}_\alpha = \emptyset$ for $\alpha \in \eta$;

(ii) if $a, b \in [\eta]^{<\omega}$ and $\delta \in \eta$ with $a < b < \delta$ satisfy

$$W(a,b) \cap H_\delta \neq \emptyset$$

then for every γ with $\delta < \gamma < \eta$ we have

$$W(a, b \cup \{\gamma\}) \cap H_\delta \neq \emptyset$$

as well.

We can now formulate a lemma that corresponds to Lemma 2.11 for weakly loose sequences.

Lemma 2.15.

(A) If $\kappa \le d(X)$ and $\mathcal{H} = \{H_\alpha : \alpha \in \kappa\} \subset \mathcal{P}(X)$ then there is a weakly loose sequence of length κ over $\mathcal{H}$.

(B) If qkappa $\le \pi(X)$ and $\mathcal{H} = \{H_\alpha : \alpha \in \kappa\} \subset \tau(X)$ then there is a weakly loose sequence of length κ over $\mathcal{H}$.

Proof. The proof is almost the same as that of Lemma 2.11, only the de?nition of $\mathcal{W}_\alpha$ needs to be modified in it as follows:

$$\mathcal{W}_\alpha = \{W(a, b) \cap H_\delta : a, b \in [\alpha]^{<\omega} \text{ and } \delta < \alpha\} \setminus \{\emptyset\}.$$

$\square$

Now, the proof of Theorem 2.8 is again quite similar to that of Theorem 2.9, the difference is in using weakly loose sequences instead of loose ones.

Proof of Theorem 2.8. Assume $\kappa \le d(X)$. Apply Lemma 2.15 (A) to get a weakly loose sequence $\vec{s} = \langle\langle U_\alpha, V_\alpha\rangle : \alpha \in \kappa\rangle$ over $\{C_\alpha : \alpha \in \kappa\}$. Since $\kappa \in \mathrm{Cal}(X)$, there is a set $I \in [\kappa]^\kappa$ and an ordinal $\delta \in \kappa$ such that $\delta < I$ and $\bigcap\{U_\alpha : \alpha \in I\} \cap C_\delta \ne \emptyset$.

The compactness of C_δ together with the fact that $\vec{s}$ is weakly loose over $\{C_\alpha : \alpha \in \kappa\}$ now easily imply that

$$\bigcap\{\overline{U}_\xi : \xi \in (\alpha + 1) \cap I\} \cap \bigcap\{\overline{V}_\xi : \xi \in I \setminus (\alpha + 1)\} \cap C_\delta \ne \emptyset,$$

for all $\alpha \in I$. Thus Lemma 2.12 applied to the sequence

$$\big\langle\langle \overline{U}_\alpha \cap C_\delta, \overline{V}_\alpha \cap C_\delta\big\rangle : \alpha \in I\big\rangle$$

gives us a free sequence of length κ in C_δ, a contradiction. $\square$

Perhaps the most interesting particular case of Theorem 2.8 is the following: If X is the union at most ω_1 compact subsets of countable tightness and $\omega_1 \in \mathrm{Cal}(X)$ then X is separable. This result seems to be new even for the case in which X is σ-compact and countably tight.

Now we formulate a result that on one hand generalizes this last observation, and on the other relates to Theorem 2.8 in the same way as Theorem 2.4 does to Theorem 2.2.

Theorem 2.16. *Let X be a space and ϱ be a cardinal such that $T(X) \le \varrho$, and $n > 0$ be a natural number such that $X = \bigcup \mathcal{C}$ with $|\mathcal{C}| \le \varrho^{(+n)}$, where each $C \in \mathcal{C}$ is compact, moreover $\varrho^{(+i)} \in \mathrm{Cal}(X)$ whenever $0 < i \le n$. Then $d(X) \le \varrho$.*

Proof. First, note that for every $C \in \mathcal{C}$ we have

$$\widehat{F}(C) \leq t(C)^+ = T(C)^+ \leq \varrho^+ \leq \varrho^{(+n)},$$

hence Theorem 2.8 can be applied with $\kappa = \varrho^{(+n)}$ to conclude that $d(X) < \varrho^{(+n)}$. But then from $T(X) \leq \varrho$ and $\varrho^{(+i)} \in \mathrm{Cal}(X)$ for $0 < i < n$ it follows, just like in the proof of Theorem 2.4, that $d(X) \leq \varrho$. □

Finally, we shall give an example which shows that the results of this section cannot be strengthened in the direction of Corollary 2.13, i.e., we cannot replace in them $d(X)$ by $\pi(X)$ or even by $\delta(X)$. Let us recall that

$$\delta(X) = \sup\{d(Y) : \overline{Y} = X\},$$

hence one clearly has

$$d(X) \leq \delta(X) \leq \pi(X)$$

for any space X.

Example 2.17. *There is a space X which is the union of countably many compact sets of countable tightness, moreover X is separable, consequently every κ with $\mathrm{cf}(\kappa) > \omega$ is a caliber of X, but X has a dense subspace Y with $d(Y) = \delta(X) = \mathfrak{c} = 2^\omega$.*

Indeed, let S be a countable dense subset of the Cantor cube $2^\mathfrak{c}$ and let Y be the σ-product in $2^\mathfrak{c}$, i.e.,

$$Y = \{y \in 2^\mathfrak{c} : |\{\alpha : y(\alpha) = 1\}| < \omega\}.$$

Then $Y = \bigcup\{Y_n : n \in \omega\}$, where

$$Y_n = \{y \in 2^\mathfrak{c} : |\{\alpha : y(\alpha) = 1\}| \leq n\}$$

is both compact and countably tight. Consequently the space $X = S \cup Y$, as a subspace of $2^\mathfrak{c}$, is clearly as required.

Of course, if we only want an example with $\pi(X) > \omega$ then a countable space with uncountable π-weight will do. Let us also remark that if X itself has countable tightness then its separability, i.e., $d(X) = \omega$, implies $\delta(X) = \omega$.

3 On order of π-bases

3.1 Projective π-character bounds the order of a π-base

Let us start by recalling a few definitions and basic facts. A π-base $\mathcal{B}$ of a space X (resp. a local π-base at a point $x \in X$) is a family of non-empty open sets such that every non-empty open set (resp. every neighbourhood of x) includes some member of $\mathcal{B}$. The π-weight $\pi(X)$ of X is the smallest infinite cardinal such that X has a π-base of at most that cardinality. The π-character $\pi\chi(x, X)$ of x in X is the smallest cardinality of a local π-base at $x \in X$ and

$$\pi\chi(X) = \sup\{\pi\chi(x, X) : x \in X\}$$

is the π-character of the space X. Finally, the local tightness at $x \in X$ is the smallest cardinal κ such that if x belongs to the closure $\overline{A}$ of a set A then there is $B \subset A$ with $|B| \leq \kappa$ and $x \in \overline{B}$; moreover

$$t(X) = \sup\{t(x, X) : x \in X\}$$

is the tightness of the space X.

Šhapirovskii proved the following two important results concerning these cardinal functions for compacta: If X is compact then $\pi\chi(X) \leq t(X)$, moreover X has a π-base $\mathcal{B}$ of order $\leq t(X)$, i.e. every point of X is contained in at most $t(X)$-many members of $\mathcal{B}$. (A trivial consequence is that if $t(X)^+$ is a caliber of X, i.e. among $t(X)^+$-many open sets there always are $t(X)^+$-many with non-empty intersection, then X has a π-base of cardinality at most $t(X)$.) The first result was proved in [54], alternative proofs were given in [5] and [25]. The second result first appeared in [56] and then in [58]. A very short and elegant new proof (using a variant of Shapirovskii's "algebraic" approach to free sequences) was presented in [67].

Arhangel'skii has recently introduced in [6] the concept of a space of *countable projective π-character* and noticed that any compact space of countable tightness has countable projective π-character. Then he showed that a compact space of countable projective π-character that has ω_1 as a caliber is separable (or equivalently: has a countable π-base), thereby strengthening the above consequence of Shapirovskii's result for countably tight compacta.

In this paper we introduce the general concept of projective π-character and give the following significant generalization of Shapirovskii's full result: Any Tychonov space has a π-base of order at most the projective π-character of the space. Not only is this result stronger for compacta, because it replaces tightness with projective π-character that is smaller, but somewhat surprisingly it extends to all Tychonov spaces.

Let φ be any cardinal function defined on a class $\mathcal{C}$ of topological spaces. We define the projective version $p\varphi$ of φ on $\mathcal{C}$ as follows. For any $X \in \mathcal{C}$ we let $p\varphi(X)$ be the

the supremum of the values $\varphi(Y)$ where Y ranges over all continuous images of X belonging to $\mathcal{C}$. In particular, we shall consider the case in which $\varphi = \pi\chi$, the π-character defined on the class of Tychonov spaces. It is easy to show that then a Tychonov space X has countable projective π-character in the sense of [6] iff $p\,\pi\chi(X) \le \omega$.

Also, as was already mentioned before, if X is compact Hausdorff then we have $p\,\pi\chi(X) \le t(X)$. In fact, this follows because $t(Y) \le t(X)$ for any continuous image of X and, by Shapirovskii's first result above, $\pi\chi(Y) \le t(Y)$ for every compact Y. But are $p\,\pi\chi(X)$ and $t(X)$ really different? Arhangel'skii asked, more specifically, if there is a compactum of countable projective π-character that is not countably tight, see [6], problem 7. Our next example yields such a compactum.

Example 3.1. *Let X be a compactification of ω whose remainder is (homeomorphic to) the ordinal $\omega_1 + 1$. Then $p\,\pi\chi(X) \le \omega < t(X)$.*

Proof. It is obvious that $t(\omega_1, X) = t(X) = \omega_1$. To see $p\,\pi\chi(X) \le \omega$, consider any continuous surjection $f : X \to Y$. If $f(\omega_1) = p$ is an isolated point in Y then there is an $\alpha < \omega_1$ such that f is constant on the interval $[\alpha, \omega_1]$, hence Y is countable and compact and so, trivially, $\pi\chi(Y) \le w(Y) = \omega$.

If, however, p is not isolated then Y has a countable dense subset S with $p \notin S$. So there is a closed G_δ set F such that $p \in F \subset Y \setminus S$ and again we can find an $\alpha < \omega_1$ such that $f[\alpha, \omega_1] \subset F$. But then $G = Y \setminus F$ is countable and dense open in Y, moreover $w(G) = \omega$ because every countable and locally compact space is second countable. So we have $\pi\chi(Y) \le \pi(Y) = w(G) = \omega$. □

We recall from [25] that $\pi sw(X)$ denotes the π-*separating weight* of a space X, that is the minimum order of a π-base of X, see p. 74 of [25].

With this we may now formulate our main result in this section as follows.

Theorem 3.2. *For any Tychonov space X we have $\pi sw(X) \le p\,\pi\chi(X)$. In particular, any Tychonov space of countable projective π-character has a point-countable π-base.*

Our proof of theorem 3.2 will go along similar lines as Shapirovskii's proof of the weaker result $\pi sw(X) \le t(X)$ for compact spaces. The main idea of that was to show that the compactum X admits an irreducible map onto a subspace of the $\Sigma_{t(X)}$-power of the unit interval. The role of irreducible maps in our proof will be played by a new, more general, type of maps that we shall call π-irreducible. So we shall first define and deal with these maps. (The referee has pointed out to us that [58] is an excellent source concerning Shapirovskii's original method.)

Definition 3.3. Let f be a continuous map of X *onto* Y. We say that the map f is π-irreducible if for every proper closed subset $F \subset X$ its image $f[F]$ is not dense in Y.

Clearly, an onto map f is π-irreducible iff the f-image of a non-dense set is non-dense. Also, it is obvious that a closed map is π-irreducible iff it is irreducible, consequently the two concepts coincide for maps between compact Hausdorff spaces.

The following proposition will be used in the proof of theorem 3.2 and explains our terminology.

Proposition 3.4. *Let f be a continuous map of X onto Y. Then the following five statements (1)–(5) are equivalent.*

(1) f is π-irreducible ;

(2) for every π-base $\mathcal{B}$ of X and for every $B \in \mathcal{B}$ the f-image of its complement, $f[X \setminus B]$, is not dense in Y;

(3) there is a π-base $\mathcal{B}$ of X such that for every $B \in \mathcal{B}$ the f-image $f[X \setminus B]$ is not dense in Y;

(4) for every π-base $\mathcal{C}$ of Y the family $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a π-base of X ;

(5) there is a π-base $\mathcal{C}$ of Y such that $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a π-base of X .

Proof. We shall show (3)⇒(4) and (5)⇒(1) only because the other three implications of the cycle are trivial.

So, let $\mathcal{B}$ be as in (3) and $\mathcal{C}$ be any π-base of Y. For every non-empty open set U in X choose $B \in \mathcal{B}$ with $B \subset U$. Then there is a $C \in \mathcal{C}$ such that $C \cap f[X \setminus B] = \emptyset$, and hence $f^{-1}(C) \subset B \subset U$.

Now, let $\mathcal{C}$ be as in (5) and F be a proper closed subset of X. Then there is a $C \in \mathcal{C}$ with $F \cap f^{-1}(C) = \emptyset$, consequently we have $f[F] \cap C = \emptyset$ and so $f[F]$ is not dense in Y. □

Corollary 3.5. *If $f : X \to Y$ is π-irreducible then $\pi(X) = \pi(Y)$.*

Proof. $\pi(X) \le \pi(Y)$ is immediate from part (4) of proposition 3.4. To see $\pi(X) \ge \pi(Y)$ first note that for any non-empty open $U \subset X$ the interior of $f[U]$ in Y is non-empty. So for any π-base $\mathcal{B}$ of X the family $\{\underset{Y}{\int}(f[B]) : B \in \mathcal{B}\}$ is a π-base of Y. Indeed, this is because if V is non-empty open in Y and $B \in \mathcal{B}$ with $B \subset f^{-1}(V)$ then $f[B] \subset V$. □

We now consider another key ingredient of the proof of our main result: certain specially embedded subspaces of Tychonov cubes. As usual, we shall denote the unit interval $[0, 1]$ by I. The members of the Tychonov cube I^κ will be construed as functions from κ to I. So if $x \in I^\kappa$ and $\alpha < \kappa$ then $x \restriction \alpha$ is the projection of x to the subproduct I^α.

Definition 3.6. We say that $Y \subset I^\kappa$ is 0-embedded in the Tychonov cube I^κ if

$$\{y \restriction \alpha : y \in Y \text{ and } y(\alpha) = 0\}$$

is dense in the projection $Y \restriction \alpha = \{y \restriction \alpha : y \in Y\}$ for every $\alpha < \kappa$.

We now present two results concerning 0-embedded subspaces of Tychonov cubes which will be crucial in the proof of our main theorem and are also interesting in themselves.

Theorem 3.7. *Assume that Y is 0-embedded in the Tychonov cube I^κ where κ is a regular cardinal and $y \in Y$ is such that $y(\alpha) > 0$ for all $\alpha < \kappa$. Then $\pi\chi(y, Y) = \kappa$.*

Proof. Of course, only $\pi\chi(y, Y) \geq \kappa$ needs to be proven. To see this, let $\mathcal{U}$ be any family of elementary open sets in I^κ such that $|\mathcal{U}| < \kappa$ and $U \cap Y \neq \emptyset$ for all $U \in \mathcal{U}$. Every elementary open set $U \in \mathcal{U}$ is supported by a finite subset of κ, hence the regularity of κ implies the existence of an ordinal $\alpha < \kappa$ such that the support of each $U \in \mathcal{U}$ is included in α.

Since Y is 0-embedded in I^κ, this implies that for every $U \in \mathcal{U}$ we may pick a point $y_U \in U \cap Y$ such that $y_U(\alpha) = 0$. But then $y(\alpha) > 0$ clearly implies that the point y is not in the closure of the set $\{y_U : U \in \mathcal{U}\}$, consequently $\mathcal{U}$ cannot be a local π-base at y in Y, completing the proof. □

From theorem 3.7 we can immediately obtain the following useful corollary about the projective π-character of 0-embedded subspaces of Tychonov cubes.

