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#### Abstract

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#### Abstract

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## Prologue

...the determination of the value of an item must not be based on its price, but rather on the utility it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.
(Daniel Bernoulli, [9])
This dissertation is based on the articles [72, 23, 78, 21, 22, 74, 76, 47]. I have included most of the proofs. Tedious, but standard arguments of advanced measure theory are sometimes omitted, I refer to the original papers instead. The purpose of the present work is to explain in detail certain novel results on optimal investment. This required a thorough rewriting of the material in the above mentioned articles so as to unify notation, to highlight the internal relationships between the dissertation's topics, to provide as streamlined a presentation as possible, and to explain the most important underlying ideas in detail. Some proofs of the earlier articles could be simplified using later developments. Some results were stated in greater generality than in the papers while in some cases I opted for a less general version for reasons of simplicity. I hope I managed to give an overview that pleases the gentle readers. In the sequel I shall employ the plural that is usual in scientific texts: "we present. . .", "we shall prove. . .", etc.

We deal with decisions under risk and consider investors acting in a financial market who wish to find the best available portfolio. Investors may have diverse preferences. The prevailing, classical approach in economic theory is to model preferences of an individual by a utility funcion $u$ which assigns a numerical value to each possible level of wealth, and to rank investments by comparing the expectations of the their future utility. Furthermore, $u$ is usually assumed concave to express investors' aversion of risk. More recent theories, based on the observed behaviour of investors, drop concavity of $u$ and calculate expectations using distorted probabilities. Mathematics have not yet caught up with these new developments. In our present work we report progress in this direction.

Our main results (Theorem 2.1, Corollary 2.20, Theorems 3.4, 3.16, 4.16, 4.18 and 5.12) establish the existence of optimal strategies in various classes of financial market models. These theorems (and some related counterexamples) delineate the types of utilities which are promising candidates for future applications and they also foreshadow the difficulties for finding efficient optimization algorithms. We significantly surpass previously available results: we investigate what happens if $u$ fails to be concave (Chapter 2); we treat illiquid markets where securities cannot be traded at fixed prices in arbitrarily large volumes (Chapter 5); moreover, investment problems with distorted probabilities will be addressed both in discrete- (Chapter 3) and in continuous-time (Chapter 4) models. The Appendix collects auxiliary results. Necessary concepts and notations will be defined along the way. Earlier chapters are prerequisites for later ones.

The author is eager to receive comments and to engage in scientific discussions on the topics treated here. The gentle readers are encouraged either to write an e-mail to

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Gödöllő, 2nd December, 2015.

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## 1 Fundamental concepts

In this introductory chapter we first briefly explain standard notions of expected utility theory and indicate their rôle in subsequent chapters. We define the mathematical models of financial markets in discrete and in continuous time which we will study in Chapters 1-4.

### 1.1 Expected utility theory

An agent is to choose between two gambles with random payoffs $X$ and $Y$. Which one should (s)he prefer? The simplest approach is to compare their expectations $E X$ and $E Y$ but this was found to be unsatisfactory already back in the 17th century. Daniel Bernoulli in [9], motivated partly by what is known as the "St. Petersburg paradox" (see [32]), proposed to compare $E u(X)$ and $E u(Y)$ instead, with an appropriate function $u: \mathbb{R} \rightarrow \mathbb{R}$. Going back to [31], the quantity $E u(X)$ had been called the "moral value/expectation" of $X$ up to the second half of the 20th century when the term "expected utility" became standard usage, $u$ being called the investor's utility function.

Bernoulli's ideas fertilized economic theory following the works [64, 102] and led to impressive developments in understanding market equilibria, see [4]. Expected utility theory (EUT) became a major subject of investigation in mathematical finance as well. Starting with [65, 89], a vast literature on optimal investment problems sprang where $E u(X)$ needed to be maximised over the possible portfolio values $X$ of the given investor. Expected utilities $E u(X)$ with various choices of $u$ are major tools for measuring the risk of a financial position $X$ and also for derivative pricing, see [26].

In mainstream economics, two properties of $u$ are universally accepted: $u$ should be increasing and concave. Monotonicity is explained by preferring more money to less. Concavity is customarily justified by the following argument: when the investor has $x>0$ pounds, (s)he is made less happy by earning one more pound than in the case where (s)he has $y>x$ pounds, i.e. the derivative $u^{\prime}$ (if exists) should be decreasing. Similarly, losing one more pound is less painful for someone with a loss of $x<0$ than for someone with a loss of $y<x$. The concavity property of $u$ is called risk aversion in the economics literature since Jensen's inequality $E u(X) \leq u(E X)$ implies that the expectation of $X$ (a deterministic number, the "riskless equivalent" of $X$ ) is preferred to the random payoff $X$ itself. Analogously, convexity of $u$ (perhaps only on a subset) is referred to as a risk-seeking attitude.

Concavity of $u$ has unquestionable mathematical advantages: a unique optimiser is found in most cases of maximising expected utility and it can often be calculated in an efficient way. The arguments for concavity presented above, however, are not entirely convincing: one can easily imagine that an investor below a desired wealth level $w$ is willing to take risks in order to ameliorate his/her position (and thus $u$ may be convex below $w$ ) and at $w$ his/her attitude may switch to risk aversion ( $u$ being concave above $w$ ). It has been demonstrated in [58, 101] (see also the references therein) that psychological experiments contradict the hypothesis of a concave $u$ in human decision-making.

In Chapter 2 we shall prove the existence of optimal strategies for investors maximising their expected utility from terminal wealth with possibly non-concave $u$, in a discrete-time model of a frictionless financial market. Two standard ways for proving such existence theorems are the Banach-Saks theorem (and its variants, e.g. the Komlós theorem, see [93]) and an indirect approach through the dual convex optimisation problem (which is easier to treat, see [57, 91]). For $u$ non-concave both methods fail but dynamic programming and direct estimates will save the day.

An even more deadly blow to EUT is the observation that investors tend to have distorted views of their chances, they exaggerate the probabilities of unlikely events, see [58, 71, 101]. In the mathematical theory this leads to nonlinear expectations which cannot be handled by the machinery of dynamic programming. Here we need an entirely different approach, see Chapters 3 and 4 for our results in this direction. In Chapter 5 we return to concave $u$ but investigate what happens in the presence of market illiquidity (i.e. when trading speed influences prices).

### 1.2 Market model in discrete time

We now expound the standard model for a market where trading takes place at finitely many time instants, see e.g. [44, 54]. We shall stay in the setting of the present section throughout Chapters 2 and 3 and also in Section 4.4.

The concatenation $x y$ of two vectors $x, y \in \mathbb{R}^{n}$ of equal dimension denotes scalar product, $|x|$ denotes the Euclidean norm of $x$. Let us fix a probability space $(\Omega, \mathcal{F}, P)$ and a time horizon $T \in \mathbb{N} \backslash\{0\}$. Let $\mathcal{F}_{t}, t=0, \ldots, T$ be a growing sequence of sub-sigma-algebras of $\mathcal{F}$, that is, a discrete-time filtration. Every sigma-algebra on $\Omega$ appearing in this dissertation is assumed to contain all sets of outer $P$-measure zero. For a probability measure $Q$ on $(\Omega, \mathcal{F})$ we will use $E_{Q} X$ to denote the expectation under $Q$. When $Q=P$ we shall write $E X$ instead of $E_{P} X$ in most cases.

We denote by $\Xi_{t}^{n}$ the set of $\mathbb{R}^{n}$-valued $\mathcal{F}_{t}$-measurable random variables. For $x \in \mathbb{R}$ we set $x^{+}:=\max \{x, 0\}$ and $x^{-}:=\max \{-x, 0\}$. Similarly, for a real-valued function $f(\cdot), f^{+}(x)$ (resp. $f^{-}(x)$ ) will denote the positive (resp. negative) part of $f(x)$.

For some $d \in \mathbb{N} \backslash\{0\}$, let $S_{t}, t=0, \ldots, T$ be a $d$-dimensional stochastic process adapted to the given filtration, describing the prices of $d$ risky securities in the given economy. For the validity of the results below we do not need to assume positivity of the prices and in certain cases (e.g. investments with possible losses) this would not be a reasonable restriction either.

We also suppose that there is a riskless asset (bond or bank account) with constant price $S_{t}^{0}:=1$, for all $t$. Incorporating a nonzero interest rate could be done in a trivial way, [44], but we refrain from doing so as it would only obscure the simplicity of the trading mechanism.

We consider an investor who has initial capital $z \in \mathbb{R}$ at time 0 in the riskless asset (and zero positions in the risky assets). His/her portfolio is rebalanced at the time moments $t=1, \ldots, T$. Mathematically speaking, a portfolio strategy $\phi_{t}, t=1, \ldots, T$ is defined to be a $d$-dimensional process, representing the portfolio position taken in the $d$ risky assets at time $t$. As the investment decision is assumed to be taken before new prices are revealed, we assume that $\phi$ is a predictable process, i.e. $\phi_{t}$ is $\mathcal{F}_{t-1}$-measurable for $t=1, \ldots, T$. We denote by $\Phi$ the family of all portfolio strategies. A negative coordinate $\phi_{t}^{i}<0$ means selling short ${ }^{1}$ $-\phi_{t}^{i}>0$ units of asset $i$. We introduce the adapted process $\phi_{t}^{0}, t=1, \ldots, T$ which describes the position in the riskless asset at time $t$. We allow borrowing money (i.e. $\phi_{t}^{0}<0$ ). We do not require the $\phi_{t}^{i}$ to be integers and do not put any bound on the available supply in the assets so they can be bought and sold in arbitrary quantities for the respective prices $S_{t}^{i}$ at time $t$, for $t=0, \ldots, T-1$ and $i=1, \ldots, d$.

The value process of the portfolio is defined to be $X_{0}^{z, \phi}:=z$ and

$$
X_{t}^{z, \phi}:=\phi_{t} S_{t}+\phi_{t}^{0} S_{t}^{0}=\phi_{t} S_{t}+\phi_{t}^{0}
$$

for $t \geq 1$. A frictionless trading mechanism is assumed: there are no transaction fees, taxes or liquidity costs. Only self-financing portfolios are considered, where no capital is injected or withdrawn during the trading period and hence the portfolio value changes uniquely because of price fluctuations and changes in the investor's positions. Mathematically speaking, we assume that, when the initial capital is $z$ and strategy $\phi \in \Phi$ is pursued,

$$
X_{t}^{z, \phi}-X_{t-1}^{z, \phi}=\phi_{t}\left(S_{t}-S_{t-1}\right)+\phi_{t}^{0}\left(S_{t}^{0}-S_{t-1}^{0}\right)=\phi_{t}\left(S_{t}-S_{t-1}\right), t=1, \ldots T
$$

In other words,

$$
\begin{equation*}
X_{t}^{z, \phi}=z+\sum_{j=1}^{t} \phi_{j} \Delta S_{j} \tag{1}
\end{equation*}
$$

where we denote $\Delta S_{j}:=S_{j}-S_{j-1}, j=1, \ldots, T$. This means that the portfolio value is uniquely determined by $z$ and $\phi \in \Phi$ and so is the position $\phi_{t}^{0}=X_{t}^{z, \phi}-\phi_{t} S_{t}$ in the riskless asset. Hence we do not need to bother with the positions $\phi_{t}^{0}, t=1, \ldots, T$ at all in what follows.

[^0]An investor with utility function $u: \mathbb{R} \rightarrow \mathbb{R}$ is considered. We assume that $u$ is nondecreasing (more money is preferred to less) and that it is continuous (small change in wealth causes a small change in satisfaction level). Investors are often required to meet certain payment obligations (e.g. delivering the value of a derivative product at the end of the trading period). Let $B$ be a $\mathcal{F}_{T}$-measurable scalar random variable representing this payment obligation (negative $B$ means receiving a payment $-B$ ). The investor with initial capital $z$ seeks to attain the highest possible expected utility from terminal wealth, $\bar{u}(z)$, where

$$
\begin{equation*}
\bar{u}(z):=\sup _{\phi \in \Phi(u, z, B)} E u\left(X_{T}^{z, \phi}-B\right) \tag{2}
\end{equation*}
$$

and $\Phi(u, z, B):=\left\{\phi \in \Phi: E u^{-}\left(X_{T}^{z, \phi}-B\right)<\infty\right\}$. We will drop $u$ and $B$ in the notation and will simply write $\Phi(z)$ henceforth. By definition of $\Phi(z)$, the expectations in (2) exist but may take the value $+\infty$. The quantity $\bar{u}(z)$ defines the indirect utility of $z$ when investment opportunities in the given market are taken into account.

Remark 1.1. The quantity $B$ admits an alternative interpretation: it may be a reference point (a "benchmark") to which the given investor compares his/her performance. For example, $B$ may be the terminal value of the portfolio of another investor (perhaps trading in a different market as well) or some functional of economic factors (such as market indices). In this case $E u\left(X_{T}^{z, \phi}-B\right)$ is a measure of portfolio performance relative to the benchmark $B$, see Chapter 3 below for more on this viewpoint.

When $\bar{u}(z)=\infty$ the investor may attain unlimited satisfaction which looks unrealistic, so we say that the optimal investment problem (2) is well-posed if $\bar{u}(z)<\infty$. It is also desirable that the domain of optimisation $\Phi(z)$ should be non-empty.

Our main concern will be to find an optimal strategy $\phi^{*}=\phi^{*}(z) \in \Phi(z)$ such that

$$
\begin{equation*}
\sup _{\phi \in \Phi(z)} E u\left(X_{T}^{z, \phi}-B\right)=E u\left(X_{T}^{z, \phi^{*}}-B\right) . \tag{3}
\end{equation*}
$$

Remark 1.2. For $z \geq 0$, one may also consider (3) with $\Phi(z)$ replaced by $\Phi_{+}(z):=\{\phi \in \Phi(z)$ : $X_{t}^{z, \phi} \geq 0$ a.s. $\left., t=0, \ldots, T\right\}$. This means that portfolios are constrained to have a non-negative value all over the trading period. As one of the main motivations for studying optimal investment problems is risk management, one should be able to analyse the possibility of (big) losses as well and in such a context the constraint in $\Phi_{+}(z)$ is unfortunate. We remark that optimisation over $\Phi_{+}(z)$ can be performed by methods which are similar to those of the present dissertation but considerably simpler, due to a convenient compactness property (Lemma 2.1 of [79]). We refer to the papers [79, 24, 77] which are not reviewed in the present dissertation due to volume limits.
Remark 1.3. One may object that it is always possible to find, for all $n \in \mathbb{N}$, a strategy $\phi(n)$ with $\bar{u}(z)-1 / n<E u\left(X_{T}^{z, \phi(n)}\right)$ and, for $n$ large enough, $\phi(n)$ should be satisfactory in practice. This argument ignores the deeper problems behind. The non-existence often comes from a lack of compactness: $\phi(n)$ may show an extreme and economically meaningless behaviour (e.g. it tends to infinity), hence the practical value of $\phi(n)$ for large $n$ is questionable. Also, some kind of compactness is a prerequisite for any numerical scheme to calculate (an approximation of) an optimiser.

Another (less common) reason for the non-existence of $\phi^{*}$ is the lack of closedness of $\left\{X_{T}^{z, \phi}\right.$ : $\phi \in \Phi\}$ in some appropriate topology. This reveals another possible pathology of the given setting: a limit point of investment payoffs not being an investment payoff itself, which is not only mathematically inconvenient but also contradicts common sense.

To sum up: if no $\phi^{*}$ exists then near-optimal strategies tend to be unintuitive. On the contrary, existence of $\phi^{*}$ normally goes together with compactness and closedness properties which look necessary for constructing numerical schemes leading to reasonable (near)optimal strategies. We will see concrete examples of the phenomena described in this remark in Example 2.11 and in Section 3.3 below.

The next notion we discuss is arbitrage (riskless profit), a central concept of economic theory. Formally, $\phi \in \Phi$ is an arbitrage strategy if $X^{0, \phi} \geq 0$ a.s. and $P\left(X_{T}^{0, \phi}>0\right)>0$.

Definition 1.4. We say that there is no arbitrage (NA) if, for all $\phi \in \Phi, X_{T}^{0, \phi} \geq 0$ a.s. implies $X_{T}^{0, \phi}=0$ a.s.

The usual justification for (NA) is that investors cannot make something out of nothing since such an opportunity would be heavily exploited, which would move prices and eventually terminate the opportunity. There is a general consensus that (NA) holds in an efficient market, [41]. The following result shows that, in the optimisation context, (NA) is a necessity, too.

Proposition 1.5. Let $u$ be strictly increasing. If (NA) fails then there exists no strategy $\phi^{*} \in \Phi$ satisfying (3).

Proof. If an optimal strategy $\phi^{*} \in \Phi(z)$ existed for the problem (2) then $\phi^{*}+\phi \in \Phi(z)$ and $E u\left(X_{T}^{z, \phi^{*}}\right)<E u\left(X_{T}^{z, \phi^{*}+\phi}\right)$ for any $\phi$ violating (NA), a contradiction with the optimality of $\phi^{*}$.

We now address the issue of redundancy of assets. Let $D_{t}(\omega)$ denote the affine hull of the support of $P\left(\Delta S_{t} \in \cdot \mid \mathcal{F}_{t-1}\right)(\omega)$, where we take a regular version for the conditional law, see [36]. For a topological space $X$ we will denote by $\mathcal{B}(X)$ the corresponding Borel sigma-algebra. By Proposition A. 1 of [78] one may assume that $\left\{(\omega, x) \in \Omega \times \mathbb{R}^{d}: x \in D_{t}(\omega)\right\} \in \mathcal{F}_{t-1} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$ and, under (NA), $D_{t}(\omega)$ is a subspace for a.e. $\omega$ (see Theorem 3 of [51]).

Intuitively, $D_{t}(\omega) \neq \mathbb{R}^{d}$ means that some of the risky assets are redundant and could be substituted by a linear combination of the other risky assets. Indeed, introduce for each $\xi \in \Xi_{t-1}^{d}$ the mapping $\widehat{\xi}: \Omega \rightarrow \mathbb{R}^{d}$ where $\widehat{\xi}(\omega)$ is the projection of $\xi(\omega)$ on $D_{t}(\omega)$. By Proposition 4.6 of [78], $\widehat{\xi}$ is an $\mathcal{F}_{t-1}$-measurable random variable and $P\left((\xi-\widehat{\xi}) \Delta S_{t}=0 \mid \mathcal{F}_{t-1}\right)=1$ a.s. by the definition of orthogonal projections, so

$$
\begin{equation*}
P\left(\xi \Delta S_{t}=\widehat{\xi} \Delta S_{t}\right)=1 \tag{4}
\end{equation*}
$$

This means that we may always replace the strategy $\phi_{t}$ at time $t$ by its orthogonal projection on $D_{t}$. Define $\widehat{\Phi}:=\left\{\phi \in \Phi: \phi_{t} \in D_{t}\right.$ a.s. $\}$ and $\operatorname{set} \widehat{\Phi}(z):=\widehat{\Phi} \cap \Phi(z)$. By (4), we may alternatively take the supremum over $\widehat{\Phi}(z)$ in (2).

Now we present a useful characterization of (NA). Define $\widehat{\Xi}_{t}^{d}:=\left\{\xi \in \Xi_{t}^{d}: \xi \in D_{t+1}\right.$ a.s. $\}$, for $t=0, \ldots, T-1$.

Proposition 1.6. (NA) holds iff there exist $\nu_{t}, \kappa_{t} \in \Xi_{t}^{1}$ with $\nu_{t}, \kappa_{t}>0$ a.s. such that for all $\xi \in \widehat{\Xi}_{t}^{d}$ :

$$
\begin{equation*}
P\left(\xi \Delta S_{t+1} \leq-\nu_{t} \mid \xi \| \mathcal{F}_{t}\right) \geq \kappa_{t} \tag{5}
\end{equation*}
$$

holds almost surely; for all $0 \leq t \leq T-1$.
Proof. Proposition 3.3 of [78] states that (NA) holds iff, on the event $G:=\left\{D_{t+1} \neq\{0\}\right\}$,

$$
P\left(\zeta \Delta S_{t+1} \leq-\nu_{t} \mid \mathcal{F}_{t}\right) \geq \kappa_{t}
$$

for all $\zeta \in \Xi_{t}^{d}$ with $|\zeta|=1$ a.s. Applying this to $\zeta:=\xi /|\xi|$ on $G \cap\{\xi \neq 0\}$ we get (5) on $G \cap\{\xi \neq 0\}$. As (5) is trivial on the complement of $G$ and on that of $\{\xi \neq 0\}$, the proof is completed.

Remark 1.7. Note that if $Q \sim P$ then (5) above implies that

$$
Q\left(\xi \Delta S_{t+1} \leq-\nu_{t}|\xi| \mid \mathcal{F}_{t}\right) \geq \kappa_{t}^{Q}
$$

for some $\kappa_{t}^{Q}>0$ a.s. At some point we will switch measures often and then this observation will be convenient. Note also that we may assume $\kappa_{t}, \nu_{t} \leq 1$ in (5).

From Proposition 1.6 we see that (NA) holds iff, at any time $t$, each one-step portfolio of unit length (i.e. $|\xi|=1$ ) and without redundancy (i.e. $\xi \in D_{t}$ ) may lead to losses of at least a prescribed size $\nu_{t}$ with (conditional) probability at least $\kappa_{t}$. This is a "quantitative" expression of the intuitive content of (NA): at any time $t$ every one-step portfolio (with 0 initial capital) should lead to a loss with positive probability.

We close this section with a more famous characterization of the (NA) property which stands at the origin of spectacular developments in financial mathematics in the 1990s. We denote by $L^{\infty}$ the set of (equivalence classes of) a.s. bounded random variables and by $\mathcal{M}$ the set of probabilities $Q \sim P$ under which $S$ is a martingale (with respect to the filtration $\mathcal{F}$.). Elements of $\mathcal{M}$ are called risk-neutral probabilities or equivalent martingale measures.

Theorem 1.8. Condition (NA) holds iff $\mathcal{M}$ is non-empty. Moreover, if (NA) holds then one can always choose $Q \in \mathcal{M}$ such that $d Q / d P \in L^{\infty}$.

The above result was well-known for finite $\Omega$ since [48]. However, the passage to infinite $\Omega$ proved to be rather challenging and was first achieved in [33]. This spawned efforts to find simpler proofs ( $[90,83,55]$ ) and also led to major advances in the continuous-time theory of arbitrage. It is outside the scope of the present dissertation to review these developments, we refer to [35]. Besides, the particular technology of the proof in [33] has had several deep applications, we only mention [43, 56, 29].

It is highly non-trivial that one can always get $d Q / d P \in L^{\infty}$ in Theorem 1.8, continuoustime models generically fail this property. We will elaborate on related issues in Section 2.7 below.

### 1.3 Market model in continuous time

We now describe the standard model for frictionless markets in continuous time, see [35] for more details. On a probability space $(\Omega, \mathcal{F}, P)$, let a continuous-time filtration $\mathcal{F}_{t}, t \in[0, T]$ be given. We assume that this filtration is right-continuous and $\mathcal{F}_{0}$ contains $P$-zero sets. A process with right-continuous trajectories that admit left-hand limits is called càdlàg (a French acronym, standard in the literature).

Let $S_{t}, t \in[0, T]$ be an $\mathbb{R}^{d}$-valued càdlàg adapted process, representing the price of $d$ risky securities. Let $\varphi_{t}, t \in[0, T]$ be an $\mathbb{R}^{d}$-valued stochastic process showing the position of the investor in the given assets. The value process of the portfolio $\varphi$ is defined by the continuoustime analogue of (1),

$$
\begin{equation*}
X_{t}^{z, \varphi}:=z+\int_{0}^{t} \varphi_{u} d S_{u}, \quad t \in[0, T] \tag{6}
\end{equation*}
$$

where $z$ is the investor's initial capital and, in order that the stochastic integral exists, we assume that $S$ is a semimartingale (w.r.t. the given filtration) and $\varphi$ is an $S$-integrable process, in particular, it is predictable ${ }^{2}$. The set of $S$-integrable processes is denoted by $\Phi$.

It turns out that, without restricting the set of admissible portfolio strategies further, there are arbitrage opportunities even in the simplest models (such as the Black-Scholes model), by the result of e.g. [39]. To avoid these, the standard class to use is $\Phi_{b}$, the set of $\varphi \in \Phi$ whose value process $X$ satisfies $X_{t}^{z, \varphi} \geq-c$ a.s. for all $t$ for some constant $c$ (which may depend on $\varphi$ but not on $t$ ). In other words, these are the strategies with a finite credit line.

As in the discrete-time setting, $\mathcal{M}$ denotes the family of probabilities equivalent to $P$ under which $S$ is a martingale. A suitable strengthening of (NA) is essentially equivalent to $\mathcal{M} \neq \emptyset$ in the present, continuous-time setting, we do not enter into the rather technical details, see [35]. One would thus be led, in the utility maximisation context, to seek $\varphi^{*} \in \Phi_{b}$ with

$$
\sup _{\varphi \in \Phi_{b}} E u\left(X_{T}^{z, \varphi}\right)=E u\left(X_{T}^{z, \varphi^{*}}\right)
$$

Unfortunately, this would be a vain enterprise to pursue as the set $X_{T}^{z, \varphi}, \varphi \in \Phi_{b}$ is not closed in any reasonable topology and a maximiser in the class $\Phi_{b}$ often fails to exist, see [91]. Various other classes of admissible strategies $\Phi_{a} \subset \Phi$ have been proposed. If $\Phi_{a}$ is chosen too large (e.g. $\Phi_{a}:=\Phi$ ) then arbitrage opportunities appear. If $\Phi_{a}$ is not large enough (e.g. $\Phi_{a}:=\Phi_{b}$ ) then it doesn't contain the optimiser $\varphi^{*}$. A reasonable compromise is the following choice: assume $\mathcal{M} \neq \emptyset$, fix $Q \in \mathcal{M}$ and define

$$
\Phi_{a}:=\Phi_{a}(Q):=\left\{\varphi \in \Phi: X^{0, \phi} \text { is a } Q \text {-martingale }\right\} .
$$

[^1]There is some arbitrariness in the choice of $Q$ but in the case of complete markets (see below) $Q$ will be unique. In a large class of standard optimisation problems the maximiser is indeed in $\Phi_{a}$ for a natural choice of $Q$, see [57, 92]

Hence we shall be looking for $\varphi^{*} \in \Phi_{a}$ satisfying

$$
\begin{equation*}
\sup _{\varphi \in \Phi_{a}} E u\left(X_{T}^{z, \varphi}\right)=E u\left(X_{T}^{z, \varphi^{*}}\right) . \tag{7}
\end{equation*}
$$

A more general objective function will be considered in Chapter 4.
We introduce market completeness at this stage. An $\mathcal{F}_{T}$-measurable random variable $B$ is said to be replicable if there exists $Q \in \mathcal{M}, \varphi \in \Phi_{a}(Q)$ and $z \in \mathbb{R}$ such that $X_{T}^{z, \phi}=B$ a.s.

We call a financial market model complete if all bounded $\mathcal{F}_{T}$-measurable $B$ are replicable.
Remark 1.9. Proposition 2.1 of [98] states that $B \in L^{\infty}$ is replicable iff it is replicable with some $\varphi_{Q} \in \Phi_{a}(Q)$, for all $Q \in \mathcal{M}$.

See Section 4.6 below for the textbook example of a complete market. We remark that though complete models are unrealistic they serve as an important model class on which new ideas are to be tested first.

Completeness is characterized by the following result which follows from [50]. A proof with the financial mathematics setting in mind comes trivially from Proposition 2.1 of [98] and Théorème 3.2 in [3].

Theorem 1.10. Assume $\mathcal{M} \neq \emptyset$. Then the market is complete iff $\mathcal{M}$ is a singleton.
For $1 \leq p<\infty$ we denote by $L^{p}(Q)$ the usual Banach space of of random variables with finite $p$ th moment under $Q$. If $Q=P$ we simply write $L^{p}$. We fix $Q \in \mathcal{M}$ and $\Phi_{a}=\Phi_{a}(Q)$.

Lemma 1.11. Let $B \in L^{1}(Q)$ such that, for all $n, B_{n}:=B \wedge n \vee(-n)$ is replicable. Then so is $B$. In particular, if in a complete market model $Q$ is the unique element of $\mathcal{M}$ then each $B \in L^{1}(Q)$ is replicable.

Proof. Let $z_{n}:=E_{Q} B_{n}$. By the definition of replicability and by Remark 1.9, there is $\varphi_{n} \in$ $\Phi_{a}(Q)$ such that $B_{n}=X_{T}^{z_{n}, \varphi_{n}}$ a.s. Clearly, $B_{n}$ tend to $B$ in $L^{1}(Q)$ and then also $z_{n} \rightarrow z:=E_{Q} B$, $n \rightarrow \infty$. It follows that the terminal values of the martingales $\int_{0}^{v} \varphi_{n}(u) d S_{u}$ converge to $B-z$ in $L^{1}(Q)$. By Yor's theorem (see [104]) there is $\varphi \in \Phi_{a}$ such that $B-z=\int_{0}^{T} \varphi(u) d S_{u}$ and the result follows.

We mention that problems of the form (7) are usually tackled using duality methods: an appropriate convex conjugate functional is minimised over $\mathcal{M}$ from which a maximiser for (7) is subsequently derived, see [57, 93]. It is clear that in our setting, where $u$ fails to be concave, one cannot pursue this route and different methods need to be developed, see Chapter 4 below.

In Chapter 5 we will consider a continuous-time model of a financial market with frictions due to illiquidity where the dynamics of the portfolio value process differs from (6).

## 2 Optimal investment under expected utility criteria

We remain in the setting of Section 1.2. We shall prove the existence of an optimiser for (3) when $u$ is not necessarily concave. No such results are available in multistep discrete-time markets. One-step models were treated in [49, 12]. First we assume $u$ to be bounded above then we also investigate the case of unbounded $u$.

This chapter is based on the papers [72, 78, 21, 23].

### 2.1 A look on the case of bounded above utility

To get a flavour of the techniques we use, we shall first treat the case where $u$ is bounded above by a constant. This assumption allows for simpler arguments and leads to the following clear-cut result which will be proved in Section 2.3 below.

Theorem 2.1. Assume (NA). Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be a nondecreasing and continuous function which is bounded above by a constant $C \geq 0$ and satisfies

$$
\begin{equation*}
\lim _{x \rightarrow-\infty} u(x)=-\infty \tag{8}
\end{equation*}
$$

Let $B$ be an arbitrary real-valued random variable with

$$
\begin{equation*}
E u(z-B)>-\infty \text { for all } z \in \mathbb{R} \tag{9}
\end{equation*}
$$

Then for all $z \in \mathbb{R}$ there exists a strategy $\phi^{*}=\phi^{*}(z) \in \Phi$ such that

$$
\begin{equation*}
\bar{u}(z)=\sup _{\phi \in \Phi} E u\left(X_{T}^{z, \phi}-B\right)=E u\left(X_{T}^{z, \phi^{*}}-B\right) . \tag{10}
\end{equation*}
$$

Furthermore, $\bar{u}$ is continuous on $\mathbb{R}$.
Note that, as $u$ is bounded above, the expectations $E u\left(X_{T}^{z, \phi}-B\right)$ exist for all $\phi \in \Phi$ hence we may use $\Phi$ instead of $\Phi(z)$ as the domain of optimization.

Continuity of $u$ is a natural requirement. Theorem 2.1 guarantees that the indirect utility $\bar{u}$ inherits the continuity property of $u$. Condition (8) means that infinite losses lead to an infinite dissatisfaction of the agent.
Remark 2.2. Most studies require $u$ to be smooth in addition to being concave. We do not need such a restriction and hence we can accomodate various loss functions: let $\ell: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$ be increasing and continuous with $\ell(0)=0, \ell(\infty)=\infty$. Set $u(x):=-\ell(-x)$ for $x \geq 0$ and $u(x)=0$ for $x>0$. Maximising $E u\left(X_{T}^{z, \phi}-B\right)$ in the present context means minimising shortfall risk (i.e. risk of performing under the benchmark $B$ ), as quantified by $\ell$.

Remark 2.3. Let us take $u$ bounded above, continuously differentiable and define $S_{0}=0$, $S_{1}= \pm 1$ with probabilities $1 / 2-1 / 2$. If

$$
\begin{equation*}
u^{\prime}(\phi)-u^{\prime}(-\phi) \geq 0 \text { for all } \phi>0 \text { and } u^{\prime}(\phi)-u^{\prime}(-\phi)>0 \text { for } \phi>0 \text { large enough, } \tag{11}
\end{equation*}
$$

(in this case $u(-\infty)>-\infty$, as easily seen) then $\phi \rightarrow E u\left(\phi \Delta S_{1}\right)$ is nondecreasing in $|\phi|$ and for large enough $|\phi|$ it is strictly increasing, which excludes the existence of an optimiser $\phi^{*}$ for (10). One may take, e.g. $u(x)=1-e^{-\alpha x}, x \geq N, u(x)=e^{\beta x}-1, x \leq-N$ and $u$ continuous and linear on $[-N, N]$. For any $\beta>\alpha>0$ and for $N$ large enough $u$ satisfies (11).

This highlights the importance of condition (8): even for very simple specifications of the price process $S$ the failure of (8) may easily cause that there is no optimizer.

### 2.2 One-step case for $u$ bounded above

In this section we consider a function $V: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for every $x, V(\omega, x)$ is a random variable and for a.e. $\omega \in \Omega$, the functions $x \rightarrow V(\omega, x)$ are non-decreasing, continuous and satisfy $\lim _{x \rightarrow-\infty} V(\omega, x)=-\infty$ and $V(\omega, x) \leq C$ for all $x \in \mathbb{R}$, with some fixed constant $C \geq 0$.

Let $\mathcal{H} \subset \mathcal{F}$ be a sigma-algebra. We assume that there is a family of real-valued random variables $M(n), n \in \mathbb{Z}$ such that $V(n) \geq M(n)$ holds a.s., for all $n$ and $m(n):=E(M(n) \mid \mathcal{H})>$ $-\infty$ a.s. We may and will assume $M(n) \leq 0$.

Let $Y$ be an $\mathbb{R}^{d}$-valued random variable. Take a regular version of $P(Y \in \cdot \mid \mathcal{H})(\omega)$. Define, for a.e. $\omega$, the multifunction $\omega \rightarrow D(\omega) \subset \mathbb{R}^{d}$ where $D(\omega)$ is the affine hull of the support of $P(Y \in \cdot \mid \mathcal{H})(\omega)$. Clearly, $D$ can also be viewed as a subset of $\Omega \times \mathbb{R}^{d}$ and we can choose $D \in \mathcal{H} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$, see Proposition A. 1 of [78].

The notation $\Xi^{n}$ will be used for the class of $\mathbb{R}^{n}$-valued $\mathcal{H}$-measurable random variables. For $\xi \in \Xi^{d}$ we will denote by $\widehat{\xi}(\omega)$ the orthogonal projection of $\xi(\omega)$ on $D(\omega)$. The function $\omega \rightarrow \widehat{\xi}(\omega)$ is then $\mathcal{H}$-measurable, by Proposition 4.6 of [78]. Define $\widehat{\Xi}^{d}:=\left\{\xi \in \Xi^{d}: \xi \in D\right.$ a.s. $\}$.
Remark 2.4. Notice that $P((\xi-\widehat{\xi}) Y=0 \mid \mathcal{H})=1$ a.s., hence $P(\xi Y=\widehat{\xi} Y)=1$ which means that we may always replace $\xi$ by $\widehat{\xi}$ when maximising $E(V(x+\xi Y) \mid \mathcal{H})$ in $\xi$.

We assume that there exist $\mathcal{H}$-measurable $\kappa, \nu>0$ such that for all $\xi \in \widehat{\Xi}^{d}$ we have

$$
\begin{equation*}
P(\xi Y \leq-\nu|\xi| \mid \mathcal{H}) \geq \kappa \text { a.s. } \tag{12}
\end{equation*}
$$

this is our "one-step no arbitrage" condition, compare to Proposition 1.6.
The next lemma allows us to work with strategies admitting a fixed bound. We denote by $1_{A}$ the indicator function of an event $A \in \mathcal{F}$ and by ess. sup $i_{i \in I} f_{i}$ the essential supremum of a family of real-valued random variables $f_{i}, i \in I$.

Lemma 2.5. There exists $v: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for all $x, v(\omega, x)$ is a version of

$$
\text { ess. } \sup _{\xi \in \Xi^{d}} E(V(x+\xi Y) \mid \mathcal{H})
$$

and, for a.e. $\omega \in \Omega$, the functions $x \rightarrow v(\omega, x)$ are non-decreasing, right-continuous, they satisfy $\lim _{x \rightarrow-\infty} v(\omega, x)=-\infty$ and $v(\omega, x) \leq C$, for all $x \in \mathbb{R}$. There exist random variables $K(n) \geq 0$, $n \in \mathbb{Z}$ such that for all $x \in \mathbb{R}$ and $\xi \in \widehat{\Xi}^{d}$,

$$
\begin{equation*}
E(V(x+\xi Y) \mid \mathcal{H}) \leq E\left(V\left(x+1_{\{|\xi| \leq K(\lfloor x\rfloor)\}} \xi Y\right) \mid \mathcal{H}\right) \tag{13}
\end{equation*}
$$

where $\lfloor x\rfloor$ denotes the largest integer $k$ with $k \leq x$. We have $m(n) \leq v(n)$ a.s., for all $n$.
Proof. We assume that $D \neq\{0\}$ as (13) is trivial on the event $\{D=\{0\}\}$.
Fix $n \in \mathbb{Z}$. Since $V(y) \rightarrow-\infty$ a.s. when $y \rightarrow-\infty$, for each $0 \leq L \in \Xi^{1}$ there is $G_{L} \in \Xi^{1}$ such that $P\left(V\left(-G_{L}\right) \leq-L \mid \mathcal{H}\right) \geq 1-\kappa / 2$ a.s. Since for all $x \in[n, n+1)$,

$$
E(V(x+\xi Y) \mid \mathcal{H}) \leq C+E\left(V(n+1-|\xi| \nu) 1_{\left\{\xi Y \leq-|\xi| \nu, V\left(-G_{L}\right) \leq-L\right\}} \mid \mathcal{H}\right)
$$

and, by (12), $P(\xi Y \leq-\nu|\xi| \mid \mathcal{H}) \geq \kappa$ a.s., we get that whenever $\nu|\xi| \geq G_{L}+n+1$, we have

$$
E(V(x+\xi Y) \mid \mathcal{H}) \leq C-L \kappa / 2
$$

Choose $L:=2(C-m(n)) / \kappa$, then $K(n):=\left(G_{L}+n+1\right) / \nu$ is such that for $|\xi| \geq K(n)$,

$$
E(V(x+\xi Y) \mid \mathcal{H}) \leq m(n) \leq E(V(x) \mid \mathcal{H})
$$

holds a.s., providing a suitable function $K(\cdot)$.
Now for each $x$, let $F(x)$ be an arbitrary version of the essential supremum in consideration. We may and will assume that $F(x) \leq C$ for all $\omega$ and $x$. Outside a negligible set $N \subset \Omega, q \rightarrow F(q)$ is non-decreasing on $\mathbb{Q}$. Define $v(x):=\inf _{q>x, q \in \mathbb{Q}} F(q)$. This function is non-decreasing and right-continuous outside $N$.

Fix $x \in \mathbb{R}$. Since $F(x) \leq F(q)$ a.s. for $x \leq q$, we clearly have $F(x) \leq v(x)$ a.s. We claim we can take $\xi_{k} \in \Xi^{d}$ with

$$
F\left(q_{k}\right)-1 / k \leq E\left(V\left(q_{k}+\xi_{k} Y\right) \mid \mathcal{H}\right),
$$

a.s. where $x<q_{k}<x+1$ and $q_{k} \in \mathbb{Q}$ decrease to $x$ as $k \rightarrow \infty$.

Indeed, as $E(V(y+\xi Y) \mid \mathcal{H}), \xi \in \Xi^{d}$ is easily seen to be directed upwards, there is a sequence $\zeta_{n} \in \Xi^{d}$ such that $E\left(V\left(y+\zeta_{n} Y\right) \mid \mathcal{H}\right)$ is a.s. nondecreasing and converges a.s. to $F(y)$. We can define $\xi_{k}:=\zeta_{l(k)}$ where

$$
l(k)(\omega):=\min \left\{l: E\left(V\left(y+\zeta_{l} Y\right) \mid \mathcal{H}\right)(\omega) \geq F(\omega, y)-1 / k\right\} .
$$

Applying this to $y:=q_{k}$ we have verified our previous claim. Remark 2.4 and (13) imply

$$
F\left(q_{k}\right)-1 / k \leq E\left(V\left(q_{k}+\bar{\xi}_{k} Y\right) \mid \mathcal{H}\right)
$$

where $\bar{\xi}_{k}:=\widehat{\xi}_{k} 1_{\left\{\left|\widehat{\xi}_{k}\right| \leq K(\lfloor x\rfloor)+K(\lfloor x+1\rfloor)\right\}}, k \in \mathbb{N}$. By Lemma 6.8 below, there is an $\mathcal{H}$-measurable random subsequence $k_{n}, n \rightarrow \infty$ such that $\bar{\xi}_{k_{n}} \rightarrow \xi^{\prime}$ a.s. with some $\xi^{\prime} \in \widehat{\Xi}^{d}$. The Fatou lemma and continuity of $V$ imply that

$$
v(x)=\lim _{n \rightarrow \infty}\left(F\left(q_{n_{k}}\right)-1 / n_{k}\right) \leq E\left(V\left(x+\xi^{\prime} Y\right) \mid \mathcal{H}\right) \leq F(x)
$$

a.s., showing that $v(x)$ is indeed a version of $F(x)$.

Finally we claim that $v(x) \rightarrow-\infty, x \rightarrow-\infty$ a.s. For each $L(n):=n$ there is $G_{L(n)} \in \Xi^{1}$ with $P\left(V\left(-G_{L(n)}\right) \leq-L(n) \mid \mathcal{H}\right) \geq 1-\kappa / 2$. Let us notice that, by (12),

$$
E\left(V\left(-G_{L(n)}+\xi Y\right) \mid \mathcal{H}\right) \leq E\left(V\left(-G_{L(n)}\right) 1_{\left\{\xi Y \leq 0, V\left(-G_{L(n)}\right) \leq-L(n)\right\}} \mid \mathcal{H}\right)+C \leq-n \kappa / 2+C
$$

which is a bound independent of $\xi$ and it tends to $-\infty$ a.s. as $n \rightarrow-\infty$, so $v\left(-G_{L(n)}\right) \rightarrow-\infty$. By the monotonicity of $v$ this implies that a.s. $\lim _{x \rightarrow-\infty} v(x)=-\infty$. Hence our claim follows. The last statement of this lemma is trivial.

Lemma 2.6. Let $H \in \Xi^{1}$. Then $v(H)$ is a version of ess. $\sup _{\xi \in \Xi^{d}} E(V(H+\xi Y) \mid \mathcal{H})$.
Proof. Working separately on the events $\{H \in[n, n+1)\}$ we may and will assume that $H \in$ $[n, n+1)$ for a fixed $n$. The statement is clearly true for constant $H$ by Lemma 2.5 and hence also for countable step functions $H$. For general $H$, let us take step functions $H_{k} \in[n, n+1$ ), $H_{k} \in \Xi^{1}$ decreasing to $H$ as $k \rightarrow \infty . v\left(H_{k}\right) \rightarrow v(H)$ a.s. by right-continuity. It is also clear that, for all $\xi \in \Xi^{d}, E(V(H+\xi Y) \mid \mathcal{H}) \leq E\left(V\left(H_{k}+\xi Y\right) \mid \mathcal{H}\right) \leq v\left(H_{k}\right)$ a.s. for all $k$, hence

$$
\text { ess. } \sup _{\xi \in \Xi^{d}} E(V(H+\xi Y) \mid \mathcal{H}) \leq v(H) \text {. }
$$

Choose $\xi_{k}$ with $E\left(V\left(H_{k}+\xi_{k} Y\right) \mid \mathcal{H}\right)>v\left(H_{k}\right)-1 / k$ and note that $\bar{\xi}_{k}:=\widehat{\xi}_{k} 1_{\left\{\left|\widehat{\xi}_{k}\right| \leq K(n)\right\}}$ also satisfies $E\left(V\left(H_{k}+\bar{\xi}_{k} Y\right) \mid \mathcal{H}\right)>v\left(H_{k}\right)-1 / k$ by Remark 2.4 and Lemma 2.5.

By Lemma 6.8 we can take an $\mathcal{H}$-measurable random subsequence $k_{l}, l \rightarrow \infty$ such that $\bar{\xi}_{k_{l}} \rightarrow \xi^{\dagger}$ a.s., $l \rightarrow \infty$. Fatou's lemma and continuity of $V$ imply

$$
v(H)=\lim _{l \rightarrow \infty}\left[v\left(H_{k_{l}}\right)-1 / k_{l}\right] \leq \limsup _{l \rightarrow \infty} E\left(V\left(H_{k_{l}}+\bar{\xi}_{k_{l}} Y\right) \mid \mathcal{H}\right) \leq E\left(V\left(H+\xi^{\dagger} Y\right) \mid \mathcal{H}\right)
$$

completing the proof since $E\left(V\left(H+\xi^{\dagger} Y\right) \mid \mathcal{H}\right) \leq \operatorname{ess} \sup _{\xi \in \Xi^{d}} E(V(H+\xi Y) \mid \mathcal{H})$ holds trivially.
Lemma 2.7. Outside a negligible set, the trajectories $x \rightarrow v(x, \omega)$ are continuous.
Proof. Arguing by contradiction, let us suppose that the projection $A \in \mathcal{H}$ of the set

$$
E:=\left\{(x, \omega): v(x, \omega)>\varepsilon+\sup _{q<x, q \in \mathbb{Q}} v(q, \omega)\right\} \in \mathcal{B}(\mathbb{R}) \otimes \mathcal{H}
$$

on $\Omega$ has positive probability for some $\varepsilon>0$. Let $H: \Omega \rightarrow \mathbb{R}$ be a measurable selector of $E$ (which exists by III. 44-45. of [36]) on $A$ and let it be 0 outside $A$. Let $H>H_{k}, k \in \mathbb{N}$ be $\mathbb{Q}$-valued $\mathcal{H}$-measurable step functions increasing to $H$ and choose $\zeta_{l}$ such that $v(H)-1 / l \leq$ $E\left(V\left(H+\zeta_{l} Y\right) \mid \mathcal{H}\right)$ for each $l \in \mathbb{N}$. On $A$ we have

$$
\limsup _{k \rightarrow \infty} E\left(V\left(H_{k}+\zeta_{l} Y\right) \mid \mathcal{H}\right) \leq \limsup _{k \rightarrow \infty} v\left(H_{k}\right) \leq v(H)-\varepsilon
$$

On the other hand, monotone convergence ensures $\lim _{k \rightarrow \infty} E\left(V\left(H_{k}+\zeta_{l} Y\right) \mid \mathcal{H}\right)=E(V(H+$ $\left.\left.\zeta_{l} Y\right) \mid \mathcal{H}\right) \geq v(H)-1 / l$, for all $l$. This leads to a contradiction for $l$ large enough.