Corollary 3.8. *If Y is 0-embedded in the Tychonov cube I^κ then for every non-isolated point $y \in Y$ we have*

$$p\,\pi\chi(y, Y) \geq \big|\{\alpha : y(\alpha) > 0\}\big|,$$

and if $y \in Y$ is isolated then $\{\alpha : y(\alpha) > 0\}$ is finite.

Our next result shows that every Tychonov space admits a π-irredu-cible map onto a suitable 0-embedded subspace of a Tychonov cube.

Theorem 3.9. *Let X be any Tychonov space of π-weight $\pi(X) = \kappa$. Then there is a π-irreducible map f of X onto a 0-embedded subspace Y of the Tychonov cube I^κ.*

Proof. To begin with, let us choose a π-base $\mathcal{B}$ of X with $|\mathcal{B}| = \kappa$ and fix a well-ordering $\prec$ of $\mathcal{B}$ of order-type κ.

We shall define by transfinite induction on $\alpha < \kappa$ the co-ordinate maps

$$g_\alpha = p_\alpha \circ f : X \to I,$$

where $p_\alpha(y) = y(\alpha)$ is the αth co-ordinate projection, and sets $B_\alpha \in \mathcal{B}$. So assume that $\alpha < \kappa$ and for all $\beta < \alpha$ the maps $g_\beta : X \to I$ and the sets $B_\beta \in \mathcal{B}$ have been defined.

Let $f_\alpha : X \to I^\alpha$ be the map whose βth co-ordinate map is g_β for all $\beta < \alpha$ and set $Y_\alpha = f_\alpha[X]$. Then, in view of corollary 3.5, the map $f_\alpha : X \to Y_\alpha$ cannot be π-irreducible because $\pi(Y_\alpha) < \kappa = \pi(X)$, hence using part (2) of proposition 3.4 there is a member $B \in \mathcal{B}$ for which $f_\alpha[X \setminus B]$ is dense in Y_α. Let B_α be the $\prec$-first such member of $\mathcal{B}$. We then define $g_\alpha : X \to I$ as any continuous function that is identically 0 on $X \setminus B_\alpha$ and takes the value 1 at some point in B_α. As was intended, with the induction completed we let $f : X \to I^\kappa$ be the unique map having the g_α for $\alpha < \kappa$ as its co-ordinate functions and we also set $Y = f[X]$.

Note first that if $\beta < \alpha$ then $B_\beta \prec B_\alpha$. Indeed, since we have $Y_\beta = Y_\alpha \restriction \beta$, the density of $f_\alpha[X \setminus B_\alpha]$ in Y_α implies that $f_\beta[X \setminus B_\alpha]$ is dense in Y_β, hence $B_\alpha \prec B_\beta$ would contradict the choice of B_β. Moreover, by our construction, $f_{\beta+1}[X \setminus B_\beta]$ is not dense in $Y_{\beta+1}$ and consequently $f_\alpha[X \setminus B_\beta]$ is not dense in Y_α, which implies $B_\alpha \neq B_\beta$.

Since $\mathcal{B}$ is of order type κ under $\prec$, it follows from this that for every $B \in \mathcal{B}$ there is an $\alpha < \kappa$ with $B \prec B_\alpha$. But then, by the choice of B_α we have that $f_\alpha[X \setminus B]$ is not dense in $Y_\alpha = Y \restriction \alpha$ and hence $f[X \setminus B]$ cannot be dense in Y. Using part (3) of proposition 3.4 this implies that f is indeed a π-irreducible map of X onto Y.

Finally, by our construction, for every $\alpha < \kappa$ the image $f_\alpha[X \setminus B_\alpha]$ is dense in $Y_\alpha = Y \restriction \alpha$, moreover we have

$$f_\alpha[X \setminus B_\alpha] \subset \{y \restriction \alpha : y \in Y \text{ and } y(\alpha) = 0\},$$

consequently Y is indeed 0-embedded in I^κ. □

Let us now recall that the κ-th Σ_λ-power of I, denoted by $\Sigma_\lambda(I, \kappa)$, is the subspace of I^κ consisting of all points whose support is of size at most λ. The support of a point $y \in I^\kappa$ is the set $\{\alpha < \kappa : y(\alpha) > 0\}$. Thus, from theorem 3.9 and from corollary 3.8, moreover from the trivial fact that $p\,\pi\chi(Y) \le p\,\pi\chi(X)$ if Y is any continuous image of X, we immediately obtain the following result.

Corollary 3.10. *If X is a Tychonov space such that $\pi(X) = \kappa$ and $p\,\pi\chi(X) = \lambda$ then some π-irreducible image Y of X embeds into $\Sigma_\lambda(I, \kappa)$.*

This corollary is clearly a strengthening of the following result of Shapirovskii from [56] (see also 3.22 of [25]) : If X is compact Hausdorff then some irreducible image of X embeds into a $\Sigma_{t(X)}$-power of I.

The proof of our main theorem 3.2 can now be easily established by recalling the following result of Shapirovskii from [56] (see also[25], 3.24).

Theorem (Shapirovskii)**.** *If the space Y embeds into a Σ_λ-power of I then $\pi sw(Y) \le \lambda$.*

Proof of theorem 3.2. Now, to prove theorem 3.2, consider any non-discrete Tychonov space X. By corollary 3.10 then X has a π-irreducible image Y that embeds into a Σ_λ-power of I, where $\lambda = p\,\pi\chi(X)$. By the previous theorem of Shapirovskii then the space Y has a π-base $\mathcal{C}$ of order at most λ. But by part (4) of proposition 3.4, then $\{f^{-1}(C) : C \in \mathcal{C}\}$ is a π-base of X that clearly has the same order as $\mathcal{C}$. □

The following result is then an immediate consequence of theorem 3.2.

Corollary 3.11. *Let X be any Tychonov space and $\kappa > p\pi\chi(X)$ be a cardinal such that κ is a caliber of X. Then $\pi(X) < \kappa$.*

Since $t(X) \geq p\pi\chi(X)$ for a compact Hausdorff space X, this corollary implies Shapirovskii's theorem saying that if $t(X)^+$ is a caliber of such a space X then $\pi(X) \leq t(X)$. Moreover, it also extends from compacta to all Tychonov spaces Arhangel'skii's result from [6] saying that spaces of countable projective π-character and having ω_1 as a caliber are separable.

Let us conclude this paper by pointing out that neither theorem 3.2 nor corollary 3.11 remain valid if the projective π-character $p\pi\chi$ is replaced by simple π-character $\pi\chi$ in them. In fact, it has recently been shown in [35] that there are even first countable spaces whose π-separating weight is as large as you wish. Moreover, in the same paper it was also shown that it is consistent to have first countable spaces with caliber ω_1 which have uncountable π-weight (or equivalently, density). However, since first countability implies countable tightness, none of these examples are (or could be) compact, so the following intriguing questions remain open.

Problem 3.12. *Let X be a compact Hausdorff space of countable π-character. Does X have a point-countable π-base? If, in addition, ω_1 is a caliber of X, is then X separable?*

3.2 First countable spaces without point-countable π-bases

V. Tkachuk in [65] has recently proved under CH that any first countable Hausdorff space that is Lindelöf or CCC has a point-countable π-base. (Actually, in [65] all spaces are assumed to be Tychonov, but the proof only needs Hausdorff.) Tkachuk's motivation was to extend (at least partially) Shapirovskii's celebrated ZFC result saying that any countably tight compactum has a point-countable π-base, from compact spaces to Lindelöf ones. So it was natural to ask if his use of CH was necessary. Also, 27 years after Shapirovskii's result was published, Tkachuk could not come up with even a consistent example of a first countable space not having a point-countable π-base.

Our aim here is to remedy this situation and provide ZFC (and several consistent) examples of first countable (Tychonov) spaces without point-countable π-bases, as well as examples which show that Tkachuk's CH results cannot be proved in ZFC alone. In this manner we succeeded in answering 7 of the 12 questions that were listed at the end of [65].

In what follows, we shall use the notation and terminology of [25]. In particular, $\pi sw(X)$ denotes the *π-separating weight* of X, that is the minimum order of a π-base of the space X, see p. 74 of [25]. Note that $\pi sw(X) \leq \omega$ is then equivalent to the statement: X has a point-countable π-base.

3.2.1 ZFC examples

The key to Tkachuk's above mentioned CH results in [65] was his Theorem 3.1 which says that if X has countable tightness and π-character, moreover $d(X) \le \omega_1$ then $\pi sw(X) \le \omega$. In his list of problems (Problem 4.11), Tkachuk asked if the assumption of countable tightness could be omitted here. It is immediate from our next result that this question has an affirmative answer.

Theorem 3.13. *Let X be any topological space with $d(X) \le \pi\chi(X)^+$. Then $\pi sw(X) \le \pi\chi(X)$.*

Proof. Let us set $\pi\chi(X) = \kappa$. If $d(X) \le \kappa$ then we even have $\pi(X) = \kappa$. So we may assume $d(X) = \kappa^+$ and, as is well-known, we may then fix a dense set $D = \{x_\alpha : \alpha < \kappa^+\}$ that is left-separated in this well-ordering. This means that for every $\alpha < \kappa^+$ there is a neighbourhood U_α of x_α with

$$\{x_\beta : \beta < \alpha\} \cap U_\alpha = \emptyset.$$

Let us now fix a local π-base $\mathcal{B}_\alpha$ of the point x_α such that $|\mathcal{B}_\alpha| \le \kappa$ and $B \subset U_\alpha$ whenever $B \in \mathcal{B}_\alpha$. Then $\mathcal{B} = \bigcup\{\mathcal{B}_\alpha : \alpha < \kappa^+\}$ is a π-base of X such that for every $x_\beta \in D$ we have

$$ord(x_\beta, \mathcal{B}) = |\{B \in \mathcal{B} : x_\beta \in B\}| \le \kappa.$$

We claim that then we have

$$ord(\mathcal{B}) = \sup\{ord(x, \mathcal{B}) : x \in X\} \le \kappa$$

as well. Assume, on the contrary, that $ord(x, \mathcal{B}) = \kappa^+$ for some point $x \in X$. Since $\pi\chi(x, X) \le \kappa$, this implies that there are κ^+-many members of $\mathcal{B}$ (containing x) that include a fixed non-empty open set V. This, however, is impossible because $D \cap V \neq \emptyset$. $\square$

We may now turn to our first aim that is to produce, in ZFC, first countable spaces without point-countable π-bases.

Theorem 3.14. *There is a first countable, 0-dimensional Hausdorff (hence Tychonov) space X with $\pi sw(X) \ge \aleph_\omega$.*

Proof. The underlying set of our space is $X = \prod\{\omega_n : n < \omega\}$. For $f, g \in X$ we write $f \le g$ to denote that $f(n) \le g(n)$ for all $n < \omega$. The topology τ that we shall consider on X will be generated by all sets of the form $U_n(f)$ (with $f \in X$ and $n < \omega$), where

$$U_n(f) = \{g \in X : f \le g \text{ and } f \restriction n = g \restriction n\}.$$

Note that if $g \in U_n(f)$ then $U_n(g) \subset U_n(f)$, and if $g \notin U_n(f)$ then there is $k < \omega$ such that $U_k(g) \cap U_n(f) = \emptyset$. It follows that, for any $f \in X$, the family $\{U_n(f) : n < \omega\}$

forms a clopen neighbourhood base of f with respect to the topology τ, consequently the space $\langle X, \tau\rangle$ is indeed first countable, 0-dimensional, and Hausdorff.

It is also easy to see from the definitions that if $\{U_{n_\alpha}(f_\alpha) : \alpha < \kappa\}$ is a π-base of τ then $\{f_\alpha : \alpha < \kappa\}$ must be cofinal in the partial order $\langle X, \le\rangle$. But it is well-known that the cofinality of $\langle X, \le\rangle$ is greater than $\aleph_\omega$, consequently we have $\pi(X) > \aleph_\omega$. (Actually, it is easy to see that $\pi(X) = \mathrm{cf}\,\big(\langle X, \le\rangle\big)$ but we shall not need this.)

Next we claim that, for any $k < \omega$, the pair $(\aleph_{\omega+1}, \aleph_k)$ is a *pair caliber* of the space X, i.e. among any $\aleph_{\omega+1}$ open sets one can find $\aleph_k$ whose intersection is non-empty. Without any loss of generality, it suffices to check this for a family of basic open sets of the form $\{U_n(f) : f \in F\}$ where $F \in [X]^{\aleph_{\omega+1}}$ and $n > k$ is fixed. We may also assume that $f \restriction n = \sigma$ for a fixed $\sigma \in \prod_{i<n} \omega_i$ whenever $f \in F$. Now let $G \subset F$ with $|G| = \aleph_k$ then there is $g \in X$ with $g \restriction n = \sigma$, moreover $f(i) < g(i)$ for all $i \ge n$ and $f \in G$. But then we have $g \in \bigcap\{U_n(f) : f \in G\}$.

Putting together the previous two paragraphs we conclude that the order of any π-base of $\langle X, \tau\rangle$ must be at least $\aleph_\omega$, that is we have $\pi sw(X) \ge \aleph_\omega$. $\square$

It is clear that if we replace in the above proof the sequence of cardinals $\langle \omega_n : n < \omega\rangle$ with any other strictly increasing ω-sequence of regular cardinals, say $\langle \kappa_n : n < \omega\rangle$, then we obtain a first countable, 0-dimensional space in which the order of any π-base is at least $\sum_{n<\omega} \kappa_n$.

The referee has pointed out that the method of constructing such spaces was published by TodorčevicŽ in [66], Theorem 0.5 (of course, the fact that they do not have a point-countable π-base is not mentioned there).

The cardinality of our above example is $\aleph_\omega^{\aleph_0}$ that is much larger than the optimal value $\aleph_2$ permitted by Theorem 3.13. So it is natural to raise the question if we could find other examples of smaller cardinality. As it turns out, we can do slightly better by choosing an appropriate subspace Y of the space X from Theorem 3.14. First, however, we need to fix some notation. For $f, g \in X = \prod\{\omega_n : n < \omega\}$ we write $f <^* g$ to denote that $|\{n < \omega : f(n) \ge g(n)\}|$ is finite, i.e. f is below g modulo finite. Similarly, we write $f =^* g$ to denote that $|\{n < \omega : f(n) \ne g(n)\}|$ is finite. Finally, it is well-known that there is in X a transfinite sequence of order type $\omega_{\omega+1}$ that is increasing with respect to $<^*$.

Theorem 3.15. *Let $\{f_\alpha : \alpha < \omega_{\omega+1}\} \subset X$ be an increasing sequence with respect to $<^*$ and set*

$$Y = \{f \in X : \exists\, \alpha < \omega_{\omega+1} \text{ with } f =^* f_\alpha\}.$$

Then the subspace Y of X, with the subspace topology inherited from τ, also satisfies $\pi sw(Y) \ge \aleph_\omega$.

Proof. The proof is very similar to that of Theorem 3.14. First we note that, trivially, again we have $\pi(Y) > \aleph_\omega$. Next, we show that $(\aleph_{\omega+1}, \aleph_k)$ is a pair caliber of Y for each $k < \omega$. To see this, we again consider a family $\{U_n(f) : f \in F\}$ where $F \in [Y]^{\aleph_{\omega+1}}$

and $n > k > 0$, moreover $f \restriction n = \sigma$ for a fixed $\sigma \in \prod_{i<n} \omega_i$ whenever $f \in F$. Let us choose any subset $G \subset F$ with $|G| = \aleph_k$, then there is an ordinal $\alpha < \omega_{\omega+1}$ such that $g <^* f_\alpha$ for all $g \in G$. We may find an integer $m \geq n$ such that the set

$$G^* = \{g \in G : \forall\, i \geq m\ \big(g(i) < f_\alpha(i)\big)\}$$

also has cardinality $\aleph_k$.

Note that if $n \leq j < m$ then $\{g(j) : g \in G^*\}$ is bounded in ω_j, hence we may find a function $f \in Y$ such that $f \restriction n = \sigma$, if $n \leq j < m$ then $g(j) < f(j)$ for all $g \in G^*$, moreover $f(i) = f_\alpha(i)$ whenever $m \leq i < \omega$. Clearly, then we have $f \in \bigcap\{U_n(g) : g \in G^*\} \cap Y$. □

We were unable to produce a ZFC example of a first countable space without a point-countable π-base of cardinality less than $\aleph_{\omega+1}$. This leads us to the following intriguing open question.