Lemma 2.8. For each $H \in \Xi^{1}$ there exists $\xi_{H}^{*} \in \widehat{\Xi}^{d}$ such that $E\left(V\left(H+\xi_{H}^{*} Y\right) \mid \mathcal{H}\right)=v(H)$ a.s.
Proof. The set $E(V(H+\xi Y) \mid \mathcal{H}), \xi \in \Xi^{d}$ is directed upwards hence there is a sequence $\xi_{k} \in \Xi^{d}$ such that $E\left(V\left(H+\xi_{k} Y\right) \mid \mathcal{H}\right)$ increases to ess. $\sup _{\xi \in \Xi^{d}} E(V(H+\xi Y) \mid \mathcal{H})=v(H)$, recall Lemma 2.6. Define the random variable $Z:=\sum_{n \in \mathbb{Z}} K(n) 1_{H \in[n, n+1)}$. By Remark 2.4 and Lemma 2.5, for $\bar{\xi}_{k}:=\widehat{\xi}_{k} 1_{\left\{\left|\widehat{\xi}_{k}\right| \leq Z\right\}}$,

$$
E\left(V\left(H+\xi_{k} Y\right) \mid \mathcal{H}\right) \leq E\left(V\left(H+\bar{\xi}_{k} Y\right) \mid \mathcal{H}\right)
$$

a.s. for all $k$. Since $\left|\bar{\xi}_{k}\right| \leq Z$ a.s., by Lemma 6.8 an $\mathcal{H}$-measurable random subsequence $k_{n}$ exists such that $\bar{\xi}_{k_{n}} \rightarrow \xi^{*}$ a.s., $n \rightarrow \infty$ for some $\xi^{*} \in \widehat{\Xi}^{d}$, and Fatou's lemma guarantees that

$$
E\left(V\left(H+\xi^{*} Y\right) \mid \mathcal{H}\right) \geq \limsup _{n \rightarrow \infty} E\left(V\left(H+\bar{\xi}_{k_{n}} Y\right) \mid \mathcal{H}\right) \geq \lim _{n \rightarrow \infty} E\left(V\left(H+\xi_{n} Y\right) \mid \mathcal{H}\right)=v(H)
$$

hence $\xi_{H}^{*}:=\xi^{*}$ is as required.

### 2.3 The multi-step case for $u$ bounded above

In this section all the assumptions of Theorem 2.1 will be in force. Set

$$
U_{T}(x, \omega):=u(x-B(\omega)) \text { for }(\omega, x) \in \Omega \times \mathbb{R}^{d}
$$

Lemma 2.9. For $t=0, \ldots, T-1$, there exist $U_{t}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that, for all $x, U_{t}(\omega, x)$ is a version of $\operatorname{ess} \sup _{\xi \in \Xi^{d}} E\left(U_{t+1}\left(x+\xi \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)$ and, for a.e. $\omega \in \Omega$, the functions $x \rightarrow U_{t}(\omega, x)$ are non-decreasing, continuous, they satisfy $\lim _{x \rightarrow-\infty} U_{t}(\omega, x)=-\infty$ and $U_{t}(\omega, x) \leq C$ for all $x \in \mathbb{R}$. For each $\mathcal{F}_{t}$-measurable $H_{t}$ there exists $\tilde{\xi}_{t}\left(H_{t}\right) \in \widehat{\Xi}_{t}^{d}$ such that

$$
E\left(U_{t+1}\left(H_{t}+\tilde{\xi}_{t}\left(H_{t}\right) \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)=\text { ess. } \sup _{\xi \in \Xi^{d}} E\left(U_{t+1}\left(H_{t}+\xi \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)
$$

Proof. Proceeding by backward induction, we will show the statements of this Lemma together with the existence of $M_{t}(n) \leq U_{t}(n), n \in \mathbb{Z}$ with $E M_{t}(n)>-\infty$. First apply Lemmata 2.5, 2.6, 2.7 and 2.8 to $\mathcal{H}:=\mathcal{F}_{T-1}, Y:=\Delta S_{T}, V:=U_{T}, D:=D_{T}$ and $M(n):=M_{T}(n):=$ $U_{T}(n-B)$ (the hypotheses of Section 2.2 follow from those of Theorem 2.1) and we get the statements for $T-1$. Note that (NA) implies (12) for $Y$ by Proposition 1.6. Also, defining $M_{T-1}(n):=E\left(M_{T}(n) \mid \mathcal{F}_{T-1}\right)$, this satisfies $E M_{T-1}(n)>-\infty$ by (9).

Assume that this lemma has been shown for $t+1$. Apply Lemmata 2.5, 2.6, 2.7 and 2.8 with the choice $\mathcal{H}:=\mathcal{F}_{t}, Y:=\Delta S_{t+1}, V:=U_{t+1}, D:=D_{t+1}$ and $M(n):=M_{t+1}(n)$ to get the statements for $t$ (the conditions of Section 2.2 now hold by the induction hypotheses), noting that $M_{t}(n):=m(n)=E\left(M_{t+1}(n) \mid \mathcal{F}_{t}\right)=E\left(U_{T}(n) \mid \mathcal{F}_{t}\right)$ so $E M_{t}(n)>-\infty$, by (9).

Proof of Theorem 2.1. Using the previous Lemma, define recursively $\phi_{1}^{*}:=\tilde{\xi}_{1}(z)$ and $\phi_{t+1}^{*}:=$ $\tilde{\xi}_{t+1}\left(X_{t}^{z, \phi^{*}}\right)$. For any $\phi \in \Phi$ :

$$
\begin{aligned}
E u\left(X_{T}^{z, \phi}-B\right) & =E E\left(U_{T}\left(X_{T-1}^{z, \phi}+\phi_{T} \Delta S_{T}\right) \mid \mathcal{F}_{T-1}\right) \leq E U_{T-1}\left(X_{T-1}^{z, \phi}\right) \\
& \leq \ldots \leq E U_{0}(z)
\end{aligned}
$$

by the definition of $U_{t}, t=0, \ldots, T$. Notice that there are equalities everywhere for $\phi=$ $\phi^{*}$. This finishes the proof except the continuity of $\bar{u}$. Note that $U_{0}$ is continuous outside a negligible set by Lemma 2.9. Clearly, $\bar{u}(x)=E U_{0}(x)$. Let $x_{k}, k \in \mathbb{N}$ converge to $x$ and let $n \in \mathbb{Z}$ be such that $n \leq \inf _{k} x_{k}$. Then $U_{0}\left(x_{k}\right) \rightarrow U(x)$ a.s. and $M_{0}(n) \leq U_{0}\left(x_{k}\right) \leq C$. Since $M_{0}(n)$ is integrable, dominated convergence finishes the proof.

### 2.4 The case of possibly unbounded $u$

If $u$ doesn't admit an upper bound, problem (3) may easily become ill-posed.
Example 2.10. Assume that

$$
u(x)= \begin{cases}x^{\alpha}, & x \geq 0 \\ -|x|^{\beta}, & x<0,\end{cases}
$$

with $\alpha, \beta>0$. Assume that $S_{0}=0, \Delta S_{1}= \pm 1$ with probabilities $p, 1-p$ for some $0<p<1$. Then one gets

$$
E u\left(n \Delta S_{1}\right)=p n^{\alpha}-(1-p) n^{\beta} .
$$

If $\alpha \geq \beta$ then choose $p>1 / 2$ and $E\left(U\left(n \Delta S_{1}\right)\right)$ goes to $\infty$ as $n \rightarrow \infty$. So in order to have a well-posed problem for $z=0$ one needs to assume $\beta>\alpha$. We will see later that this is indeed sufficient under further hypotheses, see Corollary 2.20 below.

We now show that the existence of the optimiser $\phi^{*}$ in (3) may fail even if (NA) holds, $u$ is concave and the supremum in (3) is finite.

Example 2.11. Define a strictly increasing concave function $u$ by setting $u(0)=0$,

$$
u^{\prime}(x):=1+1 / n^{2}, \quad x \in(n-1, n], \quad n \geq 1, u^{\prime}(x):=3-1 / n^{2}, \quad x \in(n, n+1], \quad n \leq-1 .
$$

for $n \in \mathbb{Z}$. Take $S_{0}:=0, P\left(S_{1}=1\right)=3 / 4, P\left(S_{1}=-1\right)=1 / 4$. One can calculate the expected utility of the strategy $\phi_{1}:=n$ for some $n \in \mathbb{Z}$ with initial capital $z=0$ :

$$
\begin{gathered}
E u\left(n S_{1}\right)=\frac{3 u(n)+u(-n)}{4}=\sum_{j=1}^{n} 1 / j^{2}, \quad n \geq 0 ; \\
E u\left(n S_{1}\right)=\sum_{j=1}^{-n} 1 / j^{2}+2 n, \quad n<0 .
\end{gathered}
$$

This utility tends to $\sum_{i=1}^{\infty} 1 / i^{2}=\pi^{2} / 6$ in an increasing way as $n \rightarrow \infty$. In fact, it is easy to see that the function $\phi_{1} \rightarrow E u\left(\phi_{1} S_{1}\right), \phi_{1} \in \mathbb{R}$ is increasing in $\phi_{1}$, so we may conclude that the supremum of the expected utilities is $\pi^{2} / 6$, but it is not attained by any strategy $\phi_{1}^{*}$.

Remark 2.12. In this remark we assume $u$ concave, nondecreasing and continuously differentiable. It is not known what are the precise necessary and sufficient conditions on a $u$ that guarantee the existence of $\phi^{*}$ in (3) for a reasonably large class of market models. In terms of the asymptotic elasticities $A E_{ \pm}$introduced in Section 6.3 below, the standard sufficient condition in general, continuous-time models is

$$
\begin{equation*}
A E_{+}(u)<1<A E_{-}(u), \tag{14}
\end{equation*}
$$

and this seems close to necessary as well, see [91, 68]. Note that, for $u$ non-constant with $u(\infty)>0$ one always has $0 \leq A E_{+}(u) \leq 1, A E_{-}(u) \geq 1$ (see Lemma 6.1 of [61] and Proposition 4.1 of [91]). We shall see in Remark 2.14 below that in discrete-time models (14) can be relaxed to $A E_{+}(u)<A E_{-}(u)$, i.e. to either $A E_{+}(u)<1$ or $A E_{-}(u)>1$, as already noticed in [78].

We now present conditions on $u$ which allow to assert the existence of an optimal strategy. The main novelty with respect to previous studies is that we do not require concavity of $u$.

Assumption 2.13. The function $u: \mathbb{R} \rightarrow \mathbb{R}$ is non-decreasing, continuous and there exist $c \geq 0, \underline{x}>0, \bar{x}>0, \alpha, \beta>0$ such that $\alpha<\beta$ and for any $\lambda \geq 1$,

$$
\begin{align*}
u(\lambda x) & \leq \lambda^{\alpha} u(x)+c \text { for } x \geq \bar{x}  \tag{15}\\
u(\lambda x) & \leq \lambda^{\beta} u(x) \text { for } x \leq-\underline{x}  \tag{16}\\
u(-\underline{x}) & <0, u(\bar{x}) \geq 0 \tag{17}
\end{align*}
$$

A typical $u$ satisfying Assumption 2.13 is that of Example 2.10 with $\alpha<\beta$.
Remark 2.14. As explained in Section 6.3, $A E_{ \pm}(u)$ can be defined for nonconcave and nonsmooth $u$ as well.

If $c=0$ then (15) and (16) together are equivalent to $A E_{+}(u)<A E_{-}(u)$, see Section 6.3. Hence Assumption 2.13 is a logical generalization of the condition of [78] to the nonconcave case, see Remark 2.12 above. A positive $c$ allows to incorporate bounded above utility functions as well. Note, however, that we have already obtained existence results for such $u$ in Theorem 2.1 above. That result is sharper for $u$ bounded above than Theorem 2.18 below since Assumption 2.13 implies that $u(x) \leq(|x| / \underline{x})^{\beta} u(-\underline{x})$ for $x \leq-\underline{x}$ which entails the convergence of $u(x)$ to $-\infty$ at a polynomial speed as $x \rightarrow-\infty$ while no such assumption is needed in Theorem 2.1: we may have e.g. $u(x) \sim-\ln (-x)$ or $u(x) \sim-\ln \ln (-x)$ near $-\infty$.

Since for the $u$ in Example 2.10 one trivially has $A E_{+}(u)=\alpha$ and $A E_{-}(u)=\beta$, the argument of Example 2.10 shows that the condition $\alpha<\beta$ in Assumption 2.13 is needed in order to get existence in a reasonably broad class of models, showing that Theorem 2.18 below is fairly sharp.

We will use a dynamic programming procedure and, to this end, we have to prove that the associated random functions are well-defined and a.s. finite under appropriate integrability conditions. Let $B$ be the $\mathcal{F}_{T}$-measurable random variable appearing in (3).

Proposition 2.15. Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing and left-continuous. Assume that for all $1 \leq t \leq T, x \in \mathbb{R}$ and $y \in \mathbb{R}^{d}$

$$
\begin{equation*}
E\left(u^{-}\left(x+y \Delta S_{t}-B\right) \mid \mathcal{F}_{t-1}\right)<+\infty \tag{18}
\end{equation*}
$$

holds true a.s. for all $t=1, \ldots, T$. Then the following random functions are well-defined recursively, for all $x \in \mathbb{R}$ (we omit dependence on $\omega \in \Omega$ in the notation):

$$
\begin{align*}
U_{T}(x) & :=u(x-B)  \tag{19}\\
U_{t-1}(x) & :=\quad \text { ess } \sup _{\xi \in \Xi_{t-1}} E\left(U_{t}\left(x+\xi \Delta S_{t}\right) \mid \mathcal{F}_{t-1}\right) \text { for } 1 \leq t \leq T, \tag{20}
\end{align*}
$$

and one can choose $(-\infty,+\infty]$-valued versions which are non-decreasing and left-continuous in $x$, a.s. In particular, each $U_{t}$ is $\mathcal{F}_{t} \otimes \mathcal{B}(\mathbb{R})$-measurable. Moreover, for all $0 \leq t \leq T$, almost surely for all $x \in \mathbb{R}$, we have:

$$
\begin{equation*}
U_{t}(x) \geq E\left(u(x-B) \mid \mathcal{F}_{t}\right)>-\infty \tag{21}
\end{equation*}
$$

where the right-hand side of $\geq$ also has an a.s. left-continuous and non-decreasing version $G(\omega, x)$ satisfying

$$
\begin{equation*}
G(H)=E\left(u(H-B) \mid \mathcal{F}_{t}\right) \text { a.s. } \tag{22}
\end{equation*}
$$

for every $\mathcal{F}_{t}$-measurable random variable $H$ as well.
Proof. At $t=T$, (21) holds true by definition of $U_{T}$ and

$$
U_{T}(x)=E\left(u(x-B) \mid \mathcal{F}_{T}\right)=u(x-B)
$$

clearly admit a regular version by our assumptions on $u$.
Assume now that the statements hold true for $t+1$. For $x \in \mathbb{R}$, let $F(x)$ be an arbitrary version of ess $\sup _{\xi \in \Xi} E\left(U_{t+1}\left(x+\xi \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)$. Fix any pairs of real numbers $x_{1} \leq x_{2}$. As for almost $\xi \in \Xi$ all $\omega, U_{t+1}(\omega, \cdot)$ is a nondecreasing, we get that that $F\left(x_{1}\right) \leq F\left(x_{2}\right)$ almost surely. Hence there is a negligible set $N \subset \Omega$ outside which $F(\omega, \cdot)$ is non-decreasing over $\mathbb{Q}$.

For $\omega \in \Omega \backslash N$, let us define the following left-continuous function on $\mathbb{R}$ (possibly taking the value $\infty$ ): for each $x \in \mathbb{R}$ let $U_{t}(\omega, x):=\sup _{r<x, r \in \mathbb{Q}} F(\omega, r)$. For $\omega \in N$, define $F(\omega, x)=0$ for all $x \in \mathbb{R}$. Let $r_{i}, i \in \mathbb{N}$ be an enumeration of $\mathbb{Q}$. Then $U_{t}(\omega, x)=\sup _{n \in \mathbb{N}}\left[F\left(\omega, r_{n}\right) 1_{\left\{r_{n}<x\right\}}+\right.$ $\left.(-\infty) 1_{\left\{r_{n} \geq x\right\}}\right]$ for all $x$ and for all $\omega \in \Omega \backslash N$, hence $U_{t}$ is clearly an $\mathcal{F}_{t} \otimes \mathcal{B}(\mathbb{R})$-measurable function. It remains to show that, for each fixed $x \in \mathbb{R}, U_{t}(x)$ is a version of $F(x)$.

Take $\mathbb{Q} \ni r_{n} \uparrow x, r_{n}<x, n \rightarrow \infty$. Then $F\left(r_{n}\right) \leq F(x)$ a.s. and $U_{t}(x)=\lim _{n} F\left(r_{n}\right) \leq F(x)$ a.s. On the other hand, for each $k \geq 1$, there is $\xi_{k} \in \Xi$ such that

$$
F(x)-1 / k=\underset{\xi \in \Xi}{\operatorname{ess} \sup _{\xi \in}} E\left(U_{t+1}\left(x+\xi \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)-1 / k \leq E\left(U_{t+1}\left(x+\xi_{k} \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right) \text { a.s. }
$$

By definition, $F\left(r_{n}\right) \geq E\left(U_{t+1}\left(r_{n}+\xi_{k} Y\right) \mid \mathcal{F}_{t}\right)$ a.s. for all $n$. We argue over the sets $A_{m}(k):=$ $\left\{\omega: m-1 \leq\left|\xi_{k}(\omega)\right|<m\right\}, m \geq 1$ separately and fix $m$. Provided that we can apply Fatou's lemma, we get

$$
U_{t}(x)=\lim _{n} F\left(r_{n}\right)=\liminf _{n} F\left(r_{n}\right) \geq E\left(U_{t+1}\left(x+\xi_{k} \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right) \text { a.s. on } A_{m}(k)
$$

using left-continuity of $U_{t+1}$. It follows that $U_{t}(x) \geq F(x)-1 / k$ a.s. for all $k$, hence $U_{t}(x) \geq$ $F(x)$ a.s. showing our claim.

For each function $i \in W:=\{-1,+1\}^{d}$ let us introduce the vector

$$
\begin{equation*}
\theta_{i}:=(i(1) \sqrt{d}, \ldots, i(d) \sqrt{d}) . \tag{23}
\end{equation*}
$$

Fatou's lemma works above because of (22) for $t+1$ and the estimate

$$
U_{t+1}^{-}\left(x+\xi_{k} \Delta S_{t+1}\right) \leq \max _{i \in W} U_{t+1}^{-}\left(x-m \theta_{i} \Delta S_{t+1}\right) \leq \sum_{i \in W} U_{t+1}^{-}\left(x-m \theta_{i} \Delta S_{t+1}\right) \text { a.s. }
$$

which holds on $A_{m}(k)$, for each $m, k$.
A similar but simpler argument provides a suitable version $G$ of $E\left(u(x-B) \mid \mathcal{F}_{t}\right)$. Equation (22) for step functions $H$ follows trivially and taking increasing step-function approximations $H_{k}$ of an arbitrary $\mathcal{F}_{t}$-measurable $H$ we get (22) for $H$ using left-continuity of $G$ and monotone convergence.

From now on we work with these versions of $U_{t}, E\left(u(x-B) \mid \mathcal{F}_{t}\right)$. Choosing $\xi=0$, we get that, for all $x \in \mathbb{R}$,

$$
U_{t}(x) \geq E\left(U_{t+1}(x) \mid \mathcal{F}_{t}\right) \geq E\left(u(x-B) \mid \mathcal{F}_{t}\right)>-\infty \text { a.s. }
$$

where the second inequality holds by the induction hypothesis (21). Due to the left-continuous versions, $U_{t}(x) \geq E\left(u(x-B) \mid \mathcal{F}_{t}\right)$ then holds for all $x$ simultaneously, outside a fixed negligible set, see Lemma 6.6.

In order to have a well-posed problem, we impose Assumption 2.16 below.
Assumption 2.16. Let $u$ be non-decreasing and left-continuous. For all $1 \leq t \leq T, x \in \mathbb{R}$ and $y \in \mathbb{R}^{d}$ we assume that

$$
\begin{align*}
E u^{-}(x-B)<\infty &  \tag{24}\\
E\left(u^{-}\left(x+y \Delta S_{t}-B\right) \mid \mathcal{F}_{t-1}\right) & <\infty \text { a.s. }  \tag{25}\\
E U_{0}(x) & <\infty . \tag{26}
\end{align*}
$$

Note that by, Proposition 2.15, one can state (26): $U_{0}$ is well-defined under (25).
Remark 2.17. In Assumption 2.16, condition (26) is not easy to verify. We propose in Corollaries 2.20 and 2.22 fairly general set-ups where it is valid. In contrast, (24) and (25) are straightforward integrability conditions on $S$ and $B$. For instance, if $u(x) \geq-m\left(1+|x|^{p}\right)$ for some $p, m>0, E\left(B^{+}\right)^{p}<\infty$ and $E\left|\Delta S_{t}\right|^{p}<\infty$ for all $t \geq 1$ then (25) and (24) hold.

We are now able to state a theorem asserting the existence of optimal strategies.
Theorem 2.18. Let u satisfy Assumption 2.13 and $S$ satisfy the (NA) condition. Let Assumption 2.16 hold with $B$ bounded below. Then one can choose non-decreasing, continuous in $x \in \mathbb{R}$ and $\mathcal{F}_{t}$-measurable in $\omega \in \Omega$ versions of the random functions $U_{t}$ defined in (19) and (20). Furthermore, there exists a $\mathcal{F}_{t-1} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$-measurable "one-step optimal" strategy $\tilde{\xi}_{t}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying, for all $t=1, \ldots, T$, and for each $x \in \mathbb{R}$,

$$
E\left(U_{t}\left(x+\tilde{\xi}_{t}(x) \Delta S_{t}\right) \mid \mathcal{F}_{t-1}\right)=U_{t-1}(x) \text { a.s. }
$$

Using these $\tilde{\xi} .(\cdot)$, we define recursively:

$$
\phi_{1}^{*}:=\tilde{\xi}_{1}(z), \quad \phi_{t}^{*}:=\tilde{\xi}_{t}\left(z+\sum_{j=1}^{t-1} \phi_{j}^{*} \Delta S_{j}\right), 1 \leq t \leq T .
$$

If, furthermore, $E u\left(X_{T}^{z, \phi^{*}}-B\right)$ exists then $\phi^{*} \in \Phi(z), \phi^{*}$ is an optimiser for problem (3) and

$$
\bar{u}(z):=\sup _{\phi \in \Phi(z)} E u\left(X_{T}^{z, \phi}-B\right)
$$

is continuous.
Condition (26) and the existence of $E u\left(X_{T}^{z, \phi^{*}}-B\right)$ are difficult to check (unless $u$ is bounded above, but this case has already been covered in greater generality in Theorem 2.1 above). Hence, at first sight, the above theorem looks useless: for which $S$ does it apply if $u$ is unbounded? We now state two corollaries whose proofs follow the scheme of the proof of Theorem 2.18 and which give concrete, easily verifiable conditions on $S$.

Let $\mathcal{W}$ denote the set of $\mathbb{R}$-valued random variables $Y$ such that $E|Y|^{p}<\infty$ for all $p>0$. This family is clearly closed under addition, multiplication and taking conditional expectation. With a slight abuse of notation, for a $d$-dimensional random variable $Y$, we write $Y \in \mathcal{W}$ when we indeed mean $|Y| \in \mathcal{W}$.

Assumption 2.19. For all $t \geq 1, \Delta S_{t} \in \mathcal{W}$. Furthermore, for $0 \leq t \leq T-1$, there exist $\kappa_{t}, \nu_{t} \in \Xi_{t}^{1}$ positive, satisfying $1 / \kappa_{t}, 1 / \nu_{t} \in \mathcal{W}$ such that

$$
\begin{equation*}
\text { ess. } \inf _{\xi \in \bar{\Xi}_{t}^{a}} P\left(\xi \Delta S_{t+1} \leq-\nu_{t}|\xi| \mid \mathcal{F}_{t}\right) \geq \kappa_{t} \text { a.s. } \tag{27}
\end{equation*}
$$

Corollary 2.20. Let Assumptions 2.13, 2.19 hold and assume that

$$
\begin{equation*}
u(x) \geq-m\left(|x|^{p}+1\right) \text { for all } x \in \mathbb{R}, \tag{28}
\end{equation*}
$$

holds with some $m, p>0$. Let $B \in \mathcal{W}$ be bounded below. Then there exists an optimiser $\phi^{*} \in \Phi(z)$ for problem (3) with $\phi_{t}^{*} \in \mathcal{W}$ for $1 \leq t \leq T$.
Remark 2.21. In the light of Proposition 1.6, $1 / \nu_{t}, 1 / \kappa_{t} \in \mathcal{W}$ for $0 \leq t \leq T-1$ is a certain strong form of no-arbitrage. When $S$ has independent increments and (NA) holds, then one can choose $\kappa_{t}=\kappa$ and $\nu_{t}=\nu$ in Proposition 1.6 with deterministic constants $\kappa, \nu>0$. See Section 3.6 for other concrete examples where $1 / \nu_{t}, 1 / \kappa_{t} \in \mathcal{W}$ is verified.

The assumption that $\Delta S_{t+1}, 1 / \nu_{t}, 1 / \kappa_{t} \in \mathcal{W}$ for $0 \leq t \leq T-1$ could be weakened to the existence of the $N$ th moment for $N$ large enough but this would lead to complicated bookkeeping with no essential gain in generality, which we prefer to avoid.

We provide one more result in the spirit of Corollary 2.20.
Corollary 2.22. Let Assumption 2.13 hold with $B \in L^{\infty}$ and let $\Delta S_{t}, 1 \leq t \leq T$ be a bounded process. Let (NA) hold with $\nu_{t}, \kappa_{t}$ of Proposition 1.6 being constant. Then there exists a solution $\phi^{*} \in \Phi(z)$ of problem (3) which is a bounded process.

It will be clear from the proofs of the above corollaries that one could accomodate a larger class of $u$ at the price of stronger assumptions on $S$. For instance, if $u$ behaves like $-e^{-x}$ for $x$ near $-\infty$ then well-posedness and existence holds provided that certain iterated exponential functions of $S_{t}, 1 / \kappa_{t}, 1 / \nu_{t}$ are integrable. Such extensions do not seem to be of any practical use hence we refrain from chasing a greater generality here.

We will present the proofs of Theorem 2.18 and Corollaries 2.20, 2.22 in Section 2.6.
Remark 2.23. If $u$ is concave then (16) is automatic for $\beta=1$, with some $\underline{x}$. Hence in this case one can replace Assumption 2.13 in Theorem 2.18 and in Corollaries $2.20,2.22$ by (15) with $\alpha<1$ and with some $\bar{x}>0$.

Similarly, for $u$ concave, Assumption 2.13 can also be replaced by (16) with $\beta>1$, since (15) is automatic for $\alpha=1$.

Remark 2.24. Theorem 2.18 as well as the ensuing two corollaries continue to hold if, instead of stipulating that $B$ is bounded below, we assume only the existence of $\psi \in \Phi$ and $y \in \mathbb{R}$ with

$$
\begin{equation*}
X_{T}^{y, \psi} \leq B \tag{29}
\end{equation*}
$$

The proofs work in the same way but instead of $|\xi|$ for $\xi \in \Xi_{t}^{d}$ one needs to estimate $\left|\xi-\psi_{t+1}\right|$. We opted for the above less general versions for the sake of a simple presentation.

Condition (29) has a clear economic interpretation: it means that $B$ can be sub-hedged by some portfolio, i.e. the losses incurred ( $B^{-}$) are controlled by some loss realizable by the portfolio $\psi$. In particular, if $B$ can be replicated by a portfolio then (29) holds.

### 2.5 Existence of an optimal strategy for the one-step case

First we prove the existence of an optimal strategy in the case of a one-step model. Let $Y$ be a $d$-dimensional random variable, $\mathcal{H} \subset \mathcal{F}$ a sigma-algebra and a function $V: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying the hypotheses that will be presented below.

Let $\Xi^{n}$ denote the family of $\mathcal{H}$-measurable $n$-dimensional random variables. The aim of this section is to study ess. $\sup _{\xi \in \Xi^{d}} E(V(x+\xi Y) \mid \mathcal{H})$. For each $x$, let us fix an arbitrary version $v(x)=v(\omega, x)$ of this essential supremum.

We prove in Proposition 2.38 that, under suitable assumptions, there is an optimiser $\tilde{\xi}(x)$ which attains the essential supremum in the definition of $v(x)$, i.e.

$$
\begin{equation*}
v(x)=E(V(x+\tilde{\xi}(x) Y) \mid \mathcal{H}) \tag{30}
\end{equation*}
$$

In Proposition 2.38, we even prove that the same optimal solution $\tilde{\xi}(H)$ applies if we replace $x$ by any $H \in \Xi^{1}$ in (30).

This setting will be applied in Section 2.6 with the choice $\mathcal{H}=\mathcal{F}_{t-1}, Y=\Delta S_{t} ; V(x)$ will be the maximal conditional expected utility from capital $x$ if trading begins at time $t$, i.e. $V=U_{t}$. In this case, the function $v(x)$ will represent the maximal expected utility from capital $x$ if trading begins at time $t-1$, i.e. $v=U_{t-1}$.

We start with a useful little lemma.
Lemma 2.25. Let $V(\omega, x)$ be a $\mathcal{F} \otimes \mathcal{B}(\mathbb{R})$-measurable function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that, for all $\omega, V(\omega, \cdot)$ is a nondecreasing function. The following conditions are equivalent:

1a. $E\left(V^{+}(x+y Y) \mid \mathcal{H}\right)<\infty$ a.s., for all $x \in \mathbb{R}, y \in \mathbb{R}^{d}$.
2a. $E\left(V^{+}(x+|y||Y|) \mid \mathcal{H}\right)<\infty$ a.s., for all $x, y \in \mathbb{R}$.
3a. $E\left(V^{+}(H+\xi Y) \mid \mathcal{H}\right)<\infty$ a.s., for all $H \in \Xi^{1}, \xi \in \Xi^{d}$.
The following conditions are also equivalent :
1b. $E\left(V^{-}(x+y Y) \mid \mathcal{H}\right)<\infty$ a.s., for all $x \in \mathbb{R}, y \in \mathbb{R}^{d}$.
2b. $E\left(V^{-}(x-|y||Y|) \mid \mathcal{H}\right)<\infty$ a.s., for all $x, y \in \mathbb{R}$.
3b. $E\left(V^{-}(H+\xi Y) \mid \mathcal{H}\right)<\infty$ a.s., for all $H \in \Xi^{1}, \xi \in \Xi^{d}$.
Proof. We only prove the equivalences for $V^{+}$since the ones for $V^{-}$are similar. We start with 1a. implies 2a. Let $x, y \in \mathbb{R}$. We can conclude since

$$
V^{+}(x+|y||Y|) \leq \max _{i \in W} V^{+}\left(x+|y| \theta_{i} Y\right) \leq \sum_{i \in W} V^{+}\left(x+|y| \theta_{i} Y\right),
$$

by $|Y| \leq \sqrt{d}\left(\left|Y^{1}\right|+\ldots+\left|Y^{d}\right|\right.$ ) (recall (23) for the definition of $W, \theta_{i}$ ). Next we prove that 2a. implies 3a. Let $H, \xi$ be $\mathcal{H}$-measurable random variables, define $A_{m}:=\{|H|<m,|\xi|<m\}$ for $m \geq 1$. Clearly,

$$
V^{+}(H+\xi Y) 1_{A_{m}} \leq V^{+}(m+m|Y|) 1_{A_{m}}
$$

and the $\mathcal{H}$-conditional expectation of the latter is finite by 2 a . Hence 3 a . follows from Lemma 6.1. Now 3a. trivially implies 1a.

Assumption 2.26. $V(\omega, x)$ is a function from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that for almost all $\omega, V(\omega, \cdot)$ is a nondecreasing, finite-valued, continuous function and $V(\cdot, x)$ is $\mathcal{F}$-measurable for each fixed $x$. For all $x, y \in \mathbb{R}$,

$$
\begin{align*}
& E\left(V^{-}(x-|y||Y|) \mid \mathcal{H}\right)<+\infty \quad \text { a.s. }  \tag{31}\\
& E\left(V^{+}(x+|y||Y|) \mid \mathcal{H}\right)<+\infty \quad \text { a.s.. } \tag{32}
\end{align*}
$$

Remark 2.27. Let $H, \xi$ be arbitrary $\mathcal{H}$-measurable random variables. Then, from Lemma 2.25, under Assumption 2.26 above, $E(V(H+\xi Y) \mid \mathcal{H})$ exists and it is a.s. finite.

Let us recall from Section 2.2 the random set $D \in \mathcal{H} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$ such that for a.e. $\omega \in \Omega$, $D(\omega):=\left\{x \in \mathbb{R}^{d}:(\omega, x) \in D\right\}$ is the smallest affine subspace containing the support of the conditional distribution of $Y$ with respect to $\mathcal{H}$.

Let us denote $\widehat{\Xi}^{d}:=\left\{\xi \in \Xi^{d}: \xi \in D\right.$ a.s. $\}$ and recall Remark 2.4 which shows that the essential supremum in (30) can be equivalently taken over $\Xi^{d}$ or $\widehat{\Xi}^{d}$.

Assumption 2.28. There exist $\mathcal{H}$-measurable random variables with $0<\kappa, \nu \leq 1$ a.s. such that for all $\xi \in \widehat{\Xi}^{d}$ :

$$
\begin{equation*}
P(\xi Y \leq-\nu \mid \xi \| \mathcal{H}) \geq \kappa \tag{33}
\end{equation*}
$$

As easily seen, (33) implies that $D(\cdot)$ is a.s. a linear space.
We finally impose the following growth conditions on $V$.
Assumption 2.29. There exist constants $C, g \geq 0, \beta>\alpha>0$ such that

$$
\begin{align*}
V(\lambda x) & \leq \lambda^{\alpha} V(x+g)+C \lambda^{\alpha}  \tag{34}\\
V(\lambda x) & \leq \lambda^{\beta} V(x+g)+C \lambda^{\beta} \tag{35}
\end{align*}
$$

both hold for all $x, \omega$ and $\lambda \geq 1$. There exists an $\mathcal{H}$-measurable random variable $N \geq 0$ such that

$$
\begin{equation*}
P\left(\left.V(-N) \leq-\frac{2 C}{\kappa}-1 \right\rvert\, \mathcal{H}\right) \geq 1-\kappa / 2 \quad \text { a.s. } \tag{36}
\end{equation*}
$$

where $\kappa$ is as in Assumption 2.28 and $C$ as in (34).
Remark 2.30. There is no misprint here, it is crucial that the above inequalities hold with both $\alpha$ and $\beta$ : for $x$ near $-\infty$ we will need (35) while for $x$ near $\infty$ we will need (34). In order to prove that these properties are preserved by dynamic programming, we will need to verify both properties for all $x$, see (66), (67) in Section 2.6.

In the sequel when we say that a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is polynomial then we mean that there exist $k, C \geq 0$ such that

$$
f\left(x_{1}, \ldots, x_{n}\right) \leq C\left[1+\left|x_{1}\right|^{k}+\ldots+\left|x_{n}\right|^{k}\right]
$$

We will often use the following facts without mention: for all $x, y \in \mathbb{R}$, one has

$$
\begin{aligned}
& |x+y|^{\eta} \leq|x|^{\eta}+|y|^{\eta}, \text { for } 0<\eta \leq 1 \\
& |x+y|^{\eta} \leq 2^{\eta-1}\left(|x|^{\eta}+|y|^{\eta}\right), \text { for } \eta>1
\end{aligned}
$$

Lemma 2.31. Let Assumptions 2.28, 2.26 and 2.29 hold. Let $\eta$ be such that $0<\eta<1$ and $\alpha<\eta \beta$ (recall that $\alpha<\beta$ ). Define

$$
\begin{align*}
L & =E\left(V^{+}(1+|Y|+g) \mid \mathcal{H}\right),  \tag{37}\\
K_{1}(x) & =\max \left(1, x^{+},\left(\frac{x^{+}+N+g}{\nu}\right)^{\frac{1}{1-\eta}}, \frac{x^{+}+N}{\nu},\left(\frac{6 L}{\kappa}\right)^{\frac{1}{\eta \beta-\alpha}},\left(\frac{6 C}{\kappa}\right)^{\frac{1}{\eta \beta-\alpha}}\right),  \tag{38}\\
K_{2}(x) & =\left(\frac{6[E(V(x) \mid \mathcal{H})]^{-}}{\kappa}\right)^{\frac{1}{\eta \beta}},  \tag{39}\\
\tilde{K}(x) & =\max \left\{K_{1}(\lfloor x\rfloor+1), K_{2}(\lfloor x\rfloor)\right\} . \tag{40}
\end{align*}
$$

All these random variables are $\mathcal{H}$-measurable and a.s. finite-valued. $K_{1}(\omega, x)$ (resp. $K_{2}(\omega, x)$ ) is non-decreasing (resp. non-increasing) in $x, \tilde{K}(\cdot)$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable and a.s. constant on intervals of the form $[n, n+1), n \in \mathbb{Z}$.

For $\xi \in \widehat{\Xi}^{d}$ with $|\xi| \geq \tilde{K}(x)$, we have almost surely:

$$
\begin{equation*}
E(V(x+\xi Y) \mid \mathcal{H}) \leq E(V(x) \mid \mathcal{H}) \tag{41}
\end{equation*}
$$

Assume that there exist $m, p>0$ and $0 \leq R \in \Xi^{1}$ such that $V(x) \geq-m\left(1+|x|^{p}+R\right)$ a.s. for all $x \leq 0$. Then there exists a non-negative, a.s. finite-valued $\mathcal{H}$-measurable random variable $M$ and some number $\theta>0$ such that, for a.e. $\omega$,

$$
\begin{equation*}
\tilde{K}(x) \leq M\left(|x|^{\theta}+1\right), \text { for all } x \tag{42}
\end{equation*}
$$

and $M$ is a polynomial function of $N, 1 / \nu, 1 / \kappa, R$ and $L$. We have

$$
\begin{equation*}
-\infty<v(x)=\text { ess. } \sup _{\xi \in \Xi^{d},|\xi| \leq \tilde{K}(x)} E(V(x+\xi Y) \mid \mathcal{H})<\infty \text { a.s. } \tag{43}
\end{equation*}
$$

For any positive $I \in \Xi^{1}$ there exists a positive $N^{\prime} \in \Xi^{1}$ such that $v\left(-N^{\prime}\right) \leq-I$ a.s. More precisely, $N^{\prime}$ is a polynomial function of $\frac{1}{\kappa}, N, I, R$ and $E\left(V^{+}(\bar{K}|Y|) \mid \mathcal{H}\right)$, where

$$
\begin{equation*}
\bar{K}:=\max \left(1, \frac{N+g}{\nu},\left(\frac{N}{\nu}\right)^{\frac{1}{1-\eta}},\left(\frac{8 L}{\kappa}\right)^{\frac{1}{\eta \beta-\alpha}},\left(\frac{8 C}{\kappa}\right)^{\frac{1}{\eta \beta-\alpha}}\right) . \tag{44}
\end{equation*}
$$

It follows directly from (41) that $E\left(V\left(x+\xi 1_{\{|\xi|>\tilde{K}(x)\}} Y\right) \mid \mathcal{H}\right) \leq E(V(x) \mid \mathcal{H})$ a.s. for all $\xi \in \widehat{\Xi}^{d}$, so we get that

$$
\begin{equation*}
E\left(V\left(x+\xi 1_{|\xi| \leq \tilde{K}(x)} Y\right) \mid \mathcal{H}\right) \geq E(V(x+\xi Y) \mid \mathcal{H}) \text { a.s. } \tag{45}
\end{equation*}
$$

Proof of Lemma 2.31. Fix some $x \in \mathbb{R}$ and take $\xi \in \widehat{\Xi}^{d}$ with $|\xi| \geq \max \left(1, x^{+}\right)$. By (34), we have the following estimation:

$$
\begin{aligned}
V(x+\xi Y) & =V(x+\xi Y) 1_{\{V(x+\xi Y) \geq 0\}}+V(x+\xi Y) 1_{\{V(x+\xi Y)<0\}} \\
& \leq 1_{\{V(x+\xi Y) \geq 0\}}\left(|\xi|^{\alpha} V\left(\frac{x^{+}}{|\xi|}+\frac{\xi}{|\xi|} Y+g\right)+C|\xi|^{\alpha}\right)+V(x+\xi Y) 1_{\{V(x+\xi Y)<0\}} \text { a.s. }
\end{aligned}
$$

Note that the random variable $L$ (recall (37)) is finite by (32). As $V$ is nondecreasing,

$$
E\left(\left.1_{\{V(x+\xi Y) \geq 0\}} V\left(\frac{x^{+}}{|\xi|}+\frac{\xi}{|\xi|} Y+g\right) \right\rvert\, \mathcal{H}\right) \leq E\left(\left.V^{+}\left(1+\frac{\xi}{|\xi|} Y+g\right) \right\rvert\, \mathcal{H}\right) \leq L
$$

For the estimation of the negative part, we introduce the event

$$
\begin{equation*}
B:=\left\{V(x+\xi Y)<0, \frac{\xi}{|\xi|} Y \leq-\nu, V(-N) \leq-\frac{2 C}{\kappa}-1\right\} \tag{46}
\end{equation*}
$$

Then, using (35), we obtain that a.s.

$$
\begin{aligned}
-V(x+\xi Y) 1_{\{V(x+\xi Y)<0\}} & \geq-V(x+\xi Y) 1_{B} \\
& \geq-1_{B}\left(|\xi|^{\eta \beta} V\left(\frac{x^{+}}{|\xi|^{\eta}}+\frac{\xi}{|\xi|} Y|\xi|^{1-\eta}+g\right)+C|\xi|^{\eta \beta}\right)
\end{aligned}
$$

Now, from Assumption 2.28 and (36), we have

$$
\begin{equation*}
P\left(\frac{\xi}{|\xi|} Y \leq-\nu, \left.V(-N) \leq-\frac{2 C}{\kappa}-1 \right\rvert\, \mathcal{H}\right) \geq \kappa / 2 \tag{47}
\end{equation*}
$$

It is clear that $B$ contains

$$
\left\{x^{+}-\nu|\xi| \leq-N, \frac{\xi}{|\xi|} Y \leq-\nu, V(-N) \leq-\frac{2 C}{\kappa}-1\right\}
$$

Thus if we assume that $x^{+}-\nu|\xi| \leq-N$, we get that $P(B \mid \mathcal{H}) \geq \kappa / 2$ a.s. Now assume that both $x^{+}-\nu|\xi| \leq-N$ and $\frac{x^{+}}{|\xi|^{\eta}}-|\xi|^{1-\eta} \nu+g \leq-N$ hold. This is true if $|\xi| \geq K_{0}(x):=$ $\max \left(1, x^{+},\left(\frac{x^{+}+N+g}{\nu}\right)^{\frac{1}{1-\eta}}, \frac{x^{+}+N}{\nu}\right)$. Then we have

$$
\begin{align*}
E\left(V(x+\xi Y) 1_{\{V(x+\xi Y)<0\}} \mid \mathcal{H}\right) & \leq|\xi|^{\eta \beta} E\left(1_{B} V(-N) \mid \mathcal{H}\right)+C|\xi|^{\eta \beta} E\left(1_{B} \mid \mathcal{H}\right) \\
& \leq-\frac{\kappa}{2}|\xi|^{\eta \beta} \tag{48}
\end{align*}
$$

by (36). Putting together our estimations, for $|\xi| \geq K_{0}(x)$ we have a.s.

$$
\begin{equation*}
E(V(x+\xi Y) \mid \mathcal{H}) \leq|\xi|^{\alpha} L+C|\xi|^{\alpha}-\frac{\kappa}{2}|\xi|^{\eta \beta} \tag{49}
\end{equation*}
$$

In order to get (41), it is enough to have, a.s.,

$$
\begin{equation*}
|\xi|^{\alpha} L-\frac{\kappa}{6}|\xi|^{\eta \beta} \leq 0, C|\xi|^{\alpha}-\frac{\kappa}{6}|\xi|^{\eta \beta} \leq 0,-\frac{\kappa}{6}|\xi|^{\eta \beta}-E(V(x) \mid \mathcal{H}) \leq 0 \tag{50}
\end{equation*}
$$

Since $\alpha<\eta \beta<\beta$, the first two inequalities will be satisfied as soon as $|\xi| \geq K_{1}(x)$ (recall (38)) and the last one as soon as $|\xi| \geq K_{2}(x)$ (recall (39)). Clearly, $K_{1}(x), K_{2}(x) \in \Xi^{1}$. It is also clear that $K_{1}(x)$ is non-decreasing in $x$ and $K_{2}(x)$ is non-increasing in $x$. As $[E(V(x) \mid \mathcal{H})]^{-} \leq$ $E\left(V^{-}(x) \mid \mathcal{H}\right)$ (see Corollary 6.5), by (31) $K_{2}(x)$ is a.s. finite valued.

Let $\widehat{K}(x):=\max \left(K_{1}(x), K_{2}(x)\right)$. Then (41) is satisfied if $|\xi| \geq \widehat{K}(x)$. From the monotonicity property of $K_{1}(\cdot)$ and $K_{2}(\cdot)$, we get that $\tilde{K}(x) \geq \widehat{K}(x)$. Thus (41) is also satisfied as soon as $|\xi| \geq \tilde{K}(x)$. The random function $\tilde{K}(\cdot)$ is trivially $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable.

By (37)-(40), $\tilde{K}(x)$ is dominated by a polynomial function of $(\lfloor x\rfloor+1)^{+}, N, 1 / \nu, 1 / \kappa, L$ and $[E(V(\lfloor x\rfloor) \mid \mathcal{H})]^{-}$. When $V(x) \geq{ }_{\tilde{\sim}} m\left(1+|x|^{p}+R\right)$, we have $[E(V(\lfloor x\rfloor) \mid \mathcal{H})]^{-} \leq m\left(1+\mid\left\lfloor\left. x\right|^{p}+\right.\right.$ $E(R \mid \mathcal{H}))$ a.s. So $\tilde{K}(x)$ satisfies $\tilde{K}(x) \leq M\left(|x|^{\theta}+1\right)$ a.s. for some $\theta>0$ and for some random variable $M$ which is a polynomial function of $N, 1 / \nu, 1 / \kappa, E(R \mid \mathcal{H})$ and $L$.