Problem 3.16. *Is there, in ZFC, a first countable (Tychonov) space of cardinality less than $\aleph_\omega$ that has no point-countable π-base?*

Actually, at this point we do not even have such an example of cardinality $\aleph_\omega$. We conjecture, however, that having such an example is equivalent to having one of size $< \aleph_\omega$. In fact, we could verify this conjecture under the assumption $2^{\aleph_1} < \aleph_\omega$.

Theorem 3.17. *Assume that $2^{\aleph_1} < \aleph_\omega$ and X is a first countable space of cardinality $\aleph_\omega$. If every subspace of X of cardinality $< \aleph_\omega$ has a point-countable π-base then so does X.*

Proof. Let us start by giving a (very natural) definition. A family $\mathcal{B}$ of non-empty open sets in X is said to be an *outer π-base* of a subspace $Y \subset X$ if for every open set U with $U \cap Y \neq \emptyset$ there is a member $B \in \mathcal{B}$ such that $B \subset U$. We claim that, under the assumptions of our theorem, every subspace of X of cardinality $< \aleph_\omega$ even has a point-countable outer π-base. Thus if we have $X = \bigcup_{n<\omega} Y_n$ where $|Y_n| < \aleph_\omega$ for all $n < \omega$ and $\mathcal{B}_n$ is a point-countable outer π-base of Y_n in X then $\bigcup_{n<\omega} \mathcal{B}_n$ is a point-countable π-base of X.

To prove the above claim let us consider an ω_1-closed elementary submodel M of a "universe" $H(\theta)$ with $|M| < \aleph_\omega$. (As usual, here θ is a large enough regular cardinal, $H(\theta)$ is the collection of all sets of hereditary cardinality $< \theta$, and for M to be ω_1-closed means that $[M]^{\leq\omega_1} \subset M$.) The regular cardinal θ is chosen so large that $H(\theta)$ (and also M) contains X and everything else that is relevant, e.g. a map $\mathcal{V}$ that assigns to every point $x \in X$ a countable open neighbourhood base $\mathcal{V}_x$. Now, $2^{\aleph_1} < \aleph_\omega$ implies that for every $Y \in [X]^{<\aleph_\omega}$ there is such an elementary submodel M with $Y \subset M$. Consequently, our claim will be proven if we show that $X \cap M$ has a point-countable outer π-base in X whenever M is like above.

To see this, note first that for every point $x \in X \cap M$ we have $\mathcal{V}_x \in M$ and hence $\mathcal{V}_x \subset M$ as well. Consequently $\mathcal{V}_M = \cup\{\mathcal{V}_x : x \in X \cap M\} \subset M$ is an outer base of $X \cap M$ in X, hence we may choose a subfamily $\mathcal{B} \subset \mathcal{V}_M$ such that $\mathcal{B} \restriction M = \{B \cap M : B \in \mathcal{B}\}$ is a point-countable π-base of the subspace $X \cap M$.

It suffices to show now that $\mathcal{B}$ is a point-countable outer π-base of $X \cap M$ in X. Indeed, $\mathcal{B}$ is point-countable for if $\mathcal{U} \in [\mathcal{B}]^{\omega_1}$ then $\mathcal{U} \in M$ because M is ω_1-closed, and thus $\cap\mathcal{U} \neq \emptyset$ would imply $\cap\mathcal{U} \cap M \neq \emptyset$, contradicting that $\mathcal{B} \restriction M$ is point-countable. (Here we used the fact that, by elementarity, the correspondance $B \mapsto B \cap M$ is one-to-one on $\mathcal{B} \subset M$.) Finally, $\mathcal{B}$ is an outer π-base of $X \cap M$ in X because if U is open with $x \in U \cap M \neq \emptyset$ then there is $V \in \mathcal{V}_x \subset M$ with $V \subset U$, hence if $B \in \mathcal{B}$ with $B \cap M \subset V \in M$ then we also have $B \subset V \subset U$. $\square$

3.2.2 Examples from higher Suslin lines

We start this section by giving a theorem that, quite naturally, will turn out to be very useful in finding (first countable) spaces without point-countable π-bases.

Theorem 3.18. *Assume that X is a topological space which has a π-base $\mathcal{B}$ such that $ord(\mathcal{B})^+ < d(X)$. Then X has a discrete subspace D with $|D| \geq d(X)$.*

Proof. Let us first choose a point $x_B \in B$ from each $B \in \mathcal{B}$. , then the set $S = \{x_B : B \in \mathcal{B}\}$ is dense in X, hence we have

$$|S| \geq d(X) > ord(\mathcal{B})^+.$$

We now define a set mapping F on S by the following stipulation: For any point $x \in S$ let us put

$$F(x) = \{x_B \in S : x \in B\} \in [S]^{\leq ord(\mathcal{B})}.$$

By Hajnal's set mapping theorem (see [22]) then there is a free set $D \subset S$ for the set mapping F with $|D| = |S|$. This means that for every $x \in D$ we have $D \cap F(x) \subset \{x\}$. But every member of D is of the form x_B for some $B \in \mathcal{B}$, and we claim that for this point we have $B \cap D = \{x_B\}$. Indeed, $x_B \in B \cap D$ is obvious, and if $x \in D$ is different from x_B then $x_B \notin F(x)$ implies $x \notin B$. Consequently, D is as required. $\square$

The referee has pointed out to us that Theorem 3.18 is an easy consequence of the following result of Shapirovskii , see [25], 3.26: If $\mathcal{B}$ is any family of non-empty open sets in a space X with $ord(\mathcal{B}) \leq \kappa$ then there are discrete subspaces $\{D_\alpha : \alpha < \kappa^+\}$ of X such that $\bigcup\{D_\alpha : \alpha < \kappa^+\} \cap B \neq \emptyset$ for any $B \in \mathcal{B}$. Since our above proof of Theorem 3.18 is quite different and very short, for the reader's convenience we decided to keep it.

It is an immediate consequence of Theorem 3.18 that a space X satisfying $d(X) \geq \omega_2$ and $\widehat{s}(X) \leq d(X)$ cannot have a point-countable π-base. Unfortunately, we do not know if there is in ZFC a first countable Tychonov space like that. (Recall that solving Tkachuk's problems from [65] requires Tychonov examples.) If, however, we are satisfied with Hausdorff examples then we are much better off. In fact it was shown in [23] that there is a natural left-separated refinement σ of the euclidean topology τ on the real line $\mathbb{R}$ that is first countable and hereditarily Lindelöf. Consequently, by Theorem 3.18, a subspace of $\langle \mathbb{R}, \sigma \rangle$ which is left-separated in order-type ω_2, and thus of density ω_2, has no point-countable π-base. This shows that, at least for Hausdorff spaces, Tkachuk's CH results mentioned in the introduction simply fail without CH, for $\langle \mathbb{R}, \sigma \rangle$ is hereditarily Lindelöf. Actually, it is very easy to show that something stronger than CCC can be established for such a subspace, namely that ω_1 is a caliber of it. For Hausdorff spaces, this settles one more question of Tkachuk from [65]. In the next section we shall produce (consistent) Tychonov examples with these properties but that will require more work.

Next we shall consider higher Suslin lines; these are ordered spaces whose spread (equal in this case with cellularity) is less than their density. More precisely, we shall consider first countable variations of them that retain this property. For different purposes, this construction had been already used in Theorem 1.1 of [32], although there CH was additionally assumed.

Let κ be an infinite cardinal. We shall call a continuous linear order $\langle L, < \rangle$, equipped with the order topology generated by $<$, a κ-Suslin line if there are no more than κ disjoint open intervals in L (i.e. $c(L) \le \kappa$), although the density $d(L)$ of L is larger than κ. (It is known that the existence of a κ-Suslin line is equivalent to the existence of a κ-Suslin tree, but this will be irrelevant for us.) Thus, an ordinary Suslin line is the same as an ω-Suslin line and by a higher Suslin line we mean a κ-Suslin line where $\kappa > \omega$.

The main result of this section is the following theorem that, in particular, yields us a consistent example of a first countable GO-space without a point-countable π-base of the minimum possible cardinality ω_2. (Recall that GO-spaces, or generalized ordered spaces, are the subspaces of linearly ordered spaces.)

Theorem 3.19. *If there is a κ-Suslin line $\langle L, < \rangle$ then there is a first countable GO-space X with $|X| = \kappa^+$ and $\pi sw(X) = \kappa$.*

Proof. Let Z be the set of all those points $x \in L$ that have left-character ω, that is the open half line $(\leftarrow, x)$ has cofinality ω with respect to $<$. Since $\langle L, < \rangle$ is continuous, Z is dense in L. It follows that $d(Z) = d(L) = \kappa^+$ because $d(L) \le c(L)^+$ holds for any linearly ordered space L, see e. g. [9]. Now let X be any dense subspace of Z (and hence of L) with $|X| = \kappa^+$.

We consider X with the left-Sorgenfrey topology σ, i.e. for any $x \in X$ the half-open intervals $(y, x]$ form a σ-local base. Then σ is finer than the order topology on X, hence the density of $\langle X, \sigma \rangle$ must be larger than κ. It is clear from the definition that σ is a first countable topology.

Also, $\langle X, \sigma\rangle$ is a GO space because it is homeomorphic to the subspace topology on $X \times \{0\}$ inherited from the order topology on $L \times 2$ taken with the lexicographic order. Finally, we have $c(X, \sigma) = c(L)$, moreover $s(X, \sigma) = c(X, \sigma)$ is known to hold for GO-spaces, see 2.23 of [25]. Consequently we have $s(X) \le \kappa < d(X)$ and so Theorem 3.18 implies $\pi sw(X) \ge \kappa$. By Theorem 3.13, then $\pi sw(X) = \kappa$. $\square$

In particular, the existence of an ω_1-Suslin line implies that of a first countable GO space of cardinality ω_2 without a point-countable π-base.

Finally, we mention here the curious fact that it is an outstanding open question of set theory whether one can find a model of ZFC that does not contain any higher Suslin line. Consequently there is a chance that Theorem 3.19 yields us a ZFC example of a first countable GO space with no point-countable π-base.

3.2.3 Examples from subfamilies of $\mathcal{P}(\omega)$

In this section we are going to introduce a (quite simple but apparently new) way of constructing first countable, 0-dimensional Hausdorff topologies on subfamilies of $\mathcal{P}(\omega)$, the power set of ω. Then we shall use some of the spaces obtained in this manner to present examples that demonstrate the necessity of the use of CH in Tkachuk's results mentioned in the introduction.

We start with fixing some notation and terminology. We shall use Examples from subfamilies of $\mathcal{P}(\omega)$ to denote the family of all co-finite subsets of ω. For a given family $\mathcal{I} \subset \mathcal{P}(\omega)$ and for $I \in \mathcal{I}$ and $U \in \mathcal{U}$ we put

$$[I, U)_{\mathcal{I}} = \{J \in \mathcal{I} : I \subset J \subset U\}.$$

If $\mathcal{I} = \mathcal{P}(\omega)$ then we shall omit the subscript.

Finally, we say that the family $\mathcal{I} \subset \mathcal{P}(\omega)$ is *stable* if $I \in \mathcal{I}$ and $I =^* J$ for $J \subset \omega$ imply $J \in \mathcal{I}$ as well. (Of course, here $I =^* J$ means that I and J are equal mod finite, i.e. their symmetric difference $I \Delta J$ is finite.)

Definition 3.20. Let us fix a family $\mathcal{I} \subset \mathcal{P}(\omega)$. We shall denote by $\tau_{\mathcal{I}}$ the topology on $\mathcal{I}$ generated by all sets of the form $[I, U)_{\mathcal{I}}$, where $I \in \mathcal{I}$ and $U \in \mathcal{U}$, and by $X_{\mathcal{I}}$ the space $\langle \mathcal{I}, \tau_{\mathcal{I}}\rangle$.

Of course, $X_{\mathcal{I}}$ is identical with the appropriate subspace of the maximal such space $X_{\mathcal{P}(\omega)}$. A few basic (pleasant) properties of the spaces $X_{\mathcal{I}}$ are given by the following proposition.

Proposition 3.21. *The spaces $X_{\mathcal{I}}$ are first countable, 0-dimensional and Hausdorff.*

Proof. It suffices to show this for $\mathcal{I} = \mathcal{P}(\omega)$ because all three properties are inherited by subspaces.

Observe first that if $J \in [I,U) \cap [I',U')$ then

$$J \in [J, U \cap U') \subset [I,U) \cap [I',U'),$$

hence the "intervals" $[I,U)$ form an open basis of $\tau_{\mathcal{P}(\omega)}$, moreover $\{[I,U) : I \subset U \in \mathcal{U}\}$ forms a countable neighbourhood base of the point I of $X_{\mathcal{I}}$.

Next, if $J \notin [I,U)$ then either $J \setminus U \neq \emptyset$ and then $[J,\omega) \cap [I,U) = \emptyset$, or $J \subset U$ and $I \setminus J \neq \emptyset$. In the latter case we may pick $n \in I \setminus J$ and then we have $J \subset U \setminus \{n\}$, moreover $[J, U \setminus \{n\}) \cap [I,U) = \emptyset$ because $n \in I$. This means that all basic open sets $[I,U)$ are also closed, hence $X_{\mathcal{P}(\omega)}$ is indeed 0-dimensional.

Finally, for every $I \in \mathcal{P}(\omega)$ we have

$$\bigcap\{[I,U) : I \subset U \in \mathcal{U}\} = \{I\},$$

implying that $X_{\mathcal{P}}(\omega)$ is also Hausdorff. $\square$

For any family $\mathcal{I} \subset \mathcal{P}(\omega)$ we shall denote by $\operatorname{cof}(\mathcal{I})$ the cofinality of the partial order $\langle \mathcal{I}, \subset \rangle$. Also, we say that a cardinal number κ is a *set caliber* of $\mathcal{I}$ if for every subfamily $\mathcal{J} \in [\mathcal{I}]^\kappa$ there are $\mathcal{K} \in [\mathcal{J}]^\kappa$ and $I \in \mathcal{I}$ such that $\cup\mathcal{K} \subset I$ or, less formally, among any κ-many members of $\mathcal{I}$ there are κ-many that have an upper bound in $\mathcal{I}$. We now connect these concepts concerning $\mathcal{I}$ with properties of the associated space $X_{\mathcal{I}}$.

Proposition 3.22. *For any subfamily $\mathcal{I} \subset \mathcal{P}(\omega)$ we have*

(i) $d(X_{\mathcal{I}}) = \operatorname{cof}(\mathcal{I}) \cdot \omega$ *;*

(ii) if $\mathcal{I}$ is stable and κ is a cardinal with $\operatorname{cf}(\kappa) > \omega$ then κ is a caliber of the space $X_{\mathcal{I}}$ if and only if κ is a set caliber of the family $\mathcal{I}$.

Proof. The proof of (i) and the left-to-right direction of (ii) follows immediately from the fact that $\mathcal{K} \subset \mathcal{I}$ has an upper bound in $\mathcal{I}$ iff $\bigcap\{[I,\omega)_{\mathcal{I}} : I \in \mathcal{K}\} \neq \emptyset$. To see the other direction, assume that κ is a set caliber of the family $\mathcal{I}$ and consider a family $\mathcal{B}$ of κ-many basic open sets. Since $\operatorname{cf}(\kappa) > \omega$ we may assume that $\mathcal{B} = \{[I,U) : I \in \mathcal{J}\}$ for $\mathcal{J} \in [\mathcal{I}]^\kappa$ and a fixed $U \in \mathcal{U}$. By our assumption there is a $\mathcal{K} \in [\mathcal{J}]^\kappa$ which has an upper bound $K \in \mathcal{I}$. Then $K \cap U \in \mathcal{I}$ as $\mathcal{I}$ is stable and

$$K \cap U \in \bigcap \big\{[I,U) : I \in \mathcal{K}\big\}.$$

$\square$

After these preparatory propositions we can now present a result that will yield us further nice examples of first countable spaces without point-countable π-bases.