Equality in (43) follows immediately from (45). We now show that $v$ is finite. Let $\xi \in \widehat{\Xi}^{d}$, $|\xi| \leq \tilde{K}(x)$,

$$
-E\left(V^{-}(-|x|-\tilde{K}(x)|Y|) \mid \mathcal{H}\right) \leq E(V(x+\xi Y) \mid \mathcal{H}) \leq E\left(V^{+}(|x|+\tilde{K}(x)|Y|) \mid \mathcal{H}\right) \text { a.s. }
$$

and we conclude by Assumption 2.26.
Looking carefully at the estimations (48), (49) above we find that, if $x \leq 0$ and

$$
|\xi| \geq \max \left(1,\left(\frac{N+g}{\nu}\right)^{\frac{1}{1-\eta}}, \frac{N}{\nu}\right)
$$

then

$$
\begin{equation*}
E\left(1_{\{V(x+\xi Y) \geq 0\}} V(x+\xi Y) \mid \mathcal{H}\right)+\frac{1}{2} E\left(1_{\{V(x+\xi Y)<0\}} V(x+\xi Y) \mid \mathcal{H}\right) \leq 0 \text { a.s. } \tag{51}
\end{equation*}
$$

provided that $L|\xi|^{\alpha}+C|\xi|^{\alpha}-\frac{\kappa}{4}|\xi|^{\eta \beta} \leq 0$. So (51) holds true provided that $L|\xi|^{\alpha}-\frac{\kappa}{8}|\xi|^{\eta \beta} \leq 0$, and $C|\xi|^{\alpha}-\frac{\kappa}{8}|\xi|^{\eta \beta} \leq 0$, i.e.

$$
|\xi| \geq \max \left(1, \frac{N}{\nu},\left(\frac{N+g}{\nu}\right)^{\frac{1}{1-\eta}},\left(\frac{8 L}{\kappa}\right)^{\frac{1}{\eta \beta-\alpha}},\left(\frac{8 C}{\kappa}\right)^{\frac{1}{\eta \beta-\alpha}}\right)=\bar{K}
$$

Let $0 \leq I \in \Xi^{1}$. It remains to show that there exists $0 \leq N^{\prime} \in \Xi^{1}$ satisfying $v\left(-N^{\prime}\right) \leq-I$ a.s. From now on we work on the event $\{x \leq-N\}$.

$$
\begin{aligned}
-E\left(1_{\{V(x+\xi Y)<0\}} V(x+\xi Y) \mid \mathcal{H}\right) & \geq-E\left(\left.1_{\left\{\frac{\xi}{|\xi|} Y \leq-\nu, V(-N) \leq-\frac{2 C}{\kappa}-1\right\}} V(x) \right\rvert\, \mathcal{H}\right) \\
& \geq\left(\left(1+\frac{2 C}{\kappa}\right)\left(\frac{x}{-N}\right)^{\beta}-C\left(\frac{x}{-N}\right)^{\beta}\right) \frac{\kappa}{2} \\
& \geq \frac{\kappa}{2}\left(\frac{x}{-N}\right)^{\beta},
\end{aligned}
$$

where we used (35), (47) and the fact that $\kappa \leq 1$ (see Assumption 2.28). Thus, if $|\xi| \leq \bar{K}$, we obtain on $\{x \leq-N\}$,

$$
\begin{equation*}
E(V(x+\xi Y) \mid \mathcal{H}) \leq E\left(V^{+}(\bar{K}|Y|) \mid \mathcal{H}\right)-\frac{\kappa}{2}\left(\frac{x}{-N}\right)^{\beta} \text { a.s. } \tag{52}
\end{equation*}
$$

Recall the definition of $\bar{K}$ and (51): if $|\xi| \geq \bar{K}$ then we get that

$$
\begin{equation*}
E(V(x+\xi Y) \mid \mathcal{H}) \leq \frac{1}{2} E\left(1_{\{V(x+\xi Y)<0\}} V(x+\xi Y) \mid \mathcal{H}\right) \leq-\frac{\kappa}{4}\left(\frac{x}{-N}\right)^{\beta} \text { a.s. } \tag{53}
\end{equation*}
$$

The right-hand sides of both (52) and (53) are smaller than $-I$ if

$$
\begin{equation*}
\left(\frac{x}{-N}\right)^{\beta} \geq \frac{4}{\kappa}\left(I+E\left(V^{+}(\bar{K}|Y|) \mid \mathcal{H}\right)\right) \text { a.s. } \tag{54}
\end{equation*}
$$

We may and will assume that $I \geq 1 / 4$ which implies $4 I / \kappa \geq 1$. So there exists an $\mathcal{H}$ measurable random variable

$$
\begin{equation*}
N^{\prime}:=N\left(\frac{4}{\kappa}\left(I+E\left(V^{+}(\bar{K}|Y|) \mid \mathcal{H}\right)\right)\right)^{\frac{1}{\beta}} \geq N \text { a.s. } \tag{55}
\end{equation*}
$$

such that, as soon as $x \leq-N^{\prime}, E(V(x+\xi Y) \mid \mathcal{H}) \leq-I$ a.s. and, taking the supremum over all $\xi, v(x) \leq-I$ a.s. holds. From (55), one can see that $N^{\prime}$ is a polynomial function of $\frac{1}{\kappa}, N, I$ and $E\left(V^{+}(\bar{K}|Y|) \mid \mathcal{H}\right)$.

Remark 2.32. A predecessor of Lemma 2.31 above is Lemma 4.8 of [78] whose arguments, however, are considerably simpler since $V$ is assumed concave in [78]. We remark that most of the literature on the case of concave $u$ proves the existence of the maximiser through the dual problem. The only papers using a direct approach are [94, 95, 96, 78]. In the present context, due to the non-concavity of $u$, a dual approach does not look feasible and we are forced to take the primal route which has the advantage of providing explicit bounds on the optimal strategies via Lemma 2.31.

Lemma 2.33. Let Assumption 2.26 hold. There exists a version $G(\omega, x, y)$ of $E(V(x+y Y) \mid \mathcal{H})(\omega)$ for $(\omega, x, y) \in \Omega \times \mathbb{R} \times \mathbb{R}^{d}$ such that
(i) for a.e $\omega \in \Omega,(x, y) \rightarrow G(\omega, x, y)$ is continuous and nondecreasing in $x$;
(ii) for all $(x, y) \in \mathbb{R} \times \mathbb{R}^{d}$, the function $\omega \rightarrow G(\omega, x, y)$ is $\mathcal{H}$-measurable;
(iii) for each $X \in \Xi^{1}$ and for each $\xi \in \Xi^{d}$,

$$
\begin{equation*}
G(\cdot, X, \xi)=E(V(X+\xi Y) \mid \mathcal{H}), \text { a.s. } \tag{56}
\end{equation*}
$$

Remark 2.34. Note that, in particular, $G$ is $\mathcal{H} \otimes \mathcal{B}\left(\mathbb{R}^{d+1}\right)$-measurable.
Proof of Lemma 2.33. It is enough to construct $G(\omega, x, y)$ for $(x, y) \in[-N, N]^{d+1}$ for each $N \in$ $\mathbb{N}$. Let us fix $N$ and note that, $\sup _{(q, r) \in[-N, N]^{d+1}}|V(q+r Y)| \leq V^{-}(-N-N|Y|)+V^{+}(N+N|Y|)=$ : $O$ and there are $A_{j} \in \mathcal{H}, j \in \mathbb{N}$ such that $E\left(1_{A_{j}} O\right)<\infty$, by (31), (32). It is enough to carry out this construction on each $A_{j}$ separately, so we may and will assume $E(O)<\infty$.

Since $V(\omega, \cdot \cdot \cdot)$ is in the separable Banach space $C\left([-N, N]^{d+1}\right)$ and it is integrable by $E O<\infty$, Lemma 6.12 implies the existence of $G: \Omega \rightarrow C\left([-N, N]^{d+1}\right)$ such that, for each $x, y$, $G(\omega, x, y)$ is a version of $E(V(x+y Y) \mid \mathcal{H})$. For each $y, G(\omega, \cdot, y)$ is clearly a.s. non-decreasing on $\mathbb{Q}$ and this extends to $\mathbb{R}$ by continuity of $G$. As for assertion (iii), (56) is clear for $\mathcal{H}$ measurable step functions and we may assume $X, \xi$ bounded. Now ( $X, \xi$ ) can be approximated by a (bounded) sequence of $\mathcal{H}$-measurable step functions $\left(X_{n}, \varsigma_{n}\right)$ and we can conclude using continuity of $G$ and the conditional Lebesgue theorem. A more tedious but direct proof can also be given, see [23].

$$
A(\omega, x)=\sup _{y \in \mathbb{Q}^{d}} G(\omega, x, y), B(\omega, x):=\sup _{y \in \mathbb{Q}^{d},|y| \leq \tilde{K}(\omega, x)} G(\omega, x, y)
$$

for $(\omega, x) \in \Omega \times \mathbb{R}$ where $\tilde{K}(\omega, x)$ is defined in (40). Then we get that, on a set of full measure, (i) the function $x \rightarrow B(\omega, x), x \in \mathbb{R}$ is non-decreasing and continuous,
(ii) $B(\omega, x)=A(\omega, x)$ for all $x \in \mathbb{R}$.
(iii) For each $x \in \mathbb{R}, v(x)=A(x)$ a.s.

Remark 2.36. By (iii) above, for each $x, A(x)$ is a version of $v(x)$ and hence, from this point on we may choose this version replacing $v(\cdot)$ by $A(\cdot)$ : by (i) and (ii), we will work with a nondecreasing and continuous $v(\omega, \cdot)$ for a.e. $\omega$. Notice also that if $V(\omega, \cdot)$ is concave for a.e. $\omega$ then so are $G(\omega, \cdot, \cdot)$ and $v(\omega, \cdot)$.

Proof of Lemma 2.35. It is enough to prove this for $x \in[\ell, \ell+1)$ for some fixed $\ell \in \mathbb{Z}$. We remark that $B(\omega, x), A(\omega, x)$ are $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable.

We argue $\omega$-wise and fix $\omega \in \bar{\Omega}$ where $\bar{\Omega}$ is a full measure seton which conclusions of Lemma 2.33 hold. Fix also some $x \in \mathbb{R}$ such that $\ell \leq x<\ell+1$. Let $x_{n} \in[\ell, \ell+1)$ be a sequence of real numbers converging to $x$.

By definition of $B$, for all $k$, there exists some $y_{k}(\omega, x) \in \mathbb{Q}^{d},\left|y_{k}(\omega, x)\right| \leq \tilde{K}(\ell)(\omega)$ and $G\left(\omega, x, y_{k}(\omega, x)\right) \geq B(\omega, x)-1 / k$. Moreover, one has that $B\left(\omega, x_{n}\right) \geq G\left(\omega, x_{n}, y_{k}(\omega, x)\right)$ for all $n$, and

$$
\liminf _{n} B\left(\omega, x_{n}\right) \geq G\left(\omega, x, y_{k}(\omega, x)\right) \geq B(\omega, x)-1 / k
$$

and letting $k$ go to infinity,

$$
\underset{n}{\liminf _{n} B\left(\omega, x_{n}\right) \geq B(\omega, x) .}
$$

Note that $B\left(\omega, x_{n}\right)$ is defined as a supremum over a precompact set. Thus there exists $y_{n}^{*}(\omega) \in \mathbb{R}^{d},\left|y_{n}^{*}(\omega)\right| \leq \tilde{K}(\ell)(\omega)$ and $B\left(\omega, x_{n}\right)=G\left(\omega, x_{n}, y_{n}^{*}(\omega)\right)$. By compactness, there exists some $y^{*}(\omega)$ such that some subsequence $y_{n_{k}}^{*}(\omega)$ of $y_{n}^{*}(\omega)$ goes to $y^{*}(\omega), k \rightarrow \infty$, and $\lim \sup _{n} B\left(\omega, x_{n}\right)=\lim _{k} B\left(\omega, x_{n_{k}}\right)$. By Lemma 2.33 (i), one gets

$$
\underset{n}{\lim \sup } B\left(\omega, x_{n}\right)=G\left(\omega, x, y^{*}(\omega)\right) \leq B(\omega, x)
$$

We claim that $A$ is monotone on $\bar{\Omega}$. Indeed, let $r_{1} \leq r_{2}$ with $r_{1}, r_{2} \in[\ell, \ell+1)$. As $G\left(\omega, r_{1}, y\right) \leq$ $G\left(\omega, r_{2}, y\right)$ for each $y$, also $A\left(\omega, r_{1}\right) \leq A\left(\omega, r_{2}\right)$.

Applying Lemma 6.7 to $F(\omega, y)=G(\omega, x, y)$ and $K=\tilde{K}(\ell)$ for some $\ell \leq x<\ell+1$ we obtain that, almost surely,

$$
\sup _{y \in \mathbb{Q}^{d},|y| \leq K(\ell)(\omega)} G(\omega, x, y)=\text { ess. } \sup _{\xi \in \Xi^{d},|\xi| \leq K(\ell)} G(\omega, x, \xi(\omega))
$$

Now applying the same Lemma 6.7 to $F(\omega, y)=G(\omega, x, y)$ and $K=\infty$, we obtain that, almost surely,

$$
\sup _{y \in \mathbb{Q}^{d}} G(\omega, x, y)=\text { ess. } \sup _{\xi \in \Xi} G(\omega, x, \xi(\omega))
$$

Now from the definition of $v, A$ and (56) we obtain for each $x \in \mathbb{R}$,

$$
v(x)=\text { ess. } \sup _{\xi \in \Xi^{d}} E(V(x+\xi Y) \mid \mathcal{H})=\text { ess. } \sup _{\xi \in \Xi^{d}} G(\cdot, x, \xi)=A(x) \text { a.s. }
$$

and (iii) is proved for all $x \in \mathbb{R}$. Using (56) and the definition of $B$, we obtain for each $\ell \leq x<\ell+1$,

$$
v(x)=\text { ess. } \sup _{\xi \in \Xi^{d},|\xi| \leq K(\ell)} E(V(x+\xi Y) \mid \mathcal{H})=\text { ess. } \sup _{\xi \in \Xi^{d},|\xi| \leq K(\ell)} G(\cdot, x, \xi)=B(x) \text { a.s. }
$$

Our considerations so far imply that the set $\{A(\cdot, q)=B(\cdot, q)$ for all $q \in \mathbb{Q} \cap[\ell, \ell+1)\}$ has probability one. Fix some $\omega_{0}$ in the intersection of this set with the one where $A$ is nondecreasing. For any $x \in[\ell, \ell+1)$, there exist some sequences $r_{n}, q_{n} \in \mathbb{Q}, n \in \mathbb{N}$ such that $q_{n} \nearrow x$ and $r_{n} \searrow x, n \rightarrow \infty$. By definition of $\omega_{0}$,

$$
\lim _{q_{n} \nearrow x} A\left(\omega_{0}, q_{n}\right)=A\left(\omega_{0}, x-\right) \text { and } \lim _{r_{n} \searrow x} A\left(\omega_{0}, r_{n}\right)=A\left(\omega_{0}, x+\right) .
$$

As $B$ is continuous on $(\ell, \ell+1)$,

$$
\lim _{q_{n} \nearrow x} B\left(\omega_{0}, q_{n}\right)=\lim _{r_{n} \searrow x} B\left(\omega_{0}, r_{n}\right)=B\left(\omega_{0}, x\right) .
$$

So by choice of $\omega_{0}, A\left(\omega_{0}, x-\right)=B\left(\omega_{0}, x\right)=A\left(\omega_{0}, x+\right)$ hence

$$
\omega_{0} \in \mathcal{A}:=\{A(\cdot, x)=B(\cdot, x) \text { for all } x \in[\ell, \ell+1)\} .
$$

Thus $P(\mathcal{A})=1$ (in particular, $\mathcal{A}$ is measurable, which was not a priori clear).
Lemma 2.37. Let Assumptions 2.26, 2.28 and 2.29 hold. There is a set of full measure $\widehat{\Omega}$ and an $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable sequence $\xi_{n}(\omega, x)$ such that for all $\omega \in \widehat{\Omega}$ and $x \in \mathbb{R}$,

$$
\begin{aligned}
\xi_{n}(\omega, x) & \in D(\omega) \\
\left|\xi_{n}(\omega, x)\right| & \leq \tilde{K}(\omega, x), \\
G\left(\omega, x, \xi_{n}(\omega, x)\right) & \rightarrow A(\omega, x), n \rightarrow \infty
\end{aligned}
$$

see (40) for the definition of $\tilde{K}(\cdot)$. Moreover, for $(\omega, x) \in \widehat{\Omega} \times \mathbb{R}$ define

$$
\begin{equation*}
E_{n}(\omega, x):=\left|G\left(\omega, x, \xi_{n}(\omega, x)\right)-A(\omega, x)\right| . \tag{57}
\end{equation*}
$$

For all $N>0$ and for all $\omega \in \widehat{\Omega}, \sup _{|x| \leq N} E_{n}(\omega, x) \rightarrow 0, n \rightarrow \infty$.
Proof. Choose $\tilde{\Omega}$ such that all the conclusions of Lemma 2.35 hold on this set. Let $q_{k}, k \in \mathbb{N}$ be an enumeration of $\mathbb{Q}^{d}$. Define $\mathbb{D}_{n}:=\left\{l / 2^{n}: l \in \mathbb{Z}\right\}$.

For all $k$, consider the projection $Q_{k}(\omega)$ of $q_{k}$ on $D(\omega)$. By Proposition 4.6 of [78], $Q_{k}$ is $\mathcal{H}$-measurable. Moreover, from Remark 2.4, $q_{k} Y=Q_{k} Y$ a.s. for all $k$ hence

$$
\begin{aligned}
G\left(\omega, x, Q_{k}(\omega)\right) & =E\left(V\left(x+Q_{k}(\omega) Y\right) \mid \mathcal{H}\right) \\
=E\left(V\left(x+q_{k} Y\right) \mid \mathcal{H}\right) & =G\left(\omega, x, q_{k}\right)
\end{aligned}
$$

almost surely for each $x \in \mathbb{Q}$ so, by path regularity of $G$, for all $x$ simultaneously.
We denote by $\widehat{\Omega}$ the intersection of $\tilde{\Omega}$ with

$$
\cap_{k \in \mathbb{N}}\left\{G\left(x, Q_{k}(\omega)\right)=G\left(x, q_{k}\right)\right\}
$$

it is again a set of full measure.
Let $C_{1}^{n}=\left\{(\omega, x) \in \widehat{\Omega} \times \mathbb{D}_{n}:\left|q_{1}\right| \leq \tilde{K}(\omega, x)\right.$ and $\left.\left|G\left(\omega, x, q_{1}\right)-A(\omega, x)\right|<1 / n\right\}$ and for all $k \geq 2$, define $C_{k}^{n}$ recursively by

$$
C_{k}^{n}=\left\{(\omega, x) \in \widehat{\Omega} \times \mathbb{D}_{n}:\left|q_{k}\right| \leq \tilde{K}(\omega, x) \text { and }\left|G\left(\omega, x, q_{k}\right)-A(\omega, x)\right|<1 / n\right\} \backslash \cup_{l=1, \ldots, k-1} C_{l}^{n}
$$

Since, from Lemma 2.31, $\tilde{K}$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable, $C_{k}^{n}$ is in $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ (recall also Remark 2.34). As from Lemma 2.35, $A(\omega, x)=B(\omega, x)=\sup _{q_{k},\left|q_{k}\right| \leq \tilde{K}(\omega, x)} G\left(\omega, x, q_{k}\right)$, one has $\cup_{k} C_{k}^{n}=$ $\widehat{\Omega} \times \mathbb{D}_{n}$. Define for $(\omega, x) \in \widehat{\Omega} \times \mathbb{R}$

$$
\begin{equation*}
\xi_{n}(\omega, x)=\sum_{k=1}^{\infty} \sum_{l=-\infty}^{\infty} Q_{k}(\omega) 1_{\left\{\left(\omega, l / 2^{n}\right) \in C_{k}^{n}\right\}}(\omega) 1_{\left\{l / 2^{n} \leq x<(l+1) / 2^{n}\right\}}(x) \tag{58}
\end{equation*}
$$

Obviously, $\xi_{n}$ is $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$-measurable. We thus have for all $n$ and $(\omega, x) \in C_{k}^{n}$

$$
\begin{aligned}
\left|\xi_{n}(\omega, x)\right|=\left|Q_{k}(\omega)\right| \leq\left|q_{k}\right| & \leq \tilde{K}(\omega, x) \\
\left|G\left(\omega, x, \xi_{n}(\omega, x)\right)-A(\omega, x)\right| & <1 / n
\end{aligned}
$$

Fix any integer $N>0$, we will prove that for all $\omega \in \widehat{\Omega}, \sup _{|x| \leq N} E_{n}(\omega, x)$ goes to zero.
Define $K(x, y):=\max \left\{K_{1}(y), K_{2}(x)\right\}$, recalling (38), (39). We argue for each fixed $\omega \in \widehat{\Omega}$. As $x \rightarrow A(\omega, x)$ is continuous from Lemma 2.35, it is uniformly continuous on $[-N, N]$. The same argument applies to $G(\omega, x, y)$ on $[-N, N] \times[-K(-N, N+1), K(-N, N+1)]^{d}$ (see (i) in Lemma 2.33). Hence for each $\epsilon>0$ there is $\eta(\omega)>0$ such that $\left|A(\omega, x)-A\left(\omega, x_{0}\right)\right|<\epsilon / 3$ and $\left|G(\omega, x, y)-G\left(\omega, x_{0}, y_{0}\right)\right|<\epsilon / 3$ if $\left|x-x_{0}\right|+\left|y-y_{0}\right|<\eta(\omega)$ and $x, x_{0} \in[-N, N], y, y_{0} \in$ $[-K(-N, N+1), K(-N, N+1)]^{d}$. Now let $d_{n}(x)$ denote the element of $\mathbb{D}_{n}$ such that $d_{n}(x) \leq$ $x<d_{n}(x)+\left(1 / 2^{n}\right)$. Then $\xi_{n}\left(\omega, d_{n}(x)\right)=\xi_{n}(\omega, x)$. Since $\left|\xi_{n}(\cdot, x)\right| \leq \tilde{K}(x) \leq K(-N, N+1)$ for all $x \in[-N, N]$, we have

$$
\begin{aligned}
\left|G\left(\omega, x, \xi_{n}(\omega, x)\right)-A(\omega, x)\right| \leq & \mid G\left(\omega, x, \xi_{n}(\omega, x)\right)-G\left(\omega, d_{n}(x), \xi_{n}\left(\omega, d_{n}(x)\right) \mid+\right. \\
& \mid G\left(\omega, d_{n}(x), \xi_{n}\left(\omega, d_{n}(x)\right)-A\left(\omega, d_{n}(x)\right) \mid+\right. \\
\leq & \left|A\left(\omega, d_{n}(x)\right)-A(\omega, x)\right| \\
\leq & \epsilon / 3+1 / n+\epsilon / 3<\epsilon,
\end{aligned}
$$

if $n$ is chosen so large that both $1 / 2^{n}<\eta(\omega)$ and $1 / n<\epsilon / 3$.
These preparations allow us to prove the existence of an optimal strategy.
Proposition 2.38. Let Assumptions 2.26, 2.28 and 2.29 hold. Then there exists an $\mathcal{H} \otimes \mathcal{B}(\mathbb{R})$ measurable $\tilde{\xi}(\omega, x) \in D(\omega)$ such that a.s., simultaneously for all $x \in \mathbb{R}$,

$$
\begin{equation*}
A(\omega, x)=G(\omega, x, \tilde{\xi}(x)) . \tag{59}
\end{equation*}
$$

Recall the definition of $\tilde{K}(x)$ from (40). We have

$$
\begin{equation*}
|\tilde{\xi}(\omega, x)| \leq \tilde{K}(\omega, x) \text { for all } x \in \mathbb{R} \text { and } \omega \in \Omega . \tag{60}
\end{equation*}
$$

The $\tilde{\xi}$ constructed satisfies

$$
\begin{equation*}
A(\omega, H)=E(V(H+\tilde{\xi}(H) Y) \mid \mathcal{H})=\text { ess. } \sup _{\xi \in \Xi} E(V(H+\xi Y) \mid \mathcal{H}) \text { a.s. }, \tag{61}
\end{equation*}
$$

for each $H \in \Xi^{1}$.
Proof. From Lemma 2.37, there exists a sequence $\xi_{n}(\omega, x) \in D$ such that $G\left(\omega, x, \xi_{n}(\omega, x)\right)$ converges to $A(\omega, x)$ for all $\omega \in \widehat{\Omega}$ for some $\widehat{\Omega}$ of full measure and for all $x \in \mathbb{R}$. Note that $\left|\xi_{n}(x)\right|$ is bounded by $\tilde{K}(x)$ for all $x \in \mathbb{R}$ and $\omega \in \widehat{\Omega}$.

Using Lemma 6.8, we can find a random subsequence $\tilde{\xi}_{k}(\omega, x):=\xi_{n_{k}}(\omega, x)$ of $\xi_{n}(\omega, x)$ converging to some $\tilde{\xi}(\omega, x)$ for all $x$ and $\omega \in \widehat{\Omega}$. On the set $\Omega \backslash \widehat{\Omega}$ we define $\tilde{\xi}(\omega, x):=0$ for all $x$. Note that this ensures $|\tilde{\xi}(\omega, x)| \leq \tilde{K}(x)$ for all $x \in \mathbb{R}$ and $\omega \in \Omega$ and (60) is proved.

Here $\tilde{\xi}_{k}(\omega, x)=\xi_{n_{k}}(\omega, x)=\sum_{l \geq k} \xi_{l}(\omega, x) 1_{\tilde{B}(l, k)}$, with $\tilde{B}(l, k)=\left\{(\omega, x): n_{k}(\omega, x)=l\right\} \in \mathcal{H} \otimes$ $\mathcal{B}(\mathbb{R})$ and $\cup_{l \geq k} \tilde{B}(l, k)=\widehat{\Omega} \times \mathbb{R}$. Fix $x \in \mathbb{R}$ and $\omega \in \widehat{\Omega}$. Define $B(l, k):=\{\omega:(\omega, x) \in \tilde{B}(l, k)\} \in \mathcal{H}$. Then we have that a.s.

$$
\begin{aligned}
G\left(\omega, x, \tilde{\xi}_{k}(x)\right) & =\sum_{l \geq k} 1_{B(l, k)} G\left(\omega, x, \xi_{l}(x)\right) \\
& \geq \sum_{l \geq k} 1_{B(l, k)}\left(A(\omega, x)-E_{l}(\omega, x)\right) \\
& \geq \sum_{l \geq k} 1_{B(l, k)}\left(A(\omega, x)-\sup _{m \geq k} E_{m}(\omega, x)\right)=A(\omega, x)-\sup _{m \geq k} E_{m}(\omega, x) .
\end{aligned}
$$

The first inequality follows from the definition of $\xi_{l}$.
Note that $E_{m}(\omega, x) \rightarrow 0, m \rightarrow \infty$ (see Lemma 2.37) also implies $\sup _{m \geq k} E_{m}(\omega, x) \rightarrow 0$, $k \rightarrow \infty$. By the continuity of $G$, we get $G(\omega, x, \tilde{\xi}(x)) \geq A(\omega, x)$. Thus (59) is proved for each $x$ since $A(x) \geq G(\omega, x, \tilde{\xi}(x))$ is trivial.

Equation (56) implies

$$
\begin{equation*}
A(\omega, H)=E(V(H+\tilde{\xi}(H) Y) \mid \mathcal{H}) \text { a.s. } \tag{62}
\end{equation*}
$$

so it remains to show

$$
\begin{equation*}
E(V(H+\xi Y) \mid \mathcal{H}) \leq A(\omega, H) \text { a.s. } \tag{63}
\end{equation*}
$$

for each fixed $\xi \in \Xi^{d}$ but this is true by (56) and by the definition of $A$.

Remark 2.39. For the proof of Theorem 2.18 it would suffice to construct, for all $H \in \Xi^{1}$, some $\xi_{H} \in \Xi^{d}$ satisfying $E\left(V\left(H+\xi_{H} Y\right) \mid \mathcal{H}\right)=A(H)$, as we did in the case of $u$ bounded above in Lemma 2.8. We have obtained a much sharper result which we shall need in Section 2.7: there is $\tilde{\xi}: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that one can choose $\xi_{H}:=\tilde{\xi}(H)$. This requires the above careful (and rather tedious) construction for $\xi$.

### 2.6 Dynamic programming

So as to perform a dynamic programming procedure, we need to establish that some crucial properties of $u$ are true for $U_{t}$ as well, i.e. they are preserved by dynamic programming. In particular, the "asymptotic elasticity"-type conditions (66) and (67), see below.

Proposition 2.40. Assume that $u$ satisfies Assumption 2.13. Then there is a constant $C \geq 0$ such that for all $x \in \mathbb{R}$ and $\lambda \geq 1$,

$$
\begin{align*}
u(\lambda x) & \leq \lambda^{\alpha} u(x)+C \lambda^{\alpha}  \tag{64}\\
u(\lambda x) & \leq \lambda^{\beta} u(x)+C \lambda^{\beta} . \tag{65}
\end{align*}
$$

Proof. Let $C:=\max (u(\bar{x}),-u(-\underline{x}))+c+|u(0)|$. Obviously, (64) holds true for $x \geq \bar{x}$ by (15). For $0 \leq x \leq \bar{x}$, as $u$ is nondecreasing, we get

$$
\begin{aligned}
u(\lambda x) & \leq u(\lambda \bar{x}) \leq \lambda^{\alpha} u(\bar{x})+c=\lambda^{\alpha} u(x)+c+\lambda^{\alpha}(u(\bar{x})-u(x)) \\
& \leq \lambda^{\alpha} u(x)+c+\lambda^{\alpha}(u(\bar{x})+|u(0)|),
\end{aligned}
$$

from (15) and (64) holds true. Now, for $-\underline{x}<x \leq 0, u(\lambda x) \leq u(0)$ and

$$
u(0) \leq \lambda^{\alpha} u(-\underline{x})+C \lambda^{\alpha} \leq \lambda^{\alpha} u(x)+C \lambda^{\alpha}
$$

so (64) holds true.
If $x \leq-\underline{x}$ then $u(x) \leq 0$, see Assumption 2.13. By (16) and $\alpha<\beta$, one has

$$
u(\lambda x) \leq \lambda^{\beta} u(x) \leq \lambda^{\alpha} u(x) \leq \lambda^{\alpha} u(x)+\lambda^{\alpha} C
$$

We now turn to the proof of (65). For $x \geq \bar{x}$, using (64), $\alpha<\beta$ and $u(x) \geq 0$ :

$$
u(\lambda x) \leq \lambda^{\alpha} u(x)+C \lambda^{\alpha} \leq \lambda^{\beta} u(x)+C \lambda^{\beta} .
$$

For $0 \leq x \leq \bar{x}$,

$$
\begin{array}{r}
u(\lambda x) \leq u(\lambda \bar{x}) \leq \lambda^{\alpha} u(\bar{x})+c= \\
\lambda^{\beta} u(x)-\lambda^{\beta} u(x)+\lambda^{\alpha} u(\bar{x})+c \leq \lambda^{\beta} u(x)+\lambda^{\beta}|u(0)|+\lambda^{\beta}[u(\bar{x})+c] .
\end{array}
$$

For $-\underline{x}<x \leq 0$

$$
u(\lambda x) \leq u(0) \leq \lambda^{\beta} u(-\underline{x})+C \lambda^{\beta} \leq \lambda^{\beta} u(x)+C \lambda^{\beta}
$$

Finally, (65) for $x \leq-\underline{x}$ follows directly from (16).
Now we establish similar estimates for $U_{T}(x)=u(x-B)$.

Proposition 2.41. Let $B$ be bounded below. Then there exist $g, C \geq 0$ such that

$$
\begin{align*}
& U_{T}(\lambda x) \leq \lambda^{\alpha} U_{T}(x+g)+C \lambda^{\alpha},  \tag{66}\\
& U_{T}(\lambda x) \leq \lambda^{\beta} U_{T}(x+g)+C \lambda^{\beta} . \tag{67}
\end{align*}
$$

Proof. Let $g \geq 0$ be such that $B^{-} \leq g$. Then

$$
U_{T}(\lambda x) \leq \lambda^{\alpha} u(x-B / \lambda)+C \lambda^{\alpha} \leq \lambda^{\alpha} u(x+g)+C \lambda^{\alpha} .
$$

The verification of (67) is analogous.
Proposition 2.42. If Assumption 2.16 holds true then for all $1 \leq t \leq T$ and $\xi \in \Xi_{t-1}^{d}$ we have for all $x$,

$$
\begin{align*}
E\left(U_{t}\left(x+\xi \Delta S_{t}\right) \mid \mathcal{F}_{t-1}\right) \leq U_{t-1}(x) & <+\infty \text { a.s. }  \tag{68}\\
E\left(U_{t}^{+}\left(x+\xi \Delta S_{t}\right) \mid \mathcal{F}_{t-1}\right) & <+\infty \text { a.s. } \tag{69}
\end{align*}
$$

Proof. By (21) and Assumption 2.16, $E U_{j}(x)$ exists for all $j$. Choosing the strategy equal to zero at the dates $1, \ldots, t-1$, we get

$$
E\left(U_{0}(x)\right) \geq E\left(E\left(U_{1}(x) \mid \mathcal{F}_{0}\right)\right)=E\left(U_{1}(x)\right) \geq \ldots \geq E\left(E\left(U_{t-1}(x) \mid \mathcal{F}_{t-2}\right)\right)=E\left(U_{t-1}(x)\right)
$$

As $E\left(U_{0}(x)\right)<\infty$, we obtain that $E\left(U_{t-1}(x)\right)<\infty$. Statements (68) and (69) are trivial from this.

Proposition 2.43. Assume that $S$ satisfies the (NA) condition and that Assumptions 2.13 and 2.16 hold true. One can choose versions of the random functions $U_{t}, 0 \leq t \leq T$, which are almost surely nondecreasing, continuous, finite and satisfy, outside a fixed negligible set,

$$
\begin{align*}
& U_{t}(\lambda x) \leq \lambda^{\alpha} U_{t}(x+g)+C \lambda^{\alpha},  \tag{70}\\
& U_{t}(\lambda x) \leq \lambda^{\beta} U_{t}(x+g)+C \lambda^{\beta}, \tag{71}
\end{align*}
$$

for all $\lambda \geq 1$ and $x \in \mathbb{R}$. Moreover, there exist positive $N_{t-1} \in \Xi_{t-1}^{1}$ such that:

$$
\begin{equation*}
P\left(\left.U_{t}\left(-N_{t-1}\right) \leq-\frac{2 C}{\kappa_{t-1}}-1 \right\rvert\, \mathcal{F}_{t-1}\right) \geq 1-\kappa_{t-1} / 2 \tag{72}
\end{equation*}
$$

here C is the same constant as in (70) and (71) above and $\kappa_{t-1}$ is as in (5). Finally, there exist $\mathcal{F}_{t} \otimes \mathcal{B}(\mathbb{R})$-measurable functions $\tilde{\xi}_{t+1}$, taking values in $D_{t+1}, 0 \leq t \leq T-1$ such that, for all $H \in \Xi_{t}^{1}$, almost surely,

$$
\begin{equation*}
U_{t}(H)=E\left(U_{t+1}\left(H+\tilde{\xi}_{t+1}(H) \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right) . \tag{73}
\end{equation*}
$$

Proof. Using backward induction from $T$ to 0 , we will apply Lemmata 2.31 and 2.35 and Proposition 2.38 with the choice $V:=U_{t}, \mathcal{H}=\mathcal{F}_{t-1}, D:=D_{t}, Y:=\Delta S_{t}, v=U_{t-1}$. Then for each $x \in \mathbb{R}$, we will choose the random function $U_{t-1}(x)$ to be $A(x)$ which provides an almost surely nondecreasing and continuous version of $U_{t-1}(x)$ (see Lemma 2.35 and Remark 2.36). We need to verify that Assumptions 2.28, 2.26 and 2.29 hold true.

We start by the ones which can be verified directly for all $t$. The price process $S$ satisfies the (NA) condition. So by Proposition 1.6, Assumption 2.28 holds true with $\nu=\nu_{t-1}$ and $\kappa=\kappa_{t-1}$. Now, by Propositions 2.15 and 2.42, (68) and (21) are valid thus (31), (32) hold true for $V=U_{t}, Y=\Delta S_{t}$ and $\mathcal{H}=\mathcal{F}_{t-1}$.

We will prove the cases $t=T$ and $t=T-1$ only since the latter is identical to the induction step. The function $U_{T}$ is continuous and non-decreasing by Assumption 2.13. Inequalities (34) and (35) for $V=U_{T}$ follow from Proposition 2.41.

Let $g$ be an upper bound for $B^{-}$. Inequality (36) (and hence also (72) for $t=T$ ) is satisfied because for any $x \geq \underline{x}+g$,

$$
U_{T}(-x) \leq u(-x+g) \leq\left(\frac{-x+g}{-\underline{x}}\right)^{\beta} u(-\underline{x})
$$

from (16), $u(-\underline{x})<0$ by (17), so we may choose

$$
N_{T-1}:=\max \left\{\underline{x}\left(\frac{-\left(2 C / \kappa_{T-1}\right)-1}{u(-\underline{x})}\right)^{\frac{1}{\beta}}+g, \underline{x}+g\right\} .
$$

Now we are able to use Proposition 2.38 and there exists a function $\tilde{\xi}_{T}$ with values in $D_{T}$ such that (73) holds for $t=T-1$. Moreover, by Lemma 2.35, we can chose for $U_{T-1}(\omega, \cdot)$ an almost surely nondecreasing and continuous version.

We now prove that Assumption 2.29 holds for $V=U_{T-1}$. For some fixed $x \in \mathbb{R}$ and $\lambda \geq 1$, almost surely

$$
\begin{aligned}
U_{T-1}(\lambda x) & =E\left(U_{T}\left(\lambda x+\tilde{\xi}_{T}(\lambda x) \Delta S_{T}\right) \mid \mathcal{F}_{T-1}\right) \\
& \leq \lambda^{\alpha}\left(E\left(U_{T}\left(x+\left(\tilde{\xi}_{T}(\lambda x) / \lambda\right) \Delta S_{T}+g\right) \mid \mathcal{F}_{T-1}\right)+C\right) \leq \lambda^{\alpha}\left(U_{T-1}(x+g)+C\right)
\end{aligned}
$$

where the first inequality follows from (70) for $t=T$. Clearly, there is a common zeroprobability set outside which this holds for all rational $x, \lambda$. Using continuity of $U_{T-1}$ this extends to all $\lambda, x$. Thus (70) holds for $t=T-1$. By the same argument, (71) also holds for $t=T-1$.

It remains to show that (72) holds for $t=T-1$ and then Assumption 2.29 will be proved for $V=U_{T-1}$. Choose $I_{T-1}=2 C / \kappa_{T-1}+1$ which is a.s. finite-valued and invoke Lemma 2.31 (with $V=U_{T}$ ) to get some non-negative, finite valued and $\mathcal{F}_{T-1}$-measurable random variable $N^{\prime}$ such that $U_{T-1}\left(-N^{\prime}\right) \leq-I_{T-1}$ a.s. Let us define the $\mathcal{F}_{T-2}$-measurable events

$$
A_{m}:=\left\{\omega: P\left(N^{\prime} \leq m \mid \mathcal{F}_{T-2}\right)(\omega) \geq 1-\kappa_{t-2}(\omega) / 2\right\}, m \in \mathbb{N} .
$$

As $P\left(N^{\prime} \leq m \mid \mathcal{F}_{T-2}\right)$ trivially tends to 1 when $m \rightarrow \infty$, the union of the sets $A_{m}$ cover a full measure set hence, after defining recursively the partition

$$
B_{1}:=A_{1}, \quad B_{m+1}:=A_{m+1} \backslash\left(\cup_{j=1}^{m} A_{j}\right)
$$

we can construct the non-negative, $\mathcal{F}_{T-2}$-measurable random variable

$$
N_{T-2}:=\sum_{m=1}^{\infty} m 1_{B_{m}}
$$

such that $P\left(N^{\prime} \leq N_{T-2} \mid \mathcal{F}_{T-2}\right) \geq 1-\kappa_{t-2} / 2$ a.s. Then a.s.

$$
P\left(U_{T-1}\left(-N_{T-2}\right)<-I_{T-1} \mid \mathcal{F}_{T-2}\right) \geq P\left(N^{\prime} \leq N_{T-2} \mid \mathcal{F}_{T-2}\right) \geq 1-\kappa_{T-2} / 2
$$

Applying Proposition 2.38 to $U_{T-1}$, (73) follows for $t=T-2$. We can continue the procedure of dynamic programming in an analogous way and get the statements of this proposition.
Proof of Theorem 2.18. We use the results of Proposition 2.43. Set $\phi_{1}^{*}:=\tilde{\xi}_{1}(z)$ and define inductively:

$$
\phi_{t}^{*}:=\tilde{\xi}_{t}\left(z+\sum_{j=1}^{t-1} \phi_{j}^{*} \Delta S_{j}\right), 1 \leq t \leq T
$$

Joint measurability of $\tilde{\xi}_{t}$ ensures that $\phi^{*}$ is a predictable process with respect to the given filtration. Proposition 2.43 implies that, for $t=1, \ldots, T$ a.s.,

$$
\begin{equation*}
E\left(U_{t}\left(X_{t}^{z, \phi^{*}}\right) \mid \mathcal{F}_{t-1}\right)=U_{t-1}\left(X_{t-1}^{z, \phi^{*}}\right) \tag{74}
\end{equation*}
$$

We will now show that if $E u\left(X_{T}^{z, \phi^{*}}-B\right)$ exists then $\phi^{*} \in \Phi(z)$ and for any strategy $\phi \in \Phi(z)$,

$$
\begin{equation*}
E\left(u\left(X_{T}^{z, \phi}-B\right)\right) \leq E\left(u\left(X_{T}^{z, \phi^{*}}-B\right)\right) \tag{75}
\end{equation*}
$$

This will complete the proof.

Let us consider first the case where $E u^{+}\left(X_{T}^{z, \phi^{*}}-B\right)<\infty$. Then by (74) and the (conditional) Jensen inequality (see Corollary 6.5),

$$
U_{T-1}^{+}\left(X_{T-1}^{z, \phi^{*}}\right) \leq E\left(U_{T}^{+}\left(X_{T}^{z, \phi^{*}}\right) \mid \mathcal{F}_{T-1}\right) \text { a.s. }
$$

Thus $E\left[U_{T-1}^{+}\left(X_{T-1}^{z, \phi^{*}}\right)\right]<\infty$ and, repeating the argument, $E\left[U_{t}^{+}\left(X_{t}^{z, \phi^{*}}\right)\right]<\infty$ for all $t$.
Now let us turn to the case where $E u^{-}\left(X_{T}^{z, \phi^{*}}-B\right)<\infty$. The same argument as above with negative parts instead of positive parts shows $E\left[U_{t}^{-}\left(X_{t}^{z, \phi^{*}}\right)\right]<\infty$ for all $t$.

It follows that, for all $t, E U_{t}\left(X_{t}^{z, \phi^{*}}\right)$ exists and so does $E\left(U_{t}\left(X_{t}^{z, \phi^{*}}\right) \mid \mathcal{F}_{t-1}\right)$, satisfying

$$
E\left(E\left(U_{t}\left(X_{t}^{z, \phi^{*}}\right) \mid \mathcal{F}_{t-1}\right)\right)=E U_{t}\left(X_{t}^{z, \phi^{*}}\right),
$$

see Lemma 6.2. Hence

$$
\begin{equation*}
E\left(U_{T}\left(X_{T}^{z, \phi^{*}}\right)\right)=E\left(E\left(U_{T}\left(X_{T}^{z, \phi^{*}}\right) \mid \mathcal{F}_{T-1}\right)\right)=E\left(U_{T-1}\left(X_{T-1}^{z, \phi^{*}}\right)\right)=\ldots=E\left(U_{0}(z)\right) . \tag{76}
\end{equation*}
$$

By (21) and (24), $-\infty<E u(z-B) \leq E U_{0}(z)$, hence $\phi^{*} \in \Phi(z)$ follows.
Let $\phi \in \Phi(z)$, then $E\left(U_{T}\left(X_{T}^{z, \phi}\right)\right)$ exists and it is finite by definition of $\Phi(z)$ so for all $t$, $E\left(U_{T}\left(X_{T}^{z, \phi}\right) \mid \mathcal{F}_{t}\right)$ exists and $E\left(E\left(U_{T}\left(X_{T}^{z, \phi}\right) \mid \mathcal{F}_{t}\right)\right)=E\left(U_{T}\left(X_{T}^{z, \phi}\right)\right)$.

We prove by induction that $E\left(U_{T}\left(X_{T}^{z, \phi}\right) \mid \mathcal{F}_{t}\right) \leq U_{t}\left(X_{t}^{z, \phi}\right)$ a.s. For $t=T$, this is trivial. Assume that it holds true for $t+1$.

As we saw during the proof of Proposition 2.43, $E\left(U_{t+1}\left(X_{t}^{z, \phi}+\phi_{t+1} \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)$ exists and it is finite. So, by the induction hypothesis, Proposition 2.38 and (73),

$$
\begin{aligned}
E\left(U_{T}\left(X_{T}^{z, \phi}\right) \mid \mathcal{F}_{t}\right) & \leq E\left(U_{t+1}\left(X_{t}^{z, \phi}+\phi_{t+1} \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right) \leq \\
E\left(U_{t+1}\left(X_{t}^{z, \phi}+\tilde{\xi}_{t+1}\left(X_{t}^{z, \phi}\right) \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right) & =U_{t}\left(X_{t}^{z, \phi}\right) .
\end{aligned}
$$

Applying this result at $t=0$, we obtain that $E\left(U_{T}\left(X_{T}^{z, \phi}\right) \mid \mathcal{F}_{0}\right) \leq U_{0}(z)$ hence also

$$
\begin{equation*}
E\left(u\left(X_{T}^{z, \phi}-B\right)\right) \leq E\left(U_{0}(z)\right) . \tag{77}
\end{equation*}
$$

Putting (76) and (77) together, we get the optimality of $\phi^{*}$.
To see continuity of $\bar{u}$, let $x_{k} \rightarrow x, k \rightarrow \infty$ with $y_{1} \leq \inf _{k} x_{k} \leq \sup _{k} x_{k} \leq y_{2}$. By continuity of $U_{0}$, we have $U_{0}\left(x_{k}\right) \rightarrow U_{0}(x)$ a.s. Lebesgue's theorem now shows $\bar{u}\left(x_{k}\right) \rightarrow \bar{u}(x)$, noting that $\bar{u}(z)=E U_{0}(z), E\left(u\left(y_{1}-B\right) \mid \mathcal{F}_{0}\right) \leq U_{0}\left(x_{k}\right) \leq U_{0}\left(y_{2}\right)$ and $E u\left(y_{1}-B\right)>-\infty, E U_{0}\left(y_{2}\right)<\infty$ by our hypotheses.