Theorem 3.23. *Assume that $\mathcal{I} \subset \mathcal{P}(\omega)$ is stable, $\operatorname{cof}(\mathcal{I}) > \omega$, and ω_1 is a set caliber of $\mathcal{I}$. Then $\pi sw(X_{\mathcal{I}}) > \omega$.*

Proof. Since $X_{\mathcal{I}}$ is first countable, and by (i) of Proposition 3.22, we have

$$\pi(X_{\mathcal{I}}) = d(X_{\mathcal{I}}) = \text{cof}(\mathcal{I}) > \omega.$$

But, in view of part (ii) of Proposition 3.22, ω_1 is a caliber of $X_{\mathcal{I}}$, consequently no π-base of $X_{\mathcal{I}}$ can be point-countable. □

Corollary 3.24. *Assume that there is a mod finite strictly increasing ω_2-sequence in $\mathcal{P}(\omega)$. Then there is a first countable, 0-dimensional and Hausdorff space of cardinality ω_2 which has ω_1 as a caliber. In particular, MA_{ω_1} implies the existence of such a space.*

Proof. Let $\{A_\alpha : \alpha < \omega_2\} \subset \mathcal{P}(\omega)$ be a mod finite strictly increasing ω_2-sequence, i.e. we have $|A_\alpha \setminus A_\beta| < \omega$ and $|A_\beta \setminus A_\alpha| = \omega$ whenever $\alpha < \beta < \omega_2$. It is obvious that the family

$$\mathcal{I} = \{I \subset \omega : \exists\, \alpha < \omega_2 \text{ with } I =^* A_\alpha\}$$

is stable and satisfies $|\mathcal{I}| = \text{cof}(\mathcal{I}) = \omega_2$. Next, we claim that ω_1 is a set caliber of $\mathcal{I}$.

To see this, consider any family $\mathcal{J} = \{I_\alpha : \alpha \in a\} \subset \mathcal{I}$ where $a \in [\omega_2]^{\omega_1}$ and $I_\alpha =^* A_\alpha$ for all $\alpha \in a$ and pick $\beta < \omega_2$ such that $a \subset \beta$. Then $|A_\alpha \setminus A_\beta| < \omega$ for all $\alpha \in a$, hence there is a fixed $s \in [\omega]^{<\omega}$ such that

$$b = \{\alpha \in a : A_\alpha \setminus A_\beta \subset s\}$$

is uncountable, while $s \cup A_\beta$ is an upper bound of $\{I_\alpha : \alpha \in b\}$ in $\mathcal{I}$. By Theorem 3.23, the space $X_{\mathcal{I}}$ is as required. □

This result takes care of Problems 4.6 and 4.7 from [65] by showing that it is consistent to have first countable Tychonov spaces with caliber ω_1 (and hence also CCC) without any point-countable π-base. With some further elaboration we shall find examples that, in addition, are also hereditarily Lindelöf, and thus provide a solution to Problem 4.3 from [65] as well.

Theorem 3.25. *Let $\{A_\alpha : \alpha < \omega_2\} \subset \mathcal{P}(\omega)$ be a mod finite strictly increasing ω_2-sequence with the additional property that in every uncountable index set $a \in [\omega_2]^{\omega_1}$ there is a pair $\{\alpha, \beta\} \in [a]^2$ such that $A_\alpha \subset A_\beta$, (i.e. A_α is really a subset of A_β, not just mod finite). Then, with $\mathcal{I}$ defined as in Corollary 3.24, the space $X_{\mathcal{I}}$ is hereditarily Lindelöf.*

Proof. Assume, on the contrary, that $X_{\mathcal{I}}$ has an uncountable right-separated subspace. Without loss of generality this may be taken of the form $\{I_\alpha : \alpha \in a\}$, right separated in the natural well-ordering of its indices, where $a \in [\omega_2]^{\omega_1}$ and $I_\alpha =^* A_\alpha$ for all $\alpha \in a$. Moreover, we may assume that we have a fixed $U \in \mathcal{U}$ such that $[I_\alpha, U)_{\mathcal{I}}$ is a right separating neighbourhood of I_α for any $\alpha \in a$.

Now, there is a fixed finite set $s \in [\omega]^{<\omega}$ such that

$$b = \{\alpha \in a : I_\alpha \Delta A_\alpha = s\}$$

is uncountable. By our assumption, there is a pair $\{\alpha, \beta\} \in [b]^2$ (with $\alpha < \beta$) for which $A_\alpha \subset A_\beta$ and hence $I_\alpha \subset I_\beta$. This, however, would imply $I_\beta \in [I_\alpha, U)_{\mathcal{I}}$, contradicting that $[I_\alpha, U)_{\mathcal{I}}$ is a right separating neighbourhood of I_α. □

Note that a space as in Theorem 3.25 is a first countable L-space, hence unlike the spaces in Corollary 3.24, it does not exist under MA_{ω_1}, see [61]. Instead, there is a "natural" forcing construction that produces mod finite strictly increasing ω_2-sequences in $\mathcal{P}(\omega)$ with the additional property required in Theorem 3.25.

Theorem 3.26. *There is a CCC forcing that, to any ground model, adds a mod finite strictly increasing sequence $\{A_\alpha : \alpha < \omega_2\} \subset \mathcal{P}(\omega)$ in any uncountable subsequence of which there are two members with proper inclusion.*

Proof. Let $\mathbb{P}$ consist of those finite functions $p \in Fn(\omega_2 \times \omega, 2)$ for which $\mathrm{dom}(p) = a \times n$ with $a \in [\omega_2]^{<\omega}$ and $n < \omega$. We define $p' \leq p$ (i.e. p' extends p) as follows: $p' \supset p$, moreover $p'(\alpha, i) = 1$ implies $p'(\beta, i) = 1$ whenever $\alpha, \beta \in a$ with $\alpha < \beta$ and $i \in n' \setminus n$ (of course, here $\mathrm{dom}(p) = a \times n$ and $\mathrm{dom}(p') = a' \times n'$). It is straightforward to show that $\langle \mathbb{P}, \leq \rangle$ is a CCC notion of forcing.

Let $G \subset \mathbb{P}$ be generic, then it follows from standard density arguments that $g = \cup G$ maps $\omega_2 \times \omega$ into 2 and if we set

$$A_\alpha = \{i < \omega : g(\alpha, i) = 1\}$$

then $\{A_\alpha : \alpha < \omega_2\}$ is mod finite strictly increasing.

To finish the proof, let us assume that $p \in \mathbb{P}$ forces that $\dot{h}$ is an order preserving injection of ω_1 into ω_2. It suffices to show that p has an extension q which forces $A_{\dot{h}(\xi)} \subset A_{\dot{h}(\eta)}$ for some $\xi < \eta < \omega_1$.

To see this, let us choose first for each $\xi < \omega_1$ a condition $p_\xi \leq p$ and an ordinal $\alpha_\xi < \omega_2$ such that $p_\xi \Vdash \dot{h}(\xi) = \alpha_\xi$. We may assume without any loss of generality that for some $n < \omega$ we have $\mathrm{dom}(p_\xi) = a_\xi \times n$ and $\alpha_\xi \in a_\xi$ for all ξ. Using standard Δ-system and counting arguments, it is easy to find then $\xi < \eta < \omega_1$ such that p_ξ and p_η are compatible as functions and for any $i < n$ we have $p_\xi(\alpha_\xi, i) = p_\eta(\alpha_\eta, i)$. But then we have $q = p_\xi \cup p_\eta \in \mathbb{P}$ and $q \leq p$, moreover it is obvious that q forces $A_{\alpha_\xi} \subset A_{\alpha_\eta}$ and hence $A_{\dot{h}(\xi)} \subset A_{\dot{h}(\eta)}$ as well. □

From Theorems 3.25 and 3.26 we immediately obtain a joint solution to Problems 4.3 and 4.7 (and hence 4.6) of Tkachuk from [65].

Corollary 3.27. *It is consistent that there exists a first countable, hereditarily Lindelöf 0-dimensional space X of size ω_2 which has no point-countable π-base while ω_1 is a caliber of X.*

Let us recall here that the failure of CH is not sufficient to produce a mod finite strictly increasing ω_2-sequence in $\mathcal{P}(\omega)$, the basic ingredient of our examples in this section. In fact, Kunen had proved (see e.g. [34]) that if one adds ω_2 Cohen reals to a model of CH then no such sequence exists in the extension. Actually, we have shown the following strengthening of this: In the same model, if ω_1 is a set caliber of a subfamily $\mathcal{I}$ of $\mathcal{P}(\omega)$ then $\mathrm{cof}(\mathcal{I}) \le \omega$. This implies that we may not use the methods of this section to find similar examples just assuming the negation of CH. The following natural problem can thus be raised.

Problem 3.28. *Does $2^\omega > \omega_1$ imply the existence of a first countable Lindelöf and/or CCC Tychonov space having no point-countable π-base?*

4 Preserving functions

Let us call a function f from a space X into a space Y *preserving* if the image of every compact subspace of X is compact in Y and the image of every connected subspace of X is connected in Y. By elementary theorems a continuous function is always preserving. Quite a few authors noticed—mostly independently from each other—that the converse is also true for real functions: a preserving function $f : \mathbb{R} \to \mathbb{R}$ is continuous. (The first – loosely related – paper we know of is [51] from 1926!)

Klee and Utz proved in [Kl] that every preserving map between metric spaces X and Y is continuous at some point p of X exactly if X is locally connected at p. Whyburn proved [74] that a preserving function from a space X into a Hausdorff space is always continuous at a first countability and local connectivity point of X. Then Evelyn R. McMillan [46] proved in 1970 that if X is Hausdorff, locally connected and Frèchet, moreover Y is Hausdorff, then any preserving function $f : X \to Y$ is continuous. This is, we think, quite a significant result that is surprisingly little known.

We shall use the notation $Pr(X, T_i)$ ($i = 1, 2, 3$ or $3\frac{1}{2}$) to denote the following statement: Every preserving function from the topological space X into any T_i space is continuous.

The organization of the section is as follows: In §1 we give some basic definitions and then treat some results that are closely connected to McMillan's theorem. §2 treats several important technical theorems that enable us to conclude that certain preserving functions are continuous. In §3 we apply these to prove that certain product spaces X satisfy $Pr(X, T_3)$; in particular, any preserving function from an arbitrary product of connected linearly ordered spaces into a regular space is continuous. In §4 we discuss some results concerning the continuity of preserving functions defined on compact and/or sequential spaces. Finally, §5 treats the relation $Pr(X, T_1)$.

4.1 Around McMillan's theorem

The first theorem of the paper (due to D. J. White, 1971) implies that (at least among $T_{3\frac{1}{2}}$ spaces) local connectivity of X is a necessary condition for $Pr(X, T_{3\frac{1}{2}})$. Of course, the assumption of local connectivity as a condition of continuity for preserving maps is very natural and, as can be seen from our brief historical sketch given above, has been noticed long ago.

Theorem 4.1. *(D. J. White [73]) If the $T_{3\frac{1}{2}}$ space X is not locally connected at a point $p \in X$, then there exists a preserving function f from X into the interval $[0, 1]$ which is not continuous at p.* $\square$

It is not a coincidence that the target space in Theorem 4.1 is the interval $[0, 1]$, because of the following result:

Lemma 4.2. *Suppose $f : X \to Y$ is a preserving function into a $T_{3\frac{1}{2}}$ space Y and f is not continuous at the point $p \in X$. Then there exists a preserving function $h : X \to [0,1]$ which is also not continuous at p.*

Proof. Since f is not continuous at p, there exists a closed set $F \subset Y$ such that $f(p) \notin F$ but p is an accumulation point of $f^{-1}(F)$. Choose a continuous function $g : Y \to [0,1]$ such that $g(f(p)) = 0$ and g is identically 1 on F. Then the composite function $h(x) = g(f(x))$ has the stated properties. □

The following Lemmas will be often used in the sequel. They all state simple properties of preserving functions.

Lemma 4.3. *If $f : X \to Y$ is a compactness preserving function, Y is Hausdorff, $M \subset X$ with $\overline{M}$ compact then for every accumulation point y of $f(M)$ there is an accumulation point x of M such that $f(x) = y$, i. e. $f(M)' \subset f(M')$.*

Proof. Let $N = M - f^{-1}(y)$ then $f(N) = f(M) - \{y\}$ and so we have $y \in \overline{f(N)} - f(N)$. But $f(\overline{N})$ is also compact, hence closed in Y and so $y \in f(\overline{N}) - f(N)$ as well. Thus there is an $x \in \overline{N} - N$ such that $f(x) = y$ and then x is as required. □

We shall often use the following immediate consequence of this lemma:

Lemma 1 (E. R. McMillan [46]). *If $f : X \to Y$ is a compactness preserving function, Y is Hausdorff, $\{x_n : n < \omega\} \subset X$ converges to $x \in X$ then either $\{f(x_n) : n < \omega\}$ converges to $f(x)$ or there is a point $y \in Y$ distinct from $f(x)$ such that $f(x_n) = y$ for infinitely many $n \in \omega$. In particular, if the image points $f(x_n)$ are all distinct then they must converge to $f(x)$.* □

Actually, to prove Lemma 1 we do not need the full force of the assumption that f is compactness preserving. It suffices to assume that the image of a convergent sequence together with its limit is compact, in other words: the image of a topological copy of $\omega + 1$ is compact. For almost all of our results given below only this very restricted special case of compactness preservation is needed.

Lemma 4.4 ([50]). *If $f : X \to Y$ preserves connectedness, Y is a T_1-space and $C \subset X$ is a connected set, then $f(\overline{C}) \subset \overline{f(C)}$.*

Proof. If $x \in \overline{C}$ then $C \cup \{x\}$ is connected. Thus $f(C \cup \{x\}) = f(C) \cup \{f(x)\}$ is also connected and hence $f(x) \in \overline{f(C)}$. □

The next lemma will also play a crucial role in some theorems of the paper. A weaker form of it appears in [46].

Definition 4.5. We shall say that $f : X \to Y$ is *locally constant* at the point $x \in X$ if there is a neighbourhood U of x such that f is constant on U.

Lemma 4.6. *Let f be a connectivity preserving function from a locally connected space X into a T_1-space Y. If $F \subset Y$ is closed and $p \in \overline{f^{-1}(F)} - f^{-1}(F)$ then p is also in the closure of the set*

$$\{x \in f^{-1}(F) : f \text{ is not locally constant at } x\}.$$

Proof. Let G be a connected open neighbourhood of p and C be a component of the non-empty subspace $G \cap f^{-1}(F)$. Then C has a boundary point x in the connected subspace G because $\emptyset \neq C \neq G$. Clearly, $f(x) \in F$ by Lemma 4.4. If $V \subset G$ is any connected neighbourhood of x then $V \cup C$ is connected and $V - C \neq \emptyset$ because x is a boundary point of C hence V is not contained in $f^{-1}(F)$, so f is not locally constant at x. □

Lemma 4.7. *Let $f : X \to Y$ be a connectivity preserving function into the T_1-space Y. Suppose that X is locally connected at the point $p \in X$ and f is not locally constant at p. Then $f(U) \cap V$ is infinite for every neighbourhood U of p and for every neighbourhood V of $f(p)$.*

Proof. Choose any connected neighbourhood U of x; then $f(U)$ is connected and has at least two points. Thus if V is any open subset of Y containing $f(p)$ then $f(U) \cap V$ can not be finite because otherwise $f(p)$ would be an isolated point of the non-singleton connected set $f(U)$. □

The following result is a slight strengthening of McMillan's theorem in that no separation axiom is assumed on X. Its proof is based upon the same ideas as her original proof, although, we think, it is much simpler. We included it here mainly to make the paper self-contained. She needed the assumption that X be Hausdorff because originally she got her result for spaces having the hereditary K property instead of the Frèchet property and the equivalence of these two properties is only known for Hausdorff spaces.

Theorem 4.8 (E. R. McMillan [46]). *If X is a locally connected and Frèchet space, then $Pr(X, T_2)$ holds.*

Proof. Assume Y is T_2 and $f : X \to Y$ is preserving but not (sequentially) continuous at the point $p \in X$. Then by Lemma 1 there is a sequence $x_n \to p$ such that $f(x_n) = y \neq f(p)$ for all $n < \omega$. Using Lemma 4.6 with $F = \{y\}$ we can also assume that f is not locally constant at the points x_n.

As Y is T_2, there is an open set $V \subset Y$ such that $y \in V$ but $f(p) \notin \overline{V}$. By Lemma 4.7 the image of every neighbourhood of each point x_n contains infinitely many points (different from y) from V.