Proof of Corollary 2.20. We follow the proof of Proposition 2.43 but with certain refinements. Claim : We have $N_{t-1} \in \mathcal{W}$, where $N_{t-1}$ equals $N$ in (36) for the choice $V=U_{t}, \mathcal{H}:=\mathcal{F}_{t-1}$ and there exist non-negative, adapted random variables $C_{t}, J_{t-1}, M_{t-1}, R_{t}$ belonging to $\mathcal{W}$ (i.e. $C_{t}, R_{t}$ are $\mathcal{F}_{t}$-measurable and $J_{t-1}$ and $M_{t-1}$ are $\mathcal{F}_{t-1}$-measurable) and numbers $\theta, \tilde{m}>0$ such that, for a.e. $\omega$,

$$
\begin{align*}
U_{t}(x) & \geq-\tilde{m}\left(|x|^{p}+R_{t}+1\right), \text { for all } x  \tag{78}\\
U_{t}^{+}(x) & \leq C_{t}\left(\mid x x^{\alpha}+1\right), \text { for all } x,  \tag{79}\\
\tilde{K}_{t-1}(x) & \leq M_{t-1}\left(|x|^{\theta}+1\right) \text { for all } x . \tag{80}
\end{align*}
$$

In addition, for all $x, y \in \mathbb{R}$,

$$
\begin{equation*}
E\left(U_{t}^{+}\left(x+|y|\left|\Delta S_{t}\right|\right) \mid \mathcal{F}_{t-1}\right) \leq J_{t-1}\left(|x|^{\alpha}+|y|^{\alpha}+1\right)<\infty \text {, a.s. } \tag{81}
\end{equation*}
$$

where the $\mathcal{F}_{t-1}$-measurable random variable $\tilde{K}_{t-1}(x)$ is just $\tilde{K}(x)$ defined in (40) for the choice $V=U_{t}, Y=\Delta S_{t}$ and $\mathcal{H}:=\mathcal{F}_{t-1}$.

Inequality (78) is trivial from (21) for all $t$ with $R_{t}:=E\left(|B|^{p} \mid \mathcal{F}_{t}\right)$ (we may and will assume $p \geq 1$ ) thus (31) follows from (28) and (21).

We proceed by backward induction starting at $t=T$. We will only do steps $t=T$ and $t=T-1$ since the induction step is identical to the latter. Choosing

$$
N_{T-1}:=\max \left\{\underline{x}\left(\frac{-\left(2 C / \kappa_{T-1}\right)-1}{u(-\underline{x})}\right)^{\frac{1}{\beta}}+g, \underline{x}+g\right\},
$$

just like in the proof of Proposition 2.43, we can see that $N_{T-1} \in \mathcal{W}$.
We estimate, using Assumption 2.13,

$$
\begin{equation*}
U_{T}(x) \leq u(x+g) \leq \frac{|x+g|^{\alpha}}{\bar{x}^{\alpha}} u(\bar{x})+c+u(\bar{x}+g) \leq C_{T}\left(|x|^{\alpha}+1\right) \tag{82}
\end{equation*}
$$

for all $x$, with some deterministic constant $C_{T}$. It is clear that (82) also holds true for $U_{T}^{+}$and thus (79) holds true. We obtain

$$
\begin{aligned}
E\left(U_{T}^{+}\left(x+|y|\left|\Delta S_{T}\right|\right) \mid \mathcal{F}_{T-1}\right) & \leq E\left(C_{T} \mid \mathcal{F}_{T-1}\right)\left(2^{\alpha}|x|^{\alpha}+1\right)+2^{\alpha}|y|^{\alpha} E\left(C_{T}\left|\Delta S_{T}\right|^{\alpha} \mid \mathcal{F}_{T-1}\right) \\
& \leq J_{T-1}\left(|x|^{\alpha}+|y|^{\alpha}+1\right)<\infty
\end{aligned}
$$

with $J_{T-1}:=2^{\alpha} E\left(C_{T}+C_{T}\left|\Delta S_{T}\right|^{\alpha} \mid \mathcal{F}_{T-1}\right)$.
It is clear that $J_{T-1}$ belongs to $\mathcal{W}$ (recall $\Delta S_{T} \in \mathcal{W}$ ) and that $J_{T-1}$ is $\mathcal{F}_{T-1}$-measurable. Thus (81) holds true (for $V=U_{T}$ ). To finish with the step $t=T$, it remains to prove (80). As (28) holds true, we can use (42) in Lemma 2.31 and we just have to prove that $M_{T-1} \in \mathcal{W}$ where $M_{T-1}$ equals the $M$ of Lemma 2.31 when $V=U_{T}$. From Lemma $2.31, M_{T-1}$ is a polynomial function of $1 / \nu_{T-1}, 1 / \kappa_{T-1}, N_{T-1}$ and $L_{T}$, where $L_{t}$ denotes $L$ from Lemma 2.31 corresponding to $V=U_{t}$. As $L_{T}=E\left(U_{T}^{+}\left(1+\left|\Delta S_{T}\right|\right) \mid \mathcal{F}_{T-1}\right) \leq 3 J_{T-1}$ we get that $L_{T} \in \mathcal{W}$ and $M_{T-1} \in \mathcal{W}$ indeed.

Let us now turn to step $t=T-1$. Recall that $U_{T-1}$ satisfies (70) by the argument of Proposition 2.43. Hence we get

$$
\begin{align*}
U_{T-1}(x) \leq U_{T-1}(|x|) & \leq|x|^{\alpha}\left[U_{T-1}(1+g)+C\right] \leq  \tag{83}\\
|x|^{\alpha}\left[E\left(U_{T}\left(1+g+\tilde{K}_{T-1}(1+g)\left|\Delta S_{T}\right|\right) \mid \mathcal{F}_{T-1}\right)+C\right] & \leq C_{T-1}\left(|x|^{\alpha}+1\right)
\end{align*}
$$

a.s. for each $x$ with some positive $\mathcal{F}_{T-1}$-measurable $C_{T-1}$, recalling (79) and (80). Thus also $U_{T-1}^{+}(x) \leq C_{T-1}\left(|x|^{\alpha}+1\right)$ a.s. As both $U_{T-1}^{+}$and $x \rightarrow C_{T-1}\left(|x|^{\alpha}+1\right)$ are continuous, $U_{T-1}^{+}(x) \leq C_{T-1}\left(|x|^{\alpha}+1\right)$ holds for all $x$ simultaneously, outside a fixed negligible set and (79) is satisfied. As $M_{T-1}$ and $C_{T}$ belong to $\mathcal{W}, C_{T-1}$ also belongs to $\mathcal{W}$. Furthermore, for all $x, y$, a.s.

$$
\begin{aligned}
E\left(U_{T-1}^{+}\left(x+|y|\left|\Delta S_{T-1}\right|\right) \mid \mathcal{F}_{T-2}\right) & \leq E\left(C_{T-1} \mid \mathcal{F}_{T-2}\right)\left(2^{\alpha}|x|^{\alpha}+1\right)+2^{\alpha}|y|^{\alpha} E\left(C_{T-1}\left|\Delta S_{T-1}\right|^{\alpha} \mid \mathcal{F}_{T-2}\right) \\
& \leq J_{T-2}\left(|x|^{\alpha}+|y|^{\alpha}+1\right)<\infty
\end{aligned}
$$

with $J_{T-2}:=2^{\alpha} E\left(C_{T-1}+C_{T-1}\left|\Delta S_{T-1}\right|^{\alpha} \mid \mathcal{F}_{T-2}\right)$.
As $J_{T-2}$ clearly belongs to $\mathcal{W}$ and $J_{T-2}$ is $\mathcal{F}_{T-2}$-measurable, (81) is proved.
We now establish the existence of $N_{T-2} \in \mathcal{W}$ such that (36) holds true with $N=N_{T-2}$ and $V=U_{T-1}$. Let us take the random variable $N^{\prime}$ constructed in the proof of Lemma 2.31 for $V=U_{T}$ which is such that $U_{T-1}\left(-N^{\prime}\right) \leq-I_{T-1}$, where $I_{T-1}:=\left(2 C / \kappa_{T-1}\right)+1$. By (55), $N^{\prime}$ is a polynomial function of $1 / \kappa_{T-1}, N_{T-1}$ (which belong to $\mathcal{W}$ ) and $E\left(U_{T}^{+}\left(\bar{K}_{T-1}\left|\Delta S_{T}\right|\right) \mid \mathcal{F}_{T-1}\right)$, where $\bar{K}_{T-1}$ is defined as $\bar{K}$ (see (44)) when $V=U_{T}$. As $\bar{K}_{T-1}$ is a polynomial function of $N_{T-1}, 1 / \nu_{T-1}, 1 / \kappa_{T-1}$ and $L_{T}$, we have $\bar{K}_{T-1} \in \mathcal{W}$ (recall from the end of step $t=T$ that $L_{T} \in \mathcal{W}$ ). As $E\left(U_{T}^{+}\left(\bar{K}_{T-1}\left|\Delta S_{T}\right|\right) \mid \mathcal{F}_{T-1}\right)$ is bounded by $J_{T-1}\left(0+\bar{K}_{T-1}^{\alpha}+1\right)$ by (81) for $t=T$, we conclude that $N^{\prime}$ belongs to $\mathcal{M}$. Let us now set

$$
N_{T-2}:=\frac{2 E\left(N^{\prime} \mid \mathcal{F}_{T-2}\right)}{\kappa_{T-2}} \in \mathcal{W}
$$

The (conditional) Markov inequality implies that a.s.

$$
P\left(N^{\prime}>N_{T-2} \mid \mathcal{F}_{T-2}\right) \leq \frac{E\left(N^{\prime} \mid \mathcal{F}_{T-2}\right)}{N_{T-2}}=\frac{\kappa_{T-2}}{2}
$$

As in the proof of Proposition 2.43, a.s.

$$
P\left(U_{T-1}\left(-N_{T-2}\right) \leq-I_{T-1} \mid \mathcal{F}_{T-2}\right) \geq P\left(N^{\prime} \leq N_{T-2} \mid \mathcal{F}_{T-2}\right) \geq 1-\kappa_{T-2} / 2
$$

showing (36) for $V=U_{T-1}$.
We now turn to (80). By (78), one can apply (42) in Lemma 2.31 (with $V=U_{T-1}$ ) and (80) is satisfied with some $M_{T-2}$ which is a polynomial function of $1 / \nu_{T-2}, 1 / \kappa_{T-2}, N_{T-2}$ and $L_{T-1}$. So we just have to prove that $M_{T-2} \in \mathcal{W}$. As

$$
L_{T-1}=E\left(U_{T-1}^{+}\left(1+\left|\Delta S_{T-1}\right|\right) \mid \mathcal{F}_{T-2}\right) \leq 3 J_{T-2}
$$

we get that $L_{T-1} \in \mathcal{W}$ and $M_{T-2} \in \mathcal{W}$ as well. This concludes the step $t=T-1$. Continuing this inductive procedure in an analogous way, the claim is proved.

Now, since by (79),

$$
E U_{0}(x) \leq E U_{0}^{+}(x) \leq\left(|x|^{\alpha}+1\right) E C_{0}<\infty
$$

(26) holds true and thus Assumption 2.16 is satisfied.

Defining $\phi^{*}$ as in the proof of Theorem 2.18, a straightforward induction shows that $\phi_{t}^{*} \in$ $\mathcal{W}$ and $X_{t}^{z, \phi^{*}} \in \mathcal{W}$ for all $t$ (and thus $\phi^{*} \in \Phi(z)$ ).

We get the optimality of $\phi^{*}$ as in Proposition 2.43 noting that $E u\left(X_{T}^{z, \phi^{*}}-B\right)$ is finite. This completes the proof.
Proof of Corollary 2.22. In this case we note that

$$
u(x-B) \geq-u^{-}(x-g), \text { for all } x \in \mathbb{R}
$$

holds instead of (28) where $g$ is a bound for $|B|$ and $u^{-}(\cdot-g)$ is a continuous, hence also locally bounded non-negative function. Thus in Lemma 2.31, assuming that $V(x) \geq-u^{-}(x-g)$ we obtain that $\tilde{K}(x)$ (see (40)) can be chosen a non-random locally bounded function of $x$ and $u^{-}(\lfloor x\rfloor-g)$. Similarly, $\bar{K}$ (see (44)) is a (non-random) constant. So one can imitate the proof of Corollary 2.20 with $C_{t}, J_{t-1}, M_{t-1}, N_{t-1}$ non-random. We get that the $\tilde{\xi}_{t}(\cdot)$ are also locally bounded. Hence, for $z$ fixed, $X_{t}^{z, \phi^{*}}$ and $\phi_{t}^{*}$ will also be bounded for all $t$ by a trivial induction argument and we can conclude.

Remark 2.44. One may try to prove a result similar to Theorem 2.18 in continuous-time models. In the light of Proposition 4.8 below (see also Theorem 3.2 of [52]), however, serious limitations are encountered soon. If we look at the particular case of no distortion there (i.e. $\delta=1$, which corresponds to the setting of this chapter), Proposition 4.8 implies that taking $U(x)=x^{\alpha}, x>0$ and $U(x)=-(-x)^{\beta}, x \leq 0$ with $0<\alpha, \beta \leq 1$ the utility maximisation problem becomes ill-posed even in the simplest Black and Scholes model.

This shows that discrete-time models behave differently from their continuous-time counterparts as far as well-posedness is concerned. In discrete-time models the terminal values of admissible portfolios form a relatively small family of random variables hence ill-posedness does not occur even in cases where it does in the continuous-time setting, where the set of attainable payoffs is much richer.

Consequently, there is a fairly limited scope for the extension of our results in the present chapter to continuous-time market models unless the set of strategies is severely restricted (as in [11], [18] and [25]). This underlines the versatility and power of discrete-time modeling. The advantageous properties present in the discrete-time setting do not always carry over to the continuous-time case which is only an idealization of the real trading mechanism. Consult Chapter 4 for more on continuous-time models.

### 2.7 On equivalent martingale measures

The present section is concerned with the construction of $Q \in \mathcal{M}$ with desirable integrability properties for both $d Q / d P$ and $d P / d Q$, under additional assumptions on $S$. Our tool is the following proposition, a slight extension of Proposition 7.1 in [78].

Proposition 2.45. Let $u$ be concave, continuously differentiable and strictly increasing. Let Assumptions 2.13 and 2.19 be in force (see Remark 2.23) and assume

$$
\begin{equation*}
u^{\prime}(x) \leq K\left(|x|^{k}+1\right), x \in \mathbb{R} \tag{84}
\end{equation*}
$$

for some $k, K \geq 0$. Assume $B=0$.
Then problem (3) is well-posed; for every initial endowment $z$ there exists strategy $\phi^{*}(z)$ such that $\phi_{t}^{*} \in \mathcal{W}, t=1, \ldots, T$,

$$
\bar{u}(z)=E u\left(X_{T}^{z, \phi^{*}}\right)
$$

and

$$
\begin{equation*}
\frac{d Q}{d P}:=\frac{u^{\prime}\left(X_{T}^{z, \phi^{*}}\right)}{E u^{\prime}\left(X_{T}^{z, \phi^{*}}\right)}, \tag{85}
\end{equation*}
$$

defines an element of $\mathcal{M}$.
Proof. We first remark that (84) clearly implies

$$
\begin{equation*}
|u(x)| \leq K^{\prime}\left(|x|^{k+1}+1\right), x \leq 0 \tag{86}
\end{equation*}
$$

for some $K^{\prime}>0$.
The assumptions of Proposition 2.43 hold by (86), hence we follow the proof of that result. However, we also verify, by backward induction that, for all $t=0, \ldots, T$ :

$$
\begin{align*}
& U_{t} \text { is continuously differentiable and concave, }  \tag{87}\\
& \quad U_{t}^{\prime}(H)=E\left(U_{t+1}^{\prime}\left(H+\tilde{\xi}_{t+1}(H) \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right)  \tag{88}\\
& E\left(U_{t+1}^{\prime}\left(H+\tilde{\xi}_{t+1}(H) \Delta S_{t+1}\right) \Delta S_{t+1} \mid \mathcal{F}_{t}\right)=0 \tag{89}
\end{align*}
$$

for all $H \in \Xi_{t}^{1}$.
Suppose that the above statements are true for $t+1$ and proceed by the induction step (the case $t=T$ follows similarly). We apply Proposition 6.13 with the choice $V:=U_{t+1}$ and $\mathcal{H}:=\mathcal{F}_{t}$. We only need to check that

$$
\sup _{(x, y) \in[-N, N]^{d+1}} U_{t+1}^{\prime}\left(x+y \Delta S_{t+1}\right)\left|\Delta S_{t+1}\right| .
$$

is integrable. For any $W \in \Xi_{t+1}^{1}, U_{t+1}^{\prime}(W)=E\left(U_{t+2}^{\prime}\left(W+\tilde{\xi}_{t+2}(W) \Delta S_{t+2}\right) \mid \mathcal{F}_{t+1}\right)$ for some (random) function $\tilde{\xi}_{t+2}(x)$ which is polynomial in $x$, involving random constants in $\mathcal{W}$. Iterating this observation and (88), we get that $U_{t+1}^{\prime}(W)=E\left(u^{\prime}(W+p(W)) \mid \mathcal{F}_{t+1}\right)$ where $p(x)$ is a polynomial function of $x$ and of elements in $\mathcal{W}$ which proves the required integrability, noting (84). It remains to notice that, as $\phi_{t}^{*}, \Delta S_{t} \in \mathcal{W}$ for all $t$ (see Proposition 2.43), we also have $u^{\prime}\left(X_{T}^{z, \phi^{*}}\right) \in \mathcal{W}$ by (84), in particular, $d Q / d P$ defines an equivalent probability. Now $E_{Q}\left(\Delta S_{t} \mid \mathcal{F}_{t-1}\right)$ is just a $\mathcal{F}_{t-1}$-measurable function times $E\left(u^{\prime}\left(X_{T}^{z, \phi^{*}}\right) \Delta S_{t} \mid \mathcal{F}_{t-1}\right)$, and the latter is equal (by repeated applications of (88)) to

$$
\begin{aligned}
E\left(U_{T}^{\prime}\left(X_{T}^{z, \phi^{*}}\right) \Delta S_{t} \mid \mathcal{F}_{t-1}\right) & =\ldots=E\left(U_{t}^{\prime}\left(X_{t}^{z, \phi^{*}}\right) \Delta S_{t} \mid \mathcal{F}_{t-1}\right)= \\
E\left(U_{t}^{\prime}\left(X_{t-1}^{z, \phi^{*}}+\tilde{\xi}\left(X_{t-1}^{z, \phi^{*}}\right) \Delta S_{t}\right) \Delta S_{t} \mid \mathcal{F}_{t-1}\right) & =0
\end{aligned}
$$

using (89). This attests $Q \in \mathcal{M}$.
Analogous arguments show the following.
Proposition 2.46. Let $u$ be concave, continuously differentiable and strictly increasing. Let Assumption 2.13 be in force and let $B=0$. Furthermore, assume that for all $0 \leq t \leq T, \Delta S_{t}$ is bounded and (NA) holds such that $\nu_{t}, \kappa_{t}$ of Proposition 1.6 are deterministic for $0 \leq t \leq T-1$.

Then problem (3) is well-posed; for every initial endowment $z$ there exists a bounded strategy $\phi^{*}(z)$ such that

$$
\bar{u}(z)=E u\left(X_{T}^{z, \phi^{*}}\right)
$$

and

$$
\begin{equation*}
\frac{d Q}{d P}:=\frac{u^{\prime}\left(X_{T}^{z, \phi^{*}}\right)}{E u^{\prime}\left(X_{T}^{z, \phi^{*}}\right)}, \tag{90}
\end{equation*}
$$

defines an element of $\mathcal{M}$.
Corollary 2.47. Under the conditions of Proposition 2.45 there exists $Q_{1} \in \mathcal{M}$ with $d Q_{1} / d P \in$ $L^{\infty}$ and $d P / d Q_{1} \in \mathcal{W}$. There exists also $Q_{2} \in \mathcal{M}$ with $d P / d Q_{2} \in L^{\infty}$ and $d Q_{2} / d P \in \mathcal{W}$. Under the conditions of Proposition 2.46 there exists $Q_{3} \in \mathcal{M}$ with $d Q_{3} / d P, d P / d Q_{3} \in L^{\infty}$.

Proof. First define $u_{1}(x)=2 \sqrt{x+1}-2$ for $x \geq 0$ and $u_{1}(x)=x$ for $x<0$. This clearly satisfies the conditions of Proposition 2.45, hence we get $Q_{1}$ from (85) with $d Q_{1} / d P$ bounded above since $u_{1}^{\prime}$ is bounded above by 1 . Since $X_{T}^{z, \phi^{*}} \in \mathcal{W}$ and $1 / u_{1}^{\prime}(x)$ is bounded by constant times $\sqrt{x+1}$ for $x \geq 0$, we get $d P / d Q_{1} \in \mathcal{W}$. Similarly, defining $u_{2}(x):=x$ for $x \geq 0$ and $u_{2}(x):=1 / 2-(1 / 2)(x-1)^{2}$ for $x<0$ we get $Q_{2}$ with $d Q_{2} / d P$ bounded below. As $u_{2}^{\prime}(x)$ is bounded by constant times $|x|+1$ for $x<0$, we also get $d Q_{2} / d P \in \mathcal{W}$. To construct $Q_{3}$, we invoke Proposition 2.46 with $u$ equal to either $u_{1}$ or $u_{2}$, and remember that $X_{T}^{z, \phi^{*}}$ is bounded in this case.

The idea of constructing $Q \in \mathcal{M}$ via (85) was first proposed in [34] and it has become a standard technique in mathematical finance since. Nonetheless Corollary 2.47 above is the only result we know of that provides integrability conditions for $d Q / d P, d P / d Q$ under suitable hypotheses which can be directly checked on $S$. The only reference with similar results seems to be [85] where $Q \in \mathcal{M}$ with various constraints on $d P / d Q$ is constructed under restrictive hypotheses (in particular, $T=1$ or $S$ is bounded). It seems that the standard functional analytic approach of e.g. [86] leads to "abstract" conditions which cannot be easily checked on $S$, one needs the finer analysis of the present chapter. We will utilize Corollary 2.47 later, in the proof of Theorem 4.16.

Remark 2.48. Formula (85) suggests that an optimal strategy $\phi^{*}$ could perhaps be characterized by the property that, for some $Q \in \mathcal{M}$, we have

$$
\begin{equation*}
\left(u^{\prime}\right)^{-1}\left(y^{*}(d Q / d P)\right)=X_{T}^{z, \phi^{*}} \tag{91}
\end{equation*}
$$

with some constant $y^{*}>0$. This can indeed be made precise under appropriate assumptions, providing various versions of an "abstract verification theorem" in this context: (under further hypotheses) (91) implies the optimality of a given $\phi^{*}$, see [91, 46]. We shall see a similar result later (Theorem 5.18), in the context of illiquid markets.

### 2.8 Further applications

The above techniques can also be applied to studying continuity properties of strategies with respect to preferences. The following result is Theorem 2.1 from [21].

Theorem 2.49. Let $u_{n}, n \in \mathbb{N}$, $u$ be continuously differentiable, strictly concave, increasing functions for which (15) holds (with the same $\alpha, \bar{x}, c$ ) such that $u_{n}(x) \rightarrow u(x)$ for all $x \in \mathbb{R}$. Assume that $D_{t}:=\mathbb{R}^{d}$ a.s. for all $t ; \mathcal{F}_{0}$ is trivial; $B$ is bounded; $S$ is a bounded process and (NA) holds with $\nu_{t}, \kappa_{t}$ of Proposition 1.6 being constant. Then, for all $z$, there exist unique optimal strategies $\phi_{n}^{*}(z), \phi^{*}(z)$ satisfying

$$
\begin{aligned}
\bar{u}_{n}(z) & :=\sup _{\phi \in \Phi(z)} E u_{n}\left(X_{T}^{z, \phi}-B\right)=E u_{n}\left(X_{T}^{z, \phi_{n}^{*}(z)}-B\right)<\infty \\
\bar{u}(z) & :=\sup _{\phi \in \Phi(z)} E u\left(X_{T}^{z, \phi}-B\right)=E u\left(X_{T}^{z, \phi^{*}(z)}-B\right)<\infty
\end{aligned}
$$

As $n \rightarrow \infty$ the convergence $\left(\phi_{n}^{*}(z)\right)_{t} \rightarrow\left(\phi^{*}(z)\right)_{t}$ takes place almost surely for $t=1, \ldots, T$ and $\bar{u}_{n}(z) \rightarrow \bar{u}(z)$, for all $z \in \mathbb{R}$.