Now we select recursively sequences $\{x^n_k : k < \omega\}$ converging to x_n for all $n < \omega$. Suppose $n < \omega$ and the points x^m_k are already defined for $m < n$ and $k < \omega$ so that $f(x^m_k) \neq y$. Then x_n is in the closure of the set

$$f^{-1}(V - (\{f(x^m_k) : m, k < n\} \cup \{y\})),$$

hence, as X is Fréchet , the new sequence $\{x_k^n : k < \omega\}$ converging to x_n can be chosen from this set.

Since the sequence $\{x_k^n : k < \omega\}$ converges to x_n and $\{x_n : n < \omega\}$ converges to the (Frèchet) point p, there is also a "diagonal" sequence $\{x_{k_l}^{n_l} : l < \omega\}$ converging to p. But then the sequence $\{n_l : l < \omega\}$ must tend to infinity so, by passing to a subsequence if necessary, we can assume that $n_{l+1} > \max(n_l, k_l)$ for all $l < \omega$. However, the sequence $\{f(x_{k_l}^{n_l}) : l < \omega\}$ does not converge to $f(p)$ because $f(p) \notin \overline{\{f(x_{k_l}^{n_l}) : l < \omega\}} \subset \overline{V}$, while the points $f(x_{k_l}^{n_l})$ are all distinct, contradicting Lemma 1. $\square$

We could prove the following semi-local version of McMillan's theorem:

Theorem 4.9. *If X is a locally connected Hausdorff space, p is a Frèchet point of X and f is a preserving function from X into a $T_{3\frac{1}{2}}$ space Y, then f is continuous at p.*

Proof. By Lemma 4.2 it suffices to prove this in the case when Y is the interval $[0, 1]$. Assume, indirectly, that f is not continuous at p then, since p is a Fréchet point and by Lemma 4.6, we can again choose a sequence $x_n \to p$ and a $y \in [0, 1]$ with $y \neq f(p)$ such that $f(x_n) = y$ and f is not locally constant at x_n for all $n < \omega$.

For each n choose a neighbourhood U_n of x_n with $p \notin \overline{U}_n$ and put $A_n = \{x \in U_n : 0 < |f(x) - y| < 1/n\}$. For any connected neighbourhood W of x_n its image $f(W)$ is a non-singleton interval containing y, hence the local connectivity of X implies that $x_n \in \overline{A}_n$ for all $n < \omega$ and so p belongs to the closure of $\bigcup\{A_n : n < \omega\}$. As p is a Frèchet point, there is a sequence $z_k \in A_{n_k}$ converging to p where n_k necessarily tends to infinity because $p \notin \overline{A}_n \subset \overline{U}_n$ for each $n < \omega$. But then $f(z_k) \to y \neq f(p)$ contradicts Lemma 1 since the set $\{f(z_k) : k < \omega\}$ is infinite because we have $f(z_k) \neq y$ for all $k \in \omega$. $\square$

Theorem 4.9 is not a full local version of Theorem 4.8 because local connectivity is assumed in it globally for X. This leads to the following natural question:

Problem 4.10. *Let X be a Hausdorff (or regular, or Tychonov) space, f be a preserving function from X into a $T_{3\frac{1}{2}}$ space Y and let X be locally connected and Frèchet at the point $p \in X$. Is it true then that f is continuous at p?*

We do not know the answer to this problem, however we can prove some partial affirmative results.

Definition 4.11 ([1])**.** A point x of a space X is called an (α_4) point if for any sequence $\{A_n : n < \omega\}$ of countably infinite sets with $A_n \to x$ for each $n < \omega$ there is a countably infinite set $B \to x$ such that $\{n < \omega : A_n \cap B \neq \emptyset\}$ is infinite.

An (α_4) and Frèchet point will be called an (α_4)-F point in X.

Theorem 4.12. *Let f be a preserving function from a topological space X into a Hausdorff space Y and let p be a point of local connectivity and an (α_4)-F point in X. Then f is continuous at p.*

Proof. Assume not. Then by the Lemma 1 there is a point $y \in Y$ such that $y \neq f(p)$ but p is in the closure of $f^{-1}(y)$. Choose an open neighbourhood V of y in Y with $f(p) \notin \overline{V}$. By Lemma 4.7 and Lemma 1 we can recursively choose pairwise distinct points $y_n \in V$ such that p is in the closure of $f^{-1}(y_n)$ for all $n \in \omega$. As the point p is an (α_4)-F point in X, there is a "diagonal" sequence $\{x_m : m \in M\}$ converging to p, where $f(x_m) = y_m$ and $M \subset \omega$ is infinite, contradicting Lemma 1. □

The next result yields a different kind of partial answer to Problem 4.10:

Theorem 4.13. *Let f be a preserving function from a topological space X into a $T_{3\frac{1}{2}}$ space Y and let p be a Frèchet point of local connectivity of X with character $\leq 2^\omega$. Then f is continuous at p.*

Proof. Assume not, f is discontinuous at the point $p \in X$. By Lemmas 4.2 and 1 we can suppose that $Y = [0,1]$, $f(p) = 0$ and every neighbourhood of p is mapped onto the whole interval $[0,1]$. Let $\mathcal{U}$ be a neighbourhood base of p of size $\leq 2^\omega$ and choose for each $U \in \mathcal{U}$ a point $x_U \in U$ such that $f(x_U) \in [1/2, 1]$ and the points $f(x_U)$ are all distinct. Put $A = \{x_U : U \in \mathcal{U}\}$, then $p \in \overline{A}$ and so there exists a sequence $\{x_n : n < \omega\} \subset A$ converging to p, contradicting Lemma 1. □

There is a variant of this result in which the assumption that Y be $T_{3\frac{1}{2}}$ is relaxed to T_3, however the assumption on the character of the point p is more stringent. Its proof will make use of the following (probably well-known) lemma:

Lemma 4.14. *Let Z be an infinite connected regular space, then any non-empty open subset G of Z is uncountable.*

Proof. Choose a point $z \in G$ and an open proper subset V of G with $z \in V \subset \overline{V} \subset G$. If G would be countable then, as a countable regular space, G would be $T_{3\frac{1}{2}}$, and so there would be a continuous function $f : G \to [0,1]$ such that $f(z) = 1$ and f is identically zero on $G - V$. Extend f to a function $\overline{f} : Z \to [0,1]$ by putting $\overline{f}(y) = 0$ if $y \in Z - G$. Then $\overline{f}$ is continuous and hence $\overline{f}(Z)$ is also connected. Consequently we have $f(G) = \overline{f}(Z) = [0,1]$ implying that $|G| \geq |[0,1]| > \omega$, and so contradicting that G is countable. □

Theorem 4.15. *Let f be a preserving function from a topological space X into a T_3 space Y and let $p \in X$ be a Frèchet point of local connectivity with character $\leq \omega_1$. Then f is continuous at p.*

Proof. Assume f is discontinuous at the point $p \in X$. As p is a a Frèchet point, there is a sequence $x_n \to p$ such that $f(x_n)$ does not converge to $f(p)$. Taking a subsequence if necessary, we can suppose by Lemma 1 that $f(x_n) = y \neq f(p)$ for all $n < \omega$.

Choose now an open neighbourhood V of the point $y \in Y$ with $f(p) \notin \overline{V}$. Let $\mathcal{U}$ be a neighbourhood base of the point p in X such that $|\mathcal{U}| \leq \omega_1$ and the elements of $\mathcal{U}$ are connected.

Choose now points x_U from the sets $U \in \mathcal{U}$ such that $f(x_U) \in V$ and the points $f(x_U)$ are all distinct. This can be accomplished by an easy transfinite recursion because for each $U \in \mathcal{U}$ the set $f(U)$ is connected and infinite, hence $f(U) \cap V$ is uncountable by the previous lemma. Put $A = \{x_U : U \in \mathcal{U}\}$. Then $p \in \overline{A}$ and so there exists a sequence $\{y_n : n < \omega\} \subset A$ converging to p, contradicting Lemma 1. $\square$

We shall now consider some further topological properties implying that preserving functions are sequentially continuous. Since in a Fréchet point sequential continuity implies continuity, these results are clearly relevant to McMillan's theorem. Their real significance, however, will only become clear in the following two sections.

Definition 4.16. A point x in a topological space X is called a *sequentially connectible (in short: SC)* point, if $x_n \in X$, $x_n \to x$ implies that there are an infinite subsequence $\langle x_{n_k} : k < \omega \rangle$ and a sequence $\langle C_k : k < \omega \rangle$ consisting of connected subsets of X such that $\{x_{n_k}, x\} \subset C_k$ for all $k < \omega$ (i.e. C_k "connects" x_{n_k} with x, this explains the terminology), moreover $C_k \to x$, i.e. every neighbourhood of the point x contains all but finitely many C_k's. A space X is called an SC space if all its points are SC points.

Remark 4.17. *It is clear that the SC property is closely related to local connectivity. Let us say that a point x in space X is a strong local connectivity point if it has a neighbourhood base $\mathcal{B}$ such that the intersection of an arbitrary (non-empty) subfamily of $\mathcal{B}$ is connected. For example, local connectivity points of countable character in an arbitrary space or any point of a connected linearly ordered space have this property.*

We claim that if x is a strong local connectivity point of X then x is an SC point in X. Indeed, assume that $x_n \to x$ and for every $n \in \omega$ let C_n denote the intersection of all those members of $\mathcal{B}$ which contain both points x_n and x. (As the sequence $\{x_n : n < \omega\}$ converges to x, we can suppose that some element $B_0 \in \mathcal{B}$ contains all the x_n's.) Then $\{x, x_n\} \subset C_n$, moreover $C_n \to x$. Indeed, the latter holds because if $x \in B \in \mathcal{B}$ then, by definition, $x_n \in B$ implies $C_n \subset B$. $\square$

The SC property does not imply local connectivity. (If every convergent sequence is eventually constant then the space is trivially SC.) However, the following simple lemma shows that if there are "many" convergent sequences then such an implication is valid.

Lemma 4.18. *Let x be a both Frèchet and SC point in a space X. Then x is also a point of local connectivity in X.*

Proof. Let G be any open set containing x and let H be the component of the point x in G. We claim that H is a neighbourhood of x. Indeed, otherwise, as x is a Frèchet point, we could choose a sequence $x_n \to x$ from the set $G - H$ while for every point $y \in G - H$ no connected set containing both x and y is a subset of G, contradicting the SC property of x. $\square$

If SC holds globally, i.e. in an SC space, then in the above result the Frèchet property can be replaced with a weaker property that will turn out to play a very important role in the sequel.

Definition 4.19. A point p in a topological space X is called an s point if for every family $\mathcal{A}$ of subsets of X such that $p \in \overline{\bigcup \mathcal{A}}$ but $p \notin \overline{A}$ for all $A \in \mathcal{A}$ there is a sequence $\langle\langle x_n, A_n\rangle : n < \omega\rangle$ such that $x_n \in A_n \in \mathcal{A}$, the sets A_n are pairwise distinct and $\{x_n\}$ converges to some point $x \in X$ (that may be different from p).

A Frèchet point is evidently an s point, moreover any point that has a sequentially compact neighbourhood is also an s point. Other examples of s points will be seen later.

Theorem 4.20. *Any s point in a T_3 and SC space is a point of local connectivity.*

Proof. Let p be an s point in the regular SC space X and let G be an open neighbourhood of p. We have to prove that the component K_0 of the point p in G is a neighbourhood of p. Assume this is false and choose an open set U such that $p \in U \subset \overline{U} \subset G$.

Put
$$\mathcal{A} = \{K \cap \overline{U} : K \text{ is a component of } G,\ K \neq K_0\}.$$

Then $p \in \overline{\bigcup \mathcal{A}}$ and $p \notin \overline{A}$ for $A \in \mathcal{A}$ (because a component of G is relatively closed in G), hence, by the definition of an s point, there exists a sequence $\{\langle x_n, A_n\rangle : n < \omega\}$ such that $x_n \in A_n \in \mathcal{A}$, $x_n \to x$ for some $x \in X$ and if $A_n = K_n \cap \overline{U}$ then the components K_n are distinct. Note that $x \in \overline{U} \subset G$. As distinct components are disjoint, we can assume that $x \notin K_n$ for all $n < \omega$. As x is an SC point, there are a connected set C and some $n < \omega$ such that $\{x, x_n\} \subset C \subset G$. However, this is impossible, because then $K_n \cup C$ would be a connected set in G larger then the component K_n. □

The significance of the SC property in our study of continuity properties of preserving functions is revealed by the following result.

Theorem 4.21. *A preserving function $f : X \to Y$ into a Hausdorff space Y is sequentially continuous at each SC point of X.*

Proof. Let $x \in X$ be an SC point and assume that $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$ for a sequence $\{x_n : n < \omega\}$ in X. We can assume by Lemma 1 that $f(x_n) = y \neq f(x)$ for all $n < \omega$. Choose an open neighbourhood V of y in Y such that $f(x) \notin \overline{V}$.

As x is an SC point in X, we can also assume that there is a sequence of connected sets C_n such that $C_n \to x$ and $x, x_n \in C_n$ for $n < \omega$. We can now define a sequence $z_n \in C_n$ such that $f(z_n) \in V$ and the points $f(z_n)$ are all distinct. Indeed, assume $n < \omega$ and the points z_i are already defined for $i < n$ in this way. As $f(C_n)$ is connected and $f(C_n) \cap V$ is its non-empty open proper subset, this intersection $f(C_n) \cap V$ is not closed and hence is infinite. Consequently there exists a point $z_n \in C_n$ with $f(z_n) \in f(C_n) \cap V - \{f(z_i) : i < n\}$. But then the sequence $\{z_n\}$ contradicts Lemma 1. □

Corollary 4.22. *Let f be a preserving function from a topological space X into a Hausdorff space Y and let p be both an SC point and a Frèchet-point in X. Then f is continuous at p.* □

The following example (due to E. R. McMillan [46]) yields a locally connected SC space with a discontinuous preserving function. (Compare this with Theorem 4.21.)

Example 4.23. *Take ω_1 copies of the interval $[0,1]$ and identify the 0 points. We get in this way a "hedgehog" $X = \{0\} \cup \{R_\xi : \xi \in \omega_1\}$, where the spikes $R_\xi = (0,1] \times \{\xi\}$ are disjoint copies of the half closed interval $(0,1]$. A basic neighbourhood of a point $x \in R_\xi$ is an open interval around x in R_ξ. A basic neighbourhood of 0 is a set of the form $\{0\} \cup \bigcup\{J_\xi : \xi < \omega_1\}$, where each J_ξ is a non-empty initial interval of R_ξ and $J_\xi = R_\xi$ holds for all but countably many ordinals ξ.*

It is easy to see that in this way we get a locally connected $T_{3\frac{1}{2}}$ SC space X. The function $f : X \to [0,1]$ defined by $f((x,\xi)) = x$, $f(0) = 0$ is preserving but not continuous at the point 0 because every neighbourhood of 0 is mapped onto $[0,1]$. However, by Theorem 1. 21 the function f is sequentially continuous. □

The next example is locally connected, hereditary Lindelof, T_6, countably tight and has a preserving function defined on it that is not even sequentially continuous. It follows that this space is not an SC space.

Example 4.24. *The underlying set of our space X consists of a point p, of a sequence of points p_n for $n < \omega$ and countably many arcs $\{I(n,m) : m < \omega\}$ with disjoint interiors connecting the points p_n and p_{n+1} for every $n < \omega$.*

If x is an inner point of some arc, then its basic neighbourhoods are the open intervals around it on the arc. The basic neighbourhoods of a point p_n are the unions of initial (or final) segments of the arcs containing p_n. Finally, basic neighbourhoods of p are the sets which contain all but finitely many p_n's together with their basic neighbourhoods and for any two consecutive p_n's in the set all but finitely many of the arcs $I(n,m)$. Note that the subspace $X - \{p\}$ can be realized as a subspace of the plane, hence it is easy to check that X has the above stated properties.

Now let $f : X \to [0,1]$ be defined as follows: $f(p) = 0$, $f(p_n) = 0$ if n is even, $f(p_n) = 1$ if n is odd, and f is continuous on each arc $I(n,m)$. Then f is not sequentially continuous at p as is shown by the sequence $\{p_n : n \text{ odd}\}$ converging to p, but it is preserving. Indeed, an infinite sequence whose members are from the interiors of different arcs is closed discrete and so a compact subset of X can meet only finitely many open arcs. It follows then that its f-image is the union of finitely many compact subsets of [0,1]. Moreover, if a connected set contains both p and some other point, then it also contains an arc $I(n,m)$, and thus its image is the whole $[0,1]$. □

In the rest of this section we shall consider a slight weakening of the sequential continuity property that comes up naturally in the proof of McMillan's theorem or Theorems 4.32 and 4.33 below.

Definition 4.25. A function $f : X \to Y$ is said to be *weakly sequentially continuous* at the point x if $f(x_n) \to f(x)$ whenever $x_n \to x$ in X and f is not locally constant at x_n for all $n < \omega$.

We shall consider below two types of points at which preserving functions turn out to be weakly sequentially continuous. In the next section then these will be used to yield "real" continuity of preserving functions on some interesting classes of spaces.

Definition 4.26. A point x in a space X is called an *inflatable point* if $x_n \to x$ with $x_n \neq x$ for all $n < \omega$ implies that there is a subsequence $\{x_{n_k} : k < \omega\}$ with neighbourhoods U_k of x_{n_k} for $k < \omega$ such that $U_k \to x$ (i.e. every neighbourhood of x contains all but finitely many U_k's). The space X is called inflatable if all its points are inflatable.

It is obvious that any GO (i. e. generalized ordered) space is inflatable.

Theorem 4.27. *Any preserving function $f : X \to Y$ from a locally connected space X into a T_2 space Y is weakly sequentially continuous at an inflatable point x.*

Proof. Assume, indirectly, that $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$, while f is not locally constant at x_n for all $n < \omega$. By Lemma 1 we can assume that $f(x_n) = y \neq f(x)$ for all $n < \omega$. Choose an open neighbourhood $V \subset Y$ of y such that $f(x) \notin \overline{V}$. As x is inflatable, we may also assume to have open sets U_n with $x_n \in U_n$ such that $U_n \to x$. Using Lemma 4.7 we can recursively choose points $z_n \in U_n$ with distinct f-images such that $f(z_n) \in V$ for all $n < \omega$, contradicting Lemma 1 again. □

The other property we consider is both a weakening of the Frèchet property and a variation on the s property.

Definition 4.28. We call a point x in a space X a *set-Fréchet* point if whenever $A = \bigcup\{A_n : n < \omega\}$ with $x \in \overline{A}$ but $x \notin \overline{A_n}$ for all $n < \omega$ then there is a sequence $\{x_n\} \subset A$ such that $x_n \to x$. Of course, a space is set-Fréchet if all its points are.

Theorem 4.29. *Let f be a preserving function from a locally connected T_2 space X into the interval $[0, 1]$. Then f is weakly sequentially continuous at every set-Fréchet point x of X.*

Proof. Assume $x_n \to x$ but $f(x_n)$ does not converge to $f(x)$, moreover f is not locally constant at each x_n for $n < \omega$. By Lemma 1, we can assume that $f(x) = 1$ and $f(x_n) = 0$ for all $n < \omega$. Note that then for any connected neighbourhood G of any point x_n the image $f(G)$ is a proper interval containing 0. For every $n < \omega$ choose an open sets U_n such that $x_n \in U_n$ and $x \notin \overline{U}_n$ and put $A_n = \{z \in U_n : 0 < f(z) < 1/(n+1)\}$. By Lemma 4.7 the conditions in the definition of a set-Fréchet point are satisfied for the sets A_n so there is a sequence of points $z_n \in A = \bigcup\{A_n : n < \omega\}$ converging to x. It is immediate that $f(z_n)$ converges to $0 \neq 1 = f(x)$ while the set $\{f(z_n) : n < \omega\}$ is infinite because $f(z_n) \neq 0$ for all n, contradicting Lemma 1. □

4.2 From sequential continuity to continuity

The aim of this section is to prove a few results saying that if a locally connected space X fulfills one of the "convergence-type" conditions of the first section (i.e. X is an SC-space or it is inflatable or set-Fréchet) and $f : X \to Y$ is a preserving function then, assuming in addition appropriate separation axioms for X and Y, f is continuous at every s-point of X. The proofs of these theorems, just like their formulations, are very similar.

Theorem 4.30. *A preserving function $f : X \to Y$ from a locally connected SC-space X into a regular space Y is continuous at every s-point of X.*

Proof. Assume indirectly that f is not continuous at the s-point $p \in X$. Then there exists a closed set $F \subset Y$ such that $p \in \overline{f^{-1}(F)}$ but $f(p) \notin F$. Choose an open set $V \subset Y$ such that $F \subset V$ and $f(p) \notin \overline{V}$.

Let $\mathcal{K}$ be the family of the components of $f^{-1}(\overline{V})$ and put $\mathcal{A} = \{K \cap f^{-1}(F) : K \in \mathcal{K}\}$. As $p \notin \overline{K}$ for $K \in \mathcal{K}$ by Lemma 4.4 and $p \in \overline{f^{-1}(F)}$, the conditions given in the definition of an s point are fulfilled for the family $\mathcal{A}$. Thus there is a sequence $\{x_n : n < \omega\} \subset f^{-1}(F)$ such that $x_n \to x$ for some $x \in X$ and if K_n is the component of x_n in $f^{-1}(\overline{V})$, then $K_m \neq K_n$ for $m \neq n$.

As the components K_n are pairwise disjoint, we can suppose that $x \notin K_n$ for all $n < \omega$. It follows that if C is a connected set which contains both x and some x_n then $C \not\subset f^{-1}(\overline{V})$, because otherwise $K_n \cup C$ would be a connected subset of $f^{-1}(\overline{V})$ strictly larger than the component K_n, a contradiction. Hence, using that x is an SC-point, we may assume to have a sequence C_n of connected sets such that $C_n \to x$ and $C_n \not\subset f^{-1}(\overline{V})$. We can choose points $z_n \in C_n - f^{-1}(\overline{V})$ for all $n < \omega$, then $z_n \to x$ and $f(z_n) \notin \overline{V}$. But by Theorem 4.21 f is sequentially continuous, consequently $f(x) = \lim f(z_n) \in Y - V$. On the other hand, $f(x_n) \in F$ for $n < \omega$ and so, using again the sequential continuity of f at the point x, we get that $f(x) = \lim f(x_n) \in F$, a contradiction. □

The proofs of the other two analogous theorems for inflatable and set-Fréchet spaces, respectively, make essential use of the following lemma:

Lemma 4.31. *Assume that $f : X \to Y$ is a preserving and weakly sequentially continuous function from a locally connected T_3 space X into a T_3 space Y and f is not continuous at some s-point of X. Then there are two sets $F \subset V \subset Y$, F closed and V open in Y and a convergent sequence $x_n \to x$ in X such that for all $n < \omega$ we have $x_n \neq x$, $f(x_n) \in F$ but $f(U) \not\subset \overline{V}$ whenever U is a neighbourhood of x_n. It follows that f is not locally constant at the points x_n and x.*

Proof. Assume f is not continuous at the s-point $p \in X$, then there exists a closed set $F \subset Y$ such that $p \in \overline{f^{-1}(F)}$ but $f(p) \notin F$. Choose an open set $V \subset Y$ such that $F \subset V$ and $f(p) \notin \overline{V}$.

Put

$$B = \{x \in f^{-1}(F) : f \text{ is not locally constant at } x\},$$

then $p \in \overline{B}$ by Lemma 4.6. Now let

$$H = \{x \in f^{-1}(F) : f(U) \not\subset \overline{V} \text{ for every neighbourhood } U \text{ of } x\},$$

clearly $H \subset B$. We assert that $p \in \overline{H}$ as well. To show this, fix a closed neighbourhood W of p. Let $\mathcal{K}$ be the family of the components of $f^{-1}(\overline{V})$ and put $\mathcal{A} = \{K \cap B \cap W : K \in \mathcal{K}\}$. Since $p \notin \overline{K}$ for $K \in \mathcal{K}$ by Lemma 1.4, the conditions in the definition of an s-point are satisfied for p and $\mathcal{A}$. Thus there is a sequence $\{y_n : n < \omega\} \subset B \cap W$ such that $y_n \to y$ for some $y \in X$ and if K_n is the component of y_n in $f^{-1}(\overline{V})$, then $K_m \neq K_n$ if $m \neq n$.

We claim that $y \in W \cap H$. As W is closed, trivially $y \in W$. The sets K_n are disjoint so we can assume that $y \notin K_n$ for all $n < \omega$. Using that f is weakly sequentially continuous and $y_n \in B$, we get that $f(y) = \lim f(y_n) \in F$, hence $y \in B$ since B is closed in $f^{-1}(F)$. We have yet to show that $f(U) \not\subset \overline{V}$ if U is any neighbourhood of y. By local connectivity, we can suppose that U is connected. But the connected set U meets (infinitely many) distinct components of $f^{-1}(\overline{V})$, so indeed $U \not\subset f^{-1}(\overline{V})$.

Now applying the s-point property of p to the family $\mathcal{A} = \{\{x\} : x \in H\}$ there is an infinite sequence of points $x_n \in H$ converging to some point x, completing the proof. □

Theorem 4.32. *A preserving function from a locally connected and inflatable T_3 space X into a T_3 space Y is continuous at every s-point of X.*

Proof. Assume, indirectly, that the preserving function $f : X \to Y$ is not continuous at an s-point of X. By Theorem 4.27 f is weakly sequentially continuous, hence we can apply the preceding lemma and choose appropriate sets F, V in Y and points x_n and x in X. As X is inflatable and locally connected, we can also assume that there is a sequence of connected open sets U_n such that $U_n \to x$ and $x_n \in U_n$ for $n < \omega$. Then for all n we have $f(U_n) - \overline{V} \neq \emptyset$, hence by Lemma 4.14 these sets are uncountable. Consequently we can recursively select another sequence of points $y_n \in U_n$ (and so converging to x) such that $f(y_n) \in Y - \overline{V}$ and the $f(y_n)$'s are pairwise distinct. But then the sequence $f(y_n)$ does not converge to $f(x)$ because $f(x) = \lim f(x_n) \in F \subset V$ by the weak sequential continuity of f, contradicting Lemma 1 again. □

Corollary 4.33. *If X is locally compact, locally connected, and monotonically normal (in particular if X is a locally connected linearly ordered space) then $Pr(X, T_3)$ holds.*

Proof. See [26, theorem 3.12] for a proof that a (locally) compact, monotonically normal space is both inflatable and radial. Consequently, it is also locally sequentially compact, and thus an s-space. □

Theorem 4.34. *A preserving function from a locally connected and set-Frèchet T_3 space X into a $T_{3\frac{1}{2}}$ space is continuous at every s-point of X.*

Proof. By Lemma 4.2, it suffices to prove this for preserving functions $f : X \to [0, 1]$. Assume, indirectly, that the function f is not continuous at some s-point $p \in X$. Then, by Theorem 4.29, f is weakly sequentially continuous, hence we may again apply Lemma 4.31 to choose appropriate sets F and V in $Y = [0, 1]$ and points x_n and x in X. There is a continuous function $g : [0, 1] \to [0, 1]$ which is identically 1 on F and 0 on $[0, 1] - V$. Then the composite function $h = gf : X \to [0, 1]$ is also preserving, $h(x_n) = 1$ for all n, and the h-image of any neighbourhood of a point x_n is the whole interval $[0, 1]$.

Let us choose open sets G_n for $n < \omega$ such that $x_n \in G_n$ and $x \notin \overline{G}_n$. If $A_n = G_n \cap f^{-1}((0, \frac{1}{n}))$ and $A = \bigcup A_n$, then $x \notin \overline{A}_n$ but $x \in \overline{A}$. So, as X is a set-Frèchet space, there is a sequence of points $y_n \in A$ converging to x. But then the set $\{h(y_n) : n < \omega\}$ is infinite and $h(y_n) \to 0 \neq 1 = f(x)$, contradicting Lemma 1. □

Corollary 4.35. *If X is a locally connected, locally countably compact, and set-Frèchet T_3 space then $Pr(X, T_{3\frac{1}{2}})$ holds.*

Proof. It is enough to note that a countably compact set-Frèchet space is also sequentially compact and so an s-space. □

4.3 Some theorems on products

The aim of this section is to prove that an arbitrary product X of certain "good" spaces has the property $Pr(X, T_3)$. To achieve this, we shall make use of Theorem 4.30. It is well-known that any product of spaces that are both connected and locally connected is locally connected, moreover a similar argument (based on the productivity of connectedness) implies that the product of countably many connected SC spaces is SC. Hence two of the assumptions of Theorem 4.30 are countably productive if the factors are also connected. Nothing like this can be expected, however, about the third assumption of Theorem 4.30, namely the s-property. To make up for this, we are going to consider a stronger property that is countably productive, and use this stronger property to establish what we want, first for Σ-products and then for arbitrary products. Now, this stronger property will require the existence of a winning strategy for player **I** in the following game.

Definition 4.36. Fix a space X and a point $p \in X$. The game $G(X, p)$ is played by two players **I** and **II** in ω rounds. In the n-th round first **I** chooses a neighbourhood U_n of p and then **II** chooses a point $x_n \in U_n$. **I** wins if the produced sequence $\{x_n : n < \omega\}$ has a convergent subsequence, otherwise **II** wins.

We shall say that X is *winnable* at the point p if **I** has a winning strategy in the game $G(X,p)$. X is *winnable* if $G(X,p)$ is winnable for all $p \in X$. Note that, formally, a winning strategy for **I** is a map $\sigma : X^{<\omega} \to \mathcal{V}(p)$, where $\mathcal{V}(p)$ is the family of neighbourhoods of p, such that if $\langle x_n : n < \omega \rangle$ is a sequence obtained in a play of the game in which player **I** followed σ, i.e. $x_n \in \sigma\langle x_0, x_1, ..., x_{n-1}\rangle$ for all $n < \omega$, then $\langle x_n : n < \omega \rangle$ has a convergent subsequence.

Lemma 4.37. *A winnable point p of a space X is always an s point.*

Proof. Let σ be a winning strategy for **I** in the game $G(X,p)$ and fix a family $\mathcal{A}$ of subsets of X with $p \in \overline{\bigcup \mathcal{A}}$ but $p \notin \overline{A}$ for all $A \in \mathcal{A}$. Let us play the game $G(X,p)$ in such a way that **I** follows σ and assume that the first n rounds of the game have been played with the points $x_i \in U_i$ and the distinct sets $A_i \in \mathcal{A}$ with $x_i \in A_i$ chosen by player **II** for $i < n$. Let $U_n = \sigma\langle x_0, x_1, ..., x_{n-1}\rangle$ be the next winning move of **I**, then **II** can choose a set $A_n \in \mathcal{A}$ with $A_n \cap (U_n - \cup_{i<n}\overline{A}_i) \neq \emptyset$ and then pick $x_n \in A_n \cap (U_n - \cup_{i<n}\overline{A}_i)$ as his next move. But player **I** wins hence, a suitable subsequence of $\{x_n : n < \omega\}$, whose members were picked from distinct elements of $\mathcal{A}$, will converge. □

In order to prove the desired product theorem for winnable spaces we shall consider *monotone strategies* for player **I**. A strategy σ of player **I** is said to be *monotone* if for every subsequence $\langle x_{i_0}, ..., x_{i_{r-1}}\rangle$ of a sequence $\langle x_0, x_1, ..., x_{n-1}\rangle$ of points in X we have

$$\sigma\langle x_0, x_1, ..., x_{n-1}\rangle \subset \sigma\langle x_{i_0}, ..., x_{i_{r-1}}\rangle.$$

Lemma 4.38. *If player* **I** *has a winning strategy in the game $G\langle X,p\rangle$ then he also has a monotone winning strategy.*

Proof. Let σ be a winning strategy for player **I**; we define a new strategy σ_0 as follows : for any sequence $s = \langle x_0, x_1, ..., x_{n-1}\rangle$ put

$$\sigma_0(s) = \bigcap\{\sigma\langle x_{i_0}, ..., x_{i_{r-1}}\rangle : 0 \le i_0 < i_1 < ... < i_{r-1} < n\}.$$

The function σ_0 is clearly a monotone winning strategy for **I**. □

It is easy to see that if **I** plays using the monotone strategy σ_0 then any infinite subsequence of the sequence chosen by player **II** is also a win for **I**, i. e. has a convergent subsequence. Another important property of a monotone winning strategy σ_0 is the following: If $\langle x_n : n < \omega \rangle$ is a sequence of points in X such that we have $x_n \in \sigma_0\langle x_0, x_1, ..., x_{n-1}\rangle$ only for $n \ge m$ for some fixed $m < \omega$ then this is still a winning sequence for **I**. Indeed, this holds because for every $n \ge m$ we have

$$x_n \in \sigma_0\langle x_0, x_1, ..., x_{n-1}\rangle \subset \sigma_0\langle x_m, x_{m+1}, ..., x_{n-1}\rangle$$

by monotonicity, hence the "tail" sequence $\langle x_n : m \leq n < \omega \rangle$ is produced by a play of the game where **I** follows the strategy σ_0.

For a family of spaces $\{X_s : s \in S\}$ and a fixed point p (called the *base point*) of the product $X = \prod\{X_s : s \in S\}$ let $T(x)$ denote the support of the point x in X: this is the set $\{s \in S : x(s) \neq p(s)\}$. Then $\Sigma(p)$ (or simply Σ if this does not lead to misunderstanding) denotes the Σ-product with base point p: it is the subspace of X of the points with countable support, i.e.

$$\Sigma = \{x \in X : |T(x)| \leq \omega\}.$$

In the proof of the next result we shall use two lemmas. The first one is an easy combinatorial fact:

Lemma 4.39. *Let $\langle H_k : k < \omega \rangle$ be a sequence of countable sets, then for every $n < \omega$ there is a finite set F_n depending only on the first n many sets $\langle H_k : k < n \rangle$ such that $F_n \subset \bigcup_{i<n} H_i$, $F_n \subset F_{n+1}$ and $\bigcup F_n = \bigcup H_n$.*

Proof. Fix an enumeration $H_i = \{x(i,j) : j < \omega\}$ of the set H_i for all $i < \omega$ and then let $F_n = \{x(i,j) : i,j < n\}$. □

The second lemma is about certain sequences which play a crucial role in the games $G\langle X, p \rangle$. Let us call a sequence $\{x_n\}$ in the space X *good* if every infinite subsequence of it has a convergent subsequence.

Lemma 4.40. *If $x_n \in X = \prod\{X_i : i < \omega\}$ for $n < \omega$ and $\{x_n(i) : n < \omega\}$ is a good sequence in X_i for all $i \in \omega$ then $\{x_n\}$ is a good sequence in X.*

Proof. We shall prove that if N is any infinite subset of ω then there is in X a convergent subsequence of $\{x_n : n \in N\}$.

We can choose by recursion on $k < \omega$ infinite sets N_k such that $N_{k+1} \subset N_k \subset N$ and $\{x_n(k) : n \in N_k\}$ converges to a point $x(k)$ in X_k. Then there is a diagonal sequence $\{n_k : k < \omega\}$ such that $n_k \in N_k$ and $n_k < n_{k+1}$ for all $k < \omega$. The sequence $\{n_k : k < \omega\}$ is eventually contained in N_i, hence $x_{n_k}(i) \to x(i)$ in X_i, for all $i < \omega$. It follows that $\{x_{n_k}\}$ is a convergent subsequence of $\{x_n : n \in N\}$ in X. □

The following result says a little more than that winnability is a countably productive property.

Lemma 4.41. *Let $p \in X = \prod\{X_s : s \in S\}$ and suppose that $G\langle X_s, p(s) \rangle$ is winnable for every $s \in S$. Then $G\langle \Sigma(p), p \rangle$ is also winnable.*

Proof.

We have to construct a winning strategy σ for player **I** in the game $G\langle \Sigma(p), p \rangle$. By Lemma 4.37, we can fix a monotone winning strategy σ_s of **I** in the game $G\langle X_s, p(s) \rangle$

for each $s \in S$. Given a sequence $\langle x_0, ..., x_{n-1}\rangle \in [\Sigma(p)]^{<\omega}$, let H_i denote the support of x_i and F_n be the finite set assigned to the sequence $\{H_i : i < n\}$ as in Lemma 4.38. Now, if π_s is the projection from the product X onto the factor X_s for $s \in S$ then set

$$\sigma\langle x_i : i < n\rangle = \bigcap\{\pi_s^{-1}(\sigma_s\langle x_i(s) : i < n\rangle) : s \in F_n\rangle\}.$$

Now let $\langle x_n : n < \omega\rangle$ be a sequence of points in $\Sigma(p)$ produced by a play of the game $G\langle\Sigma(p), p\rangle$ in which **I** followed the strategy σ. Then for every $s \in H = \bigcup H_n$ the sequence $\langle x_n(s) : n < \omega\rangle$ is a win for player **I** in the game $G\langle X_s, p(s)\rangle$ because there is an $m < \omega$ with $s \in F_m$ and then $x_n(s) \in \sigma_s\langle x_i(s) : i < n\rangle$ is valid for all $n \geq m$. Consequently, by Lemma 4.39, the sequence $\langle x_n|H : n < \omega\rangle$ has a convergent subsequence in $\prod\{X_s : s \in H\}$, while for $s \in S - H$ we have $x_n(s) = p(s)$ for all $n < \omega$, and so $\langle x_n : n < \omega\rangle$ indeed has a convergent subsequence in $\Sigma(p)$. □

The following two statements both easily follow from the fact that any product of connected spaces is connected.

Lemma 4.42. *A Σ-product of connected SC spaces is also an SC space.* □

Lemma 4.43. *A Σ-product of connected and locally connected spaces is also locally connected.* □

We now have all the necessary ingredients needed to prove our main product theorem.

Theorem 4.44. *Let $f : X = \prod\{X_s : s \in S\} \to Y$ be a preserving function from a product of connected and locally connected SC spaces into a regular space Y. If $p \in X$ and $G\langle X_s, p(s)\rangle$ is winnable for all $s \in S$ then f is continuous at the point p.*

Proof. Let Σ denote the sigma-product with base point p. Then, by Lemma 4.41, $G\langle\Sigma, p\rangle$ is winnable and so p is an s point in Σ. Moreover, by Lemmas 4.42 and 4.43, Σ is also a locally connected SC space. Hence Theorem 4.30 implies that the restriction of the function f to the subspace Σ of X is continuous at p.

To prove that f is also continuous at the point p in X, fix a neighbourhood V of $f(p)$ in Y. As the restriction $f|\Sigma$ is continuous at p, there is an elementary neighbourhood U of p in the product space X such that $f(U \cap \Sigma) \subset V$. Since the factors X_s are connected and locally connected, we can assume that U and $U \cap \Sigma$ are also connected and hence, by Lemma 4.4, we have

$$f(U) \subset f(\overline{U \cap \Sigma}) \subset \overline{f(U \cap \Sigma)} \subset \overline{V}.$$

The regularity of Y then implies that f is continuous at p. □

Corollary 4.45. *$Pr(X, T_3)$ holds whenever X is any product of connected and locally connected, winnable SC spaces. In particular, if $X = \prod\{X_s : s \in S\}$ where each factor X_s is either a connected linearly ordered space (with the order topology) or a connected and locally connected first countable space then $Pr(X, T_3)$ is valid.* □

For the proof of the next Corollary we need a general fact about the relations $Pr(X, T_i)$.

Lemma 4.46. *If $q \colon X \to Y$ is a quotient mapping of X onto Y then, for any i, $Pr(X, T_i)$ implies $Pr(Y, T_i)$.*

Proof. Let $f \colon Y \to Z$ be a preserving function into the T_i space Z. The function $fq \colon X \to Z$, as the composition of a continuous (and so preserving) and of a preserving function is also preserving, hence, by $Pr(X, T_i)$, it is continuous. But then f is continuous because q is quotient. □

Corollary 4.47. *Let $X = \prod\{X_s : s \in S\}$ where all factors X_s are compact, connected, locally connected, and monotonically normal. Then $Pr(X, T_2)$ holds.*

Proof. It follows from the recent solution by Mary Ellen Rudin of Nikiel's conjecture [52], combined with results of L.B.Treybig [68] or J.Nikiel [48], that *every compact, connected, locally connected, monotonically normal space is the continuous image of a compact, connected, linearly ordered space.* Hence our space X is the continuous image of a product of compact, connected, linearly ordered spaces. But any T_2 continuous image of a compact T_2 space is a quotient image (and T_3), hence Corollary 4.45 and Lemma 4.46 imply our claim. □

Comparing this result with Corollary 4.33, the following question is raised naturally.

Problem 4.48. *Let X be a product of locally compact,connected and locally connected monotonically normal spaces. Is then $Pr(X, T_3)$ true?*

The following is result is mentioned here mainly as a curiosity.

Corollary 4.49. *Let $X = \prod\{X_s : s \in S\}$ be a product of linearly ordered and/or first countable $T_{3\frac{1}{2}}$ spaces. Then the following are equivalent:*

a) $Pr(X, T_3)$;

b) X is locally connected;

c) the spaces X_s are all locally connected and all but finitely many of them are also connected.

Proof. a)⇒b) : Lemma 4.1.
b)⇒c) : [15, 6.3.4]
c)⇒a) : Corollary 4.44. □

Remark 4.50.
E. R. McMillan raised the following question in [46]: does $Pr(X, T_2)$ imply that X is a k-space? We do not know the (probably negative) answer to this question, however we do know that the answer is negative if T_2 is replaced by T_3 in it. Indeed, for instance $\mathbb{R}^{\omega_1}$ is not a k-space (see e.g. [15, exercise 3.3.E],), but, by Corollary 4.45, $Pr(\mathbb{R}^{\omega_1}, T_3)$ is valid.

4.4 The sequential and the compact cases

The two examples 1.23 and 1.24, given in section 1, of (locally connected) spaces on which there are non-continuous preserving functions both lack the properties of (i) sequentiality and (ii) compactness. Here (i) arises naturally as a weakening of the Frèchet property figuring in McMillan's theorem, while the significance of (ii) needs no explanation. This leads us naturally to the following problem.

Problem 4.51. *Assume that the locally connected space X is (i) sequential and/or (ii) compact. Is then $Pr(X, T_2)$ (or $Pr(X, T_{3\frac{1}{2}})$) true?*

The answer in case (i) turns out to be positive if we assume the SC property instead of local connectivity. The following result reveals why local connectivity need not be assumed in it.(Compare this also with Lemma 4.18.)

Theorem 4.52. *Any sequential SC space X is locally connected.*

Proof. We have to prove that if K is a component of an open set $G \subset X$ then K is open. Assume not, then $X - K$ is not closed, hence as X is sequential there is a sequence $\{x_n\} \subset X - K$ such that $x_n \to x \in K$. Since X is an SC space and G is a neighbourhood of x there is a connected set $C \subset G$ such that $\{x, x_n\} \subset C$ for some $n < \omega$. But this is impossible because then the connected set $K \cup C \subset G$ would be larger than the component K of G. □

Now we give the above promised partial solution to Problem 4.51 in case (i), i. e. for sequential spaces.

Theorem 4.53. *If X is a sequential SC space then $Pr(X, T_2)$ holds.*

Proof. Let $f : X \to Y$ be a preserving map into a T_2 space Y. Since X is sequential it suffices to show that the function f is sequentially continuous but this is immediate from Theorem 4.21. □

Our next result implies that a counterexample to Problem 4.51 (in either case) can not be very simple in the sense that discontinuity of a preserving function can not occur only at a single point (as it does in both examples 4.23 and 4.24). In order to prepare this result we first introduce a topological property that generalizes both sequentiality and (even countable) compactness.

Definition 4.54. X is called a *countably* k space if for any set $A \subset X$ that is not closed in X there is a countably compact subspace C of X such that $A \cap C$ is not closed in C.

This condition means that the topology of X is determined by its countably compact subspaces. All countably compact and all k (hence also all sequential) spaces are countably k. It is easy to see that the countably k property is always inherited by closed subspaces and for regular spaces by open subspaces as well.

Theorem 4.55. *Let X be countably k and locally connected and Y be T_3, moreover let $f\colon X \to Y$ be a preserving function. Then the set of points of discontinuity of f is not a singleton. So if X is also T_3 then the discontinuity set of f is dense in itself.*

Proof. Assume, indirectly, that f is not continuous at $p \in X$ but it is continuous at all other points of X. Then we can choose a closed set $F \subset Y$ with $A = f^{-1}(F)$ not closed. Evidently, then $\overline{A} - A = \{p\}$. As A is not closed in X and X is countably k, there is a countably compact set C in X such that $A \cap C$ is not closed in C; clearly then $p \in C$ and p is in the closure of $A \cap C$.

Let $V \subset Y$ be an open set with $F \subset V$ and $f(p) \notin \overline{V}$ and put $H = f^{-1}(V)$. Then H is open in X and contains the set A. For every component L of H we have $p \notin \overline{L}$ by $f(p) \notin \overline{L} \subset \overline{V}$ and by Lemma 4.4, hence $p \in \overline{A \cap C}$ implies that there are infinitely many components of H that meet $A \cap C$. Thus we may choose a sequence $\{L_n : n < \omega\}$ of distinct such components with points $x_n \in L_n \cap A \cap C$.

We claim that $x_n \to p$. As the x_n's are chosen from the countably compact set C, it is enough to prove for this that if $x \neq p$ then x is not an accumulation point of the sequence $\{x_n\}$. If $x \notin \overline{A} = A \cup \{p\}$ this is obvious so we may assume that $x \in A \subset H$. But then, as H is open and X is locally connected, the connected component of x in H is a neighbourhood of x that contains at most one of the points x_n.

The point $f(p)$ is not in the closure of the set $\{f(x_n)\colon n < \omega\} \subset F$, hence, by Lemma 1, we can suppose that $f(x_n) = y \neq f(p)$ for each $n < \omega$.

Now we choose a sequence of neighbourhoods V_n of y in Y with $V_0 = V$ and $\overline{V_{n+1}} \subset V_n$ for all $n < \omega$ and then put $U_n = f^{-1}(V_n)$. Clearly U_n is open in X and $U_0 = H$, hence, as was noted above, the closure of any connected set contained in U_0 contains at most one of the points x_n.

Next, let K_n be the component of x_n in U_n for $n < \omega$. We claim that every boundary point of K_n is mapped by f to a boundary point of V_n, i. e.

$$f(FrK_n) \subset FrV_n.$$

Indeed, $f(\overline{K_n}) \subset \overline{V_n}$ by Lemma 4.4 (or by continuity at all points distinct from p). Moreover, we have

$$FrK_n = \overline{K_n} - K_n \subset FrU_n = \overline{U_n} - U_n$$

because K_n, U_n are open and K_n is a component of U_n, and $f(FrU_n) \cap V_n = \emptyset$ because $U_n = f^{-1}(V_n)$. Thus indeed $f(FrK_n) \subset \overline{V_n} - V_n = FrV_n$.

Every connected neighbourhood of p meets all but finitely many K_n's hence also FrK_n by local connectivity, consequently $K = \bigcup\{FrK_n : n < \omega\}$ is not closed. So there exists a countably compact set D with $D \cap K$ not closed in D. But the sets FrK_n are closed, thus D must meet infinitely many of them, i. e. the set

$$N = \{n < \omega : D \cap FrK_n \neq \emptyset\}$$

is infinite. Let us choose a point z_n from each nonempty $D \cap K_n$.

Again we claim that the only accumulation point of the sequence $\{z_n : n \in N\}$ is p. Indeed, if $x \neq p$ would be such an accumulation point, then $f(z_n) \in \overline{V_n} \subset \overline{V_1}$ for all $n \in N - \{0\}$ would also also imply $f(x) \in \overline{V_1} \subset V_0$. By continuity and local connectivity at x then there is a connected neighbourhood W of x with $f(W) \subset V_0$. But then the set

$$W \cup \bigcup\{K_n : W \cap K_n \neq \emptyset\}$$

would be a connected subset of U_0 that has p in its closure, a contradiction.

Consequently, the sequence $\{z_n : n \in N\}$ must converge to p, while $\{f(z_n) : n \in N\} \subset \overline{V}$ does not converge to $f(p)$, contradicting Lemma 1 because $f(z_n) \in FrV_n$ for all $n \in N$ and the boundaries FrV_n are pairwise disjoint, hence the $f(z_n)$'s are pairwise distinct.

The last statement of the theorem now follows easily because an isolated discontinuity of f yields an open subspace of X on which the restriction of f has a single point of discontinuity, although if X is T_3 then any open subspace of X is both countably k and locally connected. □

Noting that Problem 4.51 really comprises three different questions, and having shown above that, in a certain sense, it seems to be hard to find counterexamples to any of these, we now turn our attention to the case in which both (i) and (ii) are assumed. In this case we can provide a positive answer, at least consistently and with the extra assumption that the cellularity of the space in question is "not too large". In fact, what we can prove is that if $2^\omega < 2^{\mathfrak{p}}$ then any locally connected, compact T_2 space X that is sequential and does not contain a cellular family of size $\mathfrak{p}$ satisfies $Pr(X, T_2)$. Of course, here $\mathfrak{p}$ stands for the well-known cardinal invariant of the continuum whose definition is recalled below.

A set $H \subset \omega$ is called a *pseudo intersection* of the family $\mathcal{A} \subset [\omega]^\omega$ if H is almost contained in every member of $\mathcal{A}$, i.e. $H - A$ is finite for each $A \in \mathcal{A}$. Then $\mathfrak{p}$ is the minimal cardinal κ such that there exists a family $\mathcal{A} \subset [\omega]^\omega$ of size κ which has the finite intersection property but does not have an infinite pseudo intersection. (Here the finite intersection property means that any finite subfamily of $\mathcal{A}$ has *infinite* intersection.)

It is well-known (see e.g.[69]) that the cardinal p is regular, $\omega_1 \le p \le 2^\omega$ and $2^\kappa = 2^\omega$ for $\omega \le \kappa < p$. The condition "$2^p > 2^\omega$" of our result is satisfied if $p = 2^\omega$ (hence Martin's axiom implies it), but it is also true if $2^{\omega_1} > 2^\omega$.

Now, our promised consistency result on compact sequential spaces will be a corollary of a ZFC result of somewhat technical nature. Before formulating this, however, we shall prove two lemmas that may have some independent interest in themselves. We recall that a G_κ (resp. $G_{<\kappa}$) set in a space is one that is the intersection of κ (resp. fewer than κ) many open sets.

Lemma 4.56. *Let X be a compact T_2 space of countable tightness and $f : X \to [0,1]$ be a compactness preserving map of X into the unit interval. If $x \in X$ is a point in X and $[a,b]$ is a subinterval of $[0,1]$ such that for every neighbourhood U of x we have $[a,b] \subset f(U)$ then for any $G_{<p}$ set H containing x we also have $[a,b] \subset f(H)$.*

Proof. Without loss of generality we may assume that H is closed. Now the proof will proceed by induction on κ where $\omega \le \kappa < p$ and H is a (closed) G_κ set, or equivalently, the character $\chi(H,X) = \kappa$. If $\kappa = \omega$ then we can write $H = \bigcap\{G_n : n < \omega\}$ with G_n open and $\overline{G_{n+1}} \subset G_n$ for all $n < \omega$. Fix a countable dense subset $\{c_n : n < \omega\}$ of $[a,b]$ and then pick $x_n \in G_n$ with $f(x_n) = c_n$, this is possible by our assumption. Note that then every accumulation point of the set $M = \{x_n : n < \omega\}$ is in H, hence by Lemma 4.3 we have

$$[a,b] = f(M)' \subset f(M') \subset f(H).$$

Next, if $\omega < \kappa < p$ then we have $x \in H = \bigcap\{S_\xi : \xi < \kappa\}$, where $S_\xi \supset S_\eta$ if $\xi < \eta$ and the S_ξ are closed sets of character $< \kappa$. By induction, we have $f(S_\xi) \supset [a,b]$ for all $\xi < \kappa$, and we have to prove that $f(H) \supset [a,b]$ as well. In fact, it suffices to show that $f(H) \cap [a,b] \ne \emptyset$ because applying this to all (non-singleton) subintervals of $[a,b]$ we actually get that $f(H) \cap [a,b]$ is dense in $[a,b]$ while $f(H)$ is also compact, hence closed.

We do this indirectly, i. e. we assume that $f(H) \cap [a,b] = \emptyset$.We may then easily choose points $x_\xi \in S_\xi - H$ for all $\xi < \kappa$ such that their images $f(x_\xi) \in [a,b]$ are all distinct. Let $\bar{x}$ be a complete accumulation point of the set $\{x_\xi : \xi < \kappa\}$. Then $\bar{x} \in H$ and $t(X) = \omega$ implies that there is a countable subset $A \subset \{x_\xi : \xi < \kappa\}$ such that $\bar{x} \in \overline{A} - A$. Choose now a neighbourhood base $\mathcal{B}$ of H in X of size $\kappa < p$. The family $\{A \cap B : B \in \mathcal{B}\} \subset [A]^\omega$ has the finite intersection property hence it has an infinite pseudo intersection $P \subset A$, i. e. the set $P - B$ is finite for each $B \in \mathcal{B}$. This implies that every accumulation point of P is contained in H. But $\overline{P}$ is compact, hence by Lemma 4.3 we have

$$\emptyset \ne f(P)' \subset f(P') \cap [a,b] \subset f(H) \cap [a,b],$$

which is a contradiction. □

Before we state the other lemma, let us recall that for any space X we use $\widehat{c}(X)$ to denote the smallest cardinal κ such that X does *not* contain κ disjoint open sets.

Lemma 4.57. *Let $f : X \to Y$ be a connectivity preserving map from a locally connected space X into a T_2 space Y. Then for every $x \in X$ with $\chi(f(x), Y) < \widehat{c}(X)$ there is a $G_{<\widehat{c}(X)}$ set H in X such that $x \in H$ and if $z \in H$ is any point of continuity of f then $f(z) = f(x)$.*

Proof. Let $\kappa = \widehat{c}(X)$ and fix a neighbourhood base $\mathcal{V}$ of the point $f(x)$ in Y with $|\mathcal{V}| < \kappa$. For every $V \in \mathcal{V}$ let us then set

$$G_V = \bigcup\{G : G \text{ is open in } X \text{ and } f(G) \cap V = \emptyset\}.$$

For every component K of the open set G_V we have $f(x) \notin \overline{f(K)}$ and therefore $x \notin \overline{K}$ by Lemma 4.4, moreover the components of G_V form a cellular family because X is locally connected, hence their number is less than κ. Consequently,

$$H_V = \bigcap\{X - \overline{K} : K \text{ is a component of } G_V\}$$

is a $G_{<\kappa}$ set with $x \in H_V$ and $H_V \cap G_V = \emptyset$.

The cardinal κ is regular (see e.g. [25, 4.1]), hence $H = \cap\{H_V : V \in \mathcal{V}\}$ is also a $G_{<\kappa}$ set that contains the point x. Now, suppose that z is a point of continuity of f with $f(z) \neq f(x)$. Then there is a basic neighbourhood $V \in \mathcal{V}$ of $f(x)$ and a neighbourhood W of $f(z)$ with $V \cap W = \emptyset$, and there is an open neighbourhood U of z in X with $f(U) \subset W$. But then, by definition, we have $z \in U \subset G_V$, hence $z \notin H_V \supset H$. $\square$

Theorem 4.58. *Let X be a locally connected compact T_2 space of countable tightness. If, in addition, we also have $|X| < 2^{\mathfrak{p}}$ and $\widehat{c}(X) \leq \mathfrak{p}$ then $Pr(X, T_2)$ holds.*

Proof. Using Lemma 4.2 it suffices to show that any preserving function $f : X \to [0, 1]$ is continuous. To this end, first note that if f is not continuous at a point $x \in X$ then the oscillation of f at x is positive, hence , by local connectivity at x and because f is preserving there are $0 \leq a < b \leq 1$ such that $f(U) \supset [a, b]$ holds for every neighbourhood U of x. Consequently, by Lemma 4.56 we also have $f(H) \supset [a, b]$ whenever H is any $G_{<\mathfrak{p}}$ set containing the point x. In particular, this implies that if the singleton $\{x\}$ is a $G_{<\mathfrak{p}}$ set (equivalently, if the character of x in X is less than $\mathfrak{p}$) then f is continuous at x.

On the other hand, by Lemma 4.57, for every point $x \in X$ there is a closed $G_{<\mathfrak{p}}$ set H_x with $x \in H_x$ such that for any point of continuity $z \in H_x$ of f we have $f(z) = f(x)$. We claim that f is constant on every such set H_x and then, by the above, f is continuous at every point $x \in X$.

For this it suffices to show that f has a point of continuity in every (non-empty) closed $G_{<p}$ set H. Indeed, for any point $y \in H_x$ then the intersection $H_x \cap H_y$ contains a point of continuity z for which $f(x) = f(z) = f(y)$ must hold. By the Čech-Pospišil theorem (see e.g. [25, 3.16]) and by $|H| < 2^p$ there is a point $z \in H$ with $\chi(z, H) < p$ and so $\chi(z, X) < p$ as well, for H is a $G_{<p}$ set in X. But we have seen above that then z is indeed a point of continuity of f. □

Theorem 4.59. *Assume that $2^\omega < 2^p$ and X is a locally connected and sequential compact T_2 space with $\widehat{c}(X) \le p$. Then $Pr(X, T_2)$ holds.*

Proof. By a slight strengthening of some well-known results of Shapirovski (see e.g. [25, 2.37 and 3.14]), for any compact T_2 space X we have both $\pi\chi(X) \le t(X)$ and

$$d(X) \le \pi\chi(X)^{<\widehat{c}(X)}.$$

Consequently, for our space X we have

$$d(X) \le \omega^{<p} = 2^\omega$$

and so by sequentiality $|X| \le 2^\omega$ as well. But this shows that all the conditions of Theorem 4.58 are satisfied by our space X. □

To conclude, let us emphasize again that Lemma 4.3, i.e. the full force of compactness preservation, as opposed to just the preservation of the compactness of convergent sequences, was only used in this section (cf. the remark made after 1).

4.5 The relation $Pr(X, T_1)$

The main aim of this section is to prove that if $Pr(X, T_1)$ holds and X is T_3 then X is discrete. Note the striking contrast between $Pr(X, T_1)$ and $Pr(X, T_2)$: the latter holds for large classes of (non-discrete) spaces (see Theorem 4.8 or Corollary 4.45).

Let us recall that the *cofinite topology* on an underlying set X is the coarsest T_1 topology on X: the open sets are the empty set and the complements of the finite subsets of X. It is not hard to see that such a space is hereditarily compact and any infinite subset in it is connected. Let us start with a result that gives several different characterizations of T_1 spaces X that satisfy $Pr(X, T_1)$.

Theorem 4.60. *For a T_1 space X the following conditions are equivalent:*

a) If Y is T_1 and $f : X \to Y$ is a connectedness preserving function then f is continuous.

b) If Y is T_1 and $f : X \to Y$ is a preserving function then f is continuous (i.e. $Pr(X, T_1)$ holds).

c) If Y has the cofinite topology and $f : X \to Y$ is a preserving function then f is continuous.

d) If $A \subset X$ is not closed then there exists a connected set $H \subset X$ such that $H \cap A \neq \emptyset \neq H - A$ and $H - A$ is finite.

Proof. a) $\Rightarrow$ b) and b)$\Rightarrow$ c) are obvious.

c) $\Rightarrow$ d) Assume that $A \subset X$ is not closed. Let Y denote the space with the cofinite topology on the underlying set of X. Choose a point $a_0 \in A$. (A is not closed so it is not empty, either.) Define now the function $f : X \to Y$ by

$$f(x) = \begin{cases} a_0 & \text{if } x \in A, \\ x, & \text{otherwise.} \end{cases}$$

Then f is not continuous because the inverse image of the closed set $\{a_0\}$ is the non-closed set A hence, by c), f is not preserving. As an arbitrary subset of Y is compact, f preserves compactness, so there is a connected set $H \subset X$ such that $f(H)$ is not connected. It follows that H is infinite and $f(H)$ is finite but not a singleton. As f is the identity map on $X - A$, the set $H - A$ is finite and so $H \cap A \neq \emptyset$.Finally, $H \subset A$ is impossible because $f(H)$ is not a singleton.

d) $\Rightarrow$ a) Assume $f : X \to Y$ is not continuous for a T_1 space Y, hence there is a closed set $F \subset X$ such that $A = f^{-1}(F)$ is not closed in X. By d), there is a connected set H such that $H \cap A \neq \emptyset$ and $\emptyset \neq H - A$ is finite. But then $f(H)$ is not connected because it is the the disjoint union of two non-empty relatively closed sets, namely of $f(H) \cap F$ and of the finite set $f(H) - F$. Consequently, f does not preserve connectedness. $\square$

Corollary 4.61. *If $Pr(X, T_1)$ holds for a T_1 space X then every closed subspace of X is the topological sum of its components.*

Proof. Let K be a component of the closed subset $F \subset X$. It is enough to prove that K is relatively open in F. Assume this is false; then $A = F - K$ is not closed in X, and thus, by condition d) of Theorem 4.60, there is a connected set H in X such that $H \cap A \neq \emptyset$ and $\emptyset \neq H - A$ is finite. Then $H - F$ is also finite, consequently $H \subset F$ because H is connected and F is closed. Thus H is a connected subset of F which meets the component K of F, contradicting that $H \cap A \neq \emptyset$. $\square$

Corollary 4.62. *If $Pr(X, T_1)$ holds for a T_3 space X then every closed subspace of X is locally connected.*

Proof. By 4.61 it is enough to prove that if every closed subset of a regular space X is the topological sum of its components then X is locally connected.

Let U be a closed neighbourhood of a point $x \in X$. By our assumption if K denotes the component of x in U then K is open in U, hence $K \subset U$ is a connected neighbourhood of x in X. As the closed neighbourhoods of a point form a neighbourhood base of the point in a regular space, X is locally connected. $\square$

Theorem 4.63. *If $Pr(X, T_1)$ holds for a T_3 space X then X is discrete.*

By Corollary 4.62 it is enough to prove the following result that, we think, is interesting in itself:

Theorem 4.64. *If X is T_3 and every regular closed subspace of X is locally connected then X is discrete.*

Proof. We can assume without any loss of generality that X is connected. Suppose, indirectly, that X is not a singleton and fix a (non-isolated) point x in X. By regularity, there is a sequence of non-empty open sets $\{G_n : n < \omega\}$ such that $x \notin \overline{G_n}$ and $\overline{G}_n \subset G_{n+1}$ for all $n < \omega$. Then the open set $G = \bigcup\{G_n : n < \omega\}$ can not be also closed in the connected space X, so there is a point $p \in \overline{G} - G$.

Put $U_0 = G_0$ and $U_n = G_n - \overline{G}_{n-1}$ for $0 < n < \omega$. If $H_0 = \bigcup\{U_n : n \text{ is even}\}$ and $H_1 = \bigcup\{U_n : n \text{ is odd}\}$, then $G = H_0 \cup H_1$, hence $\overline{G} = \overline{H_0} \cup \overline{H_1}$. Consequently, $p \in \overline{H}_0$ or $p \in \overline{H}_1$; assume e. g. that $p \in \overline{H}_0$. We shall show that then $\overline{H}_0$ is not locally connected at p, although it is a regular closed set, arriving at a contradiction.

Indeed, let U be any neighbourhood of p in $\overline{H_0}$ and fix an even number $n < \omega$ with $U_n \cap U \neq \emptyset$. Then $U \cap U_{n+1} \subset \overline{H_0} \cap H_1 = \emptyset$ implies $U \subset \overline{G_n} \cup (X \setminus G_{n+1})$, where $\overline{G_n}$ and $X \setminus G_{n+1}$ are disjoint closed sets both meeting U, hence U is disconnected. $\square$

With a little more effort it can also be shown that for any non-isolated point p in a T_3 space X there is a regular closed set H in X with $p \in H$ such that p is not a local connectivity point in H.

We do not know if every T_2 space X with the property $Pr(X, T_1)$ has to be discrete. Also, the following T_2 version of Theorem 4.64 seems to be open: *If X is T_2 and all closed subspaces of X are locally connected then X has to be discrete.* Note that if X has the cofinite topology then it is hereditarily locally connected and satisfies $Pr(X, T_1)$.

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