One can also use utility maximisation to price derivative financial products via (85) or by other related techniques ("utility indifference pricing"), see e.g. [34, 44, 26]. The prices corresponding to $u_{n}$ are also shown to converge to those corresponding to $u$, see Theorems 2.3 and 2.4 in [21]. Under stronger assumptions even the convergence rate of $u_{n}$ is inherited by $\phi_{n}^{*}$ and $\bar{u}_{n}$, see Theorems 2.2 and 2.3 in [21].

The risk aversion function $r_{n}(x)$ of an investor with utility function $u_{n}(x)$ can be defined for $u_{n}$ concave, $n \in \mathbb{N}$. In [19, 20] it is shown (using techniques of the present chapter) that, if $r_{n}(x)$ tends to infinity then the so-called "utility indifference price" corresponding to $u_{n}$ converges to the superreplication cost of the given derivative product. This is intuitive: infinitely risk-averse agents take no risk.

Finally, [79, 24, 77] show the existence of optimal portfolios in a setting where $u$ is defined on $(0, \infty)$ only. This corresponds to an investor for whom creating losses is prohibited. The arguments of $[79,24]$ are in the spirit of this chapter but they are somewhat simpler. Hence we chose not to review these papers but rather to focus on the case $u: \mathbb{R} \rightarrow \mathbb{R}$.

## 3 Cumulative prospect theory in multistep models

EUT has been accepted by mainstream economics as a mathematically convenient and intellectually satisfying framework for investors' decision-making, and it served as the foundation of the theory of microeconomic equilibrium, see [4].

Regardless of the general enthousiasm about EUT, dissenting views emerged from rather early on. It was demonstrated in [1] that EUT fails in human experiments. In [101, 58], Daniel Kahneman and Amos Tversky suggested an alternative: cumulative prospect theory (CPT), supported by empirical evidence. Kahneman received the Nobel prize in economics ${ }^{3}$ in 2002 for "having integrated insights from psychological research into economic science, especially concerning human judgment and decision-making under uncertainty", see [105]. While highly regarded by many, this theory is still subject of debates in economist circles. Since we are interested in its mathematical aspects we do not discuss arguments for and against in this dissertation. Our purpose is to present CPT assumptions and then to introduce new mathematical tools for tackling optimal investment problems for agents with such preferences.

Economics literature on CPT is vast (see the references of [52,25]) but it stays mostly at the rather elementary level of one-step financial markets. More complex models appeared in [ $52,25,11,18,81$ ], but all these papers assumed that the financial market in consideration was complete, i.e. any reasonable payoff could be replicated by dynamic trading (see Section 1.3). Most prominent examples of such markets are the binary tree (Cox-Ross-Rubinstein model) and geometric Brownian motion (Black-Scholes model), see e.g. [15]. Though they provide excellent textbook material, complete market models perform poorly in practice.

Most papers also make assumptions on the portfolio losses: [25] allows only portfolios whose attainable wealth is bounded from below by 0 . In [52] the portfolio may admit losses, but this loss must be bounded from below by a constant (which may depend on the chosen strategy). Recall, however, that when the (concave) utility function $u$ is defined on the whole real line, standard utility maximisation problems usually admit optimal solutions that are not bounded from below, see [91].

It is thus desirable to investigate models which are incomplete and which allow portfolio losses that can be unbounded from below. In [12] and [49], a single period model is studied. Our research concentrated on multistep discrete-time models. These are generically incomplete ${ }^{4}$ and they form a broad enough class to match arbitrary empirical data. In addition, the real trading mechanism is discrete. Our principal results (Theorems 3.4 and 3.16 below) assert the existence of optimal strategies for CPT investors in a substantial class of relevant incomplete discrete-time market models. See Chapter 4 for continuous-time models.

We remark that other theories substituting EUT have been proposed: rank-dependent utility [71] and acceptability indices [28], for instance. It seems that optimisation under such preferences can also be treated using the tools we have developed for CPT. This is not pursued in the present work.

The standard (concave) EUT machinery provides powerful tools for risk management as well as for pricing in incomplete markets, see e.g. [26]. We hope that our present results are not only of theoretical interest but also contribute to the future development of a similar framework for CPT investors.

This chapter is based on [72, 22].

### 3.1 Investors with CPT preferences

The main tenets of CPT can be summarized as follows. First, agents analyze their gains or losses with respect to a given stochastic reference point $B$. Second, potential losses are taken into account more than potential gains. So agents behave differently on gains, i.e. on $(X-B)^{+}$(where $X$ runs over possible values of admissible portfolios) and on losses, i.e. on

[^2]$-(X-B)^{-}$. Third, agents overweight events with small probabilities (like extreme events) and underweight the ones with large probabilities, i.e. they distort the probability measure using some transformation functions.

Translated into mathematics: we assume that $u_{ \pm}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$and $w_{ \pm}:[0,1] \rightarrow[0,1]$ are continuous functions such that $u_{ \pm}(0)=0, w_{ \pm}(0)=0$ and $w_{ \pm}(1)=1$. $u_{+}$will express the agent's satisfaction of gains while $u_{-}$expresses his/her dissatisfaction of losses. We fix $B$, a scalar-valued random variable. The agent's utility function will be $\tilde{u}(x):=u_{+}(x), x \geq 0$, $\tilde{u}(x):=-u_{-}(-x), x<0$. The functions $w_{+}$(resp. $w_{-}$) will represent probability distortions applied to gains (resp. losses) of the investor.

Remark 3.1. In the seminal paper [58] the authors furthermore assumed that $u_{ \pm}$are both concave, resulting in an $S$-shaped $\tilde{u}$. They also stipulated that $u_{-}$is steeper at 0 than $u_{+}$ and that the functions $w_{ \pm}$are "inverse $S$-shaped", i.e. concave up to a certain point and then convex. While these features are relevant in an adequate description of investors' behaviour, they have no importance in the mathematical analysis, only the behaviour of $u_{ \pm}$(resp. $w_{ \pm}$) near $\infty$ (resp. near 0) matters, hence we do not impose these additional assumptions here.

Ours is the first mathematical treatment of discrete-time multiperiod incomplete models in the literature. We allow for a possibly stochastic reference point $B$. More interestingly, we need no concavity or even monotonicity assumptions on $u_{+}, u_{-}$(see Assumption 3.11 and Theorem 3.16 below). Note that in e.g. [52] and [25] the functions $u_{+}, u_{-}$are assumed to be concave and the reference point is easily incorporated: as the market is complete any stochastic reference point can be replicated. This is no longer so in our incomplete setting.

In Theorems 3.4 and 3.16 below we manage to provide intuitive and easily verifiable conditions which apply to a broad class of functions $u_{+}, u_{-}$and of probability distortions (see Assumptions 3.11, 3.12 and Remark 3.13) as soon as appropriate moment conditions hold for the price process. We also provide examples highlighting the kind of parameter restrictions which are necessary for well-posedness in a multiperiod context, see Section 3.4. It turns out that multiple trading periods exhibit phenomena which are absent in the one-step case.

We define, for $\theta \in \Phi$,

$$
\begin{equation*}
V_{+}(\theta, z):=\int_{0}^{\infty} w_{+}\left(P\left(u_{+}\left(\left[X_{T}^{\theta, z}-B\right]^{+}\right) \geq y\right)\right) d y \tag{92}
\end{equation*}
$$

and

$$
\begin{equation*}
V_{-}(\theta, z):=\int_{0}^{\infty} w_{-}\left(P\left(u_{-}\left(\left[X_{T}^{\theta, z}-B\right]^{-}\right) \geq y\right)\right) d y \tag{93}
\end{equation*}
$$

Denote by $\mathcal{A}(z)$ the set of $\theta$ for which $V_{-}(\theta, z)<\infty$. Set

$$
\begin{equation*}
V(\theta, z):=V_{+}(\theta, z)-V_{-}(\theta, z) \tag{94}
\end{equation*}
$$

for $\theta \in \mathcal{A}(z)$.
We aim to find $\theta^{*} \in \mathcal{A}(z)$ with

$$
\begin{equation*}
V\left(\theta^{*}, z\right)=\sup _{\theta \in \mathcal{A}(z)} V(\theta, z) \tag{95}
\end{equation*}
$$

Note that if $w_{ \pm}(p)=p$ (that is, there is no distortion) then we have $V(\theta, z)=E \tilde{u}\left(X_{T}^{z, \phi}-B\right)$. This shows that problem (10) is a subcase of problem (95) hence it can be expected that we shall need more stringent assumptions on $S$ in order to get existence results for the more general class of optimisation problems (95).

In particular, we shall need the following technical condition on $S$ and on the information flow $\mathcal{F}_{t}, t=0, \ldots, T$. The sigma-algebra generated by a random variable $X$ will be denoted $\sigma(X)$.

Assumption 3.2. Let $\mathcal{F}_{0}$ coincide with the family of P-zero sets and let $\mathcal{F}_{t}$ be the P-completion of $\sigma\left(Z_{1}, \ldots, Z_{t}\right)$ for $t=1, \ldots, T$, where the $Z_{i}, i=1, \ldots, T$ are $\mathbb{R}^{N}$-valued independent random
variables for some $N \in \mathbb{N}, Y_{0}$ is constant and $Y_{1}=f_{1}\left(Z_{1}\right), Y_{t}=f_{t}\left(Y_{1}, \ldots, Y_{t-1}, Z_{t}\right), t=2, \ldots, T$ for some continuous functions $f_{t}: \mathbb{R}^{N+(t-1) L} \rightarrow \mathbb{R}^{L}$ for some $L \in \mathbb{N}$. We assume that $B=$ $g\left(Y_{1}, \ldots, Y_{T}\right)$ for some continuous $g$ and that $S_{t}^{i}=Y_{t}^{i}, i=1, \ldots, d$ for some $1 \leq d \leq L$ and for all $t=0, \ldots, T$.

Furthermore, for $t=1, \ldots, T$ there exists an $\mathcal{F}_{t}$-measurable uniformly distributed random variable $U_{t}$ which is independent of $\mathcal{F}_{t-1} \vee \sigma\left(Y_{t}\right)$.

We think of the $L$-dimensional process $Y$ as the economic factors present in the given market. Its first $d$ coordinates equal $S$ and they represent the prices of $d$ risky assets while the rest of the coordinates are other variables (inflation, unemployment rate, exchange rates, assets in a different market, etc.).

Remark 3.3. Stipulating the existence of the "innovations" $Z_{t}$ might look restrictive but it can be weakened to $\left(Z_{1}, \ldots, Z_{T}\right)$ having a nice enough density w.r.t. the respective Lebesgue measure, see Proposition 6.4 of [22]. In addition to the continuity conditions, the above Assumption requires that the information filtration is "large enough": at each time $t$, there should exist some randomness which is independent of both the past $\left(\mathcal{F}_{t-1}\right)$ and of the present value of the economic factors in consideration $\left(Y_{t}\right)$. Since real markets are perceived as highly incomplete and noisy, this looks a mild requirement. See Section 3.6 for models satisfying Assumption 3.2.

As in the case of EUT, we first look at the case where $u$ is bounded above. Note that in this case we may work on $\Phi$ instead of $\mathcal{A}(z)$. The following result will be shown in Section 3.2 below.

Theorem 3.4. Assume that $u_{+}$is bounded above, $u_{-}, w_{-}$are nondecreasing with $w_{-}(x)>0$ for $x>0, u_{-}(\infty)=\infty$ (that is, $\tilde{u}(-\infty)=-\infty$ ) and

$$
\begin{equation*}
V_{-}(0, z)<\infty . \tag{96}
\end{equation*}
$$

Under Assumption 3.2 and (NA), there is $\theta^{*}=\theta^{*}(z) \in \Phi$ satisfying

$$
\begin{equation*}
V\left(\theta^{*}, z\right)=\sup _{\theta \in \Phi} V(\theta, z)>-\infty \tag{97}
\end{equation*}
$$

Remark 3.5. The case of bounded above $u_{+}$is investigated in [75] in complete continuoustime models. It turns out that $u_{-}(\infty)=\infty$ is insufficient for the existence of a strategy in that setting and the distortion needs to satisfy

$$
\begin{equation*}
\liminf _{x \rightarrow 0+} w_{-}(x) u_{-}(1 / x)>0 \tag{98}
\end{equation*}
$$

Condition (98) is also essentially sufficient under additional assumptions, see [75] for details.

### 3.2 CPT with bounded above utility

We begin with a multistep extension of Proposition 1.6.
Lemma 3.6. For all $t=0, \ldots, T-1$, there exist $\pi_{t} \in \Xi_{t}^{1}, \pi_{t}>0$ a.s., $t=0, \ldots, T-1$, such that, for all $\theta \in \widehat{\Phi}$ (recall the definition of $\widehat{\Phi}$ in Section 1.2),

$$
P\left(\theta_{t+1} \Delta S_{t+1} \leq-\nu_{t}\left|\theta_{t+1}\right|, \theta_{n} \Delta S_{n} \leq 0, n=t+2, \ldots, T \mid \mathcal{F}_{t}\right) \geq \pi_{t}
$$

Proof. Define the events

$$
\begin{aligned}
A_{t+1} & :=\left\{\theta_{t+1} \Delta S_{t+1} \leq-\nu_{t}\left|\theta_{t+1}\right|\right\}, \\
A_{n} & :=\left\{\theta_{n} \Delta S_{n} \leq 0\right\}, \quad t+2 \leq n \leq T
\end{aligned}
$$

We will prove, by induction on $m=t+1, \ldots, T$, that for all $Q \sim P$, there is $\pi_{t}^{Q}(m)>0$ a.s. such that

$$
\begin{equation*}
E_{Q}\left(1_{A_{t+1}} \ldots 1_{A_{m}} \mid \mathcal{F}_{t}\right) \geq \pi_{t}^{Q}(m) \tag{99}
\end{equation*}
$$

For $m=t+1$ this is trivial for $P=Q$ by Proposition 1.6 and it follows for all $Q \sim P$ by Remark 1.7.

Let us assume that (99) has been shown for $m-1$, we will establish it for $m$.

$$
\begin{aligned}
E_{Q}\left(1_{A_{m}} \ldots 1_{A_{t+1}} \mid \mathcal{F}_{t}\right) & =E_{Q}\left(E_{Q}\left(1_{A_{m}} \mid \mathcal{F}_{m-1}\right) 1_{A_{m-1}} \ldots 1_{A_{t+1}} \mid \mathcal{F}_{t}\right) \\
& \geq E_{Q}\left(\kappa_{m-1}^{Q} 1_{A_{m-1}} \ldots 1_{A_{t+1}} \mid \mathcal{F}_{t}\right)
\end{aligned}
$$

Now define $R \sim P$ by $d R / d P:=\kappa_{m-1}^{Q} / E \kappa_{m-1}^{Q}$. It follows that

$$
\begin{aligned}
& E_{Q}\left(\kappa_{m-1}^{Q} 1_{A_{m-1}} \ldots 1_{A_{t+1}} \mid \mathcal{F}_{t}\right)=E_{Q}\left(\kappa_{m-1}^{Q}\right) E_{R}\left(1_{A_{m-1}} \ldots 1_{A_{t+1}} \mid \mathcal{F}_{t}\right) E_{Q}\left(d R / d Q \mid \mathcal{F}_{t}\right) \\
\geq & E_{Q}\left(\kappa_{m-1}^{Q}\right) E_{Q}\left(d R / d Q \mid \mathcal{F}_{t}\right) \pi_{t}^{R}(m-1)>0 \text { a.s. }
\end{aligned}
$$

showing the induction step. Finally, one can set $\pi_{t}:=\pi_{t}^{P}(T)$.
Lemma 3.7. Under the conditions of Theorem 3.4, let $\theta(n) \in \widehat{\Phi}, n \in \mathbb{N}$ such that $V(z, \theta(n)) \geq$ $-c$ for some $c \in \mathbb{R}$, for all $n$. Then the sequence of the laws of $\left(\theta_{1}(n), \ldots, \theta_{T}(n)\right)$ is tight.
Proof. It suffices to show, by induction on $m=1, \ldots, T$, that we have, for all $m$,

$$
P\left(\left|\theta_{m}(n)\right| \geq c_{n}\right) \rightarrow 0, n \rightarrow \infty
$$

for every sequence $c_{n} \rightarrow \infty, n \rightarrow \infty$, see e.g. Lemma 4.9 on p. 66 of [59]. The first step is similar to the induction step, so we omit it. Let us assume that the above statement has been shown for $m=1, \ldots, k$, we will show it for $k+1$.

Define the right-continuous and non-decreasing function

$$
w_{-}^{\dagger}(q):=\max \left\{p \in[0,1]: w_{-}(p)=q\right\}, \quad q \in[0,1]
$$

(note the continuity of $w_{-}$). Clearly, $w_{-}^{\dagger}(q) \rightarrow 0, q \rightarrow 0$.
Since $u_{+}$and hence also $V^{+}(\theta(n), z)$ are bounded above by a fixed constant $C, V(\theta(n), z) \geq$ $-c$ implies $V_{-}(\theta(n), z) \leq c+C$ for all $n$. From (93) we get $w_{-}\left(P\left(\left(u_{-}\left(X_{T}^{z, \theta(n)}-B\right)_{-}\right) \geq y\right)\right) \leq$ $(c+C) / y$ hence also $P\left(u_{-}\left(\left(X_{T}^{z, \theta(n)}-B\right)_{-}\right) \geq y\right) \leq w_{-}^{\dagger}((c+C) / y)$, for all $y>0$. This shows that

$$
\left.P\left(\left(X_{T}^{z, \theta(n)}-B\right)_{-} \geq c_{n}\right) \leq P\left(u_{-}\left(\left(X_{T}^{z, \theta(n)}-B\right)_{-}\right)\right) \geq u_{-}\left(c_{n}\right)\right) \leq w_{-}^{\dagger}\left((c+C) / u_{-}\left(c_{n}\right)\right) \rightarrow 0
$$

as $n \rightarrow \infty$, since $u_{-}(x) \rightarrow \infty, x \rightarrow \infty$ and $w_{-}^{\dagger}(q) \rightarrow 0, q \rightarrow 0$.
We claim that, for all $1 \leq j \leq k, P\left(\left|\theta_{j}(n) \Delta S_{j}\right| \geq c_{n}\right) \rightarrow 0, n \rightarrow \infty$ for any sequence $c_{n} \rightarrow \infty$, $n \rightarrow \infty$. Indeed, fix $\varepsilon>0 . P\left(\left|\Delta S_{j}\right| \leq s\right) \geq 1-\varepsilon / 2$ for $s$ large enough. Also, for $n$ large enough, $P\left(\left|\theta_{j}(n)\right| \leq c_{n} / s\right) \geq 1-\varepsilon / 2$, by the induction hypothesis. Hence $P\left(\left|\theta_{j}(n) \Delta S_{j}\right| \geq c_{n}\right) \leq \varepsilon$ for $n$ large enough, as claimed. It follows that also

$$
l(n):=P\left(\left|X_{k}^{z, \theta(n)}\right| \geq c_{n} / 3\right) \leq \sum_{j=1}^{k} P\left(\left|\theta_{j}(n) \Delta S_{j}\right| \geq c_{n} /(3 k)\right) \rightarrow 0, \quad n \rightarrow \infty
$$

Clearly, the inclusion

$$
\left\{\left(\sum_{j=k+1}^{T} \theta_{j}(n) \Delta S_{j}\right)_{-} \geq c_{n}\right\} \subset\left\{|B| \geq c_{n} / 3\right\} \cup\left\{\left|X_{k}^{z, \theta(n)}\right| \geq c_{n} / 3\right\} \cup\left\{\left(X_{T}^{z, \theta(n)}-B\right)_{-} \geq c_{n} / 3\right\}
$$

holds, which implies

$$
\begin{aligned}
P\left(|B| \geq c_{n} / 3\right)+l(n)+P\left(\left(X_{T}^{z, \theta(n)}-B\right)_{-} \geq c_{n} / 3\right) & \geq \\
P\left(\left(\sum_{j=k+1}^{T} \theta_{j}(n) \Delta S_{j}\right)_{-} \geq c_{n}\right) & \geq \\
P\left(\theta_{k+1}(n) \Delta S_{k+1} \leq-\kappa_{k}\left|\theta_{k+1}(n)\right|, \theta_{j}(n) \Delta S_{j} \leq 0, k+2 \leq j \leq T,\right. & \\
\left.\left|\theta_{k+1}(n)\right| \geq c_{n} / \kappa_{k}\right) & \geq \\
E\left[1_{\left|\theta_{k+1}(n)\right| \geq c_{n} / \kappa_{k}} \pi_{k}\right] &
\end{aligned}
$$

so $E\left[1_{\left|\theta_{k+1}(n)\right| \geq c_{n} / \kappa_{k}} \pi_{k}\right] \rightarrow 0, n \rightarrow \infty$. Define $Q \sim P$ by $d Q / d P:=\pi_{k} / E \pi_{k}$. It follows that $Q\left(\left|\theta_{k+1}(n)\right| \geq c_{n} / \kappa_{k}\right) \rightarrow 0, n \rightarrow \infty$, which implies $Q\left(\left|\theta_{k+1}(n)\right| \geq c_{n}\right) \rightarrow 0$ and hence also $P\left(\left|\theta_{k+1}(n)\right| \geq c_{n}\right) \rightarrow 0, n \rightarrow \infty$. The induction step is completed.

Proof of Theorem 3.4. Let $\theta(j) \in \Phi$ be such that $V(\theta(j), z) \geq \sup _{\theta \in \Phi} V(\theta, z)-1 / j, j \in \mathbb{N}$. As $V(z-B)>-\infty$, this supremum is $>-\infty$. By Remark 2.4 we may and will suppose that $\theta(j) \in \widehat{\Phi}$ for all $j$. By (96), the supremum is at least $V(0, z)>-\infty$ so $\inf _{j} V(\theta(j), z)>-\infty$, hence Lemma 3.7 shows the tightness of the sequence of random variables $\left(\theta_{1}(j), \ldots, \theta_{T}(j)\right)$, $j \in \mathbb{N}$. Then the sequence

$$
I_{j}:=\left(Y_{1}, \ldots, Y_{T}, \theta_{1}(j), \ldots, \theta_{T}(j)\right), j \in \mathbb{N}
$$

is also tight so a subsequence (still denoted by $j$ ) converges weakly to some probability law $\mu$ on $\mathcal{B}\left(\mathbb{R}^{T(d+L)}\right)$. We will also use the notation $I_{j}^{(k)}:=\left(S_{1}, \ldots, S_{k}, \theta_{1}(j), \ldots, \theta_{k}(j)\right)$ and denote its law on $\mathcal{B}\left(\mathbb{R}^{k(d+L)}\right)$ by $\mu_{k}(j)$. Let us take $M$ to be a $T(d+L)$-dimensional random variable with law $\mu$. Let $\mu_{k}$ be the law of

$$
\left(M^{1}, \ldots, M^{k L}, M^{T L+1}, \ldots, M^{T L+k d}\right)
$$

on $\mathcal{B}\left(\mathbb{R}^{k(d+L)}\right)$. Obviously, $\mu_{k}(j)$ weakly converges to $\mu_{k}, j \rightarrow \infty$.
We shall construct, recursively, $\theta_{i}^{*}, i=1, \ldots, T$ such that $F_{k}:=\left(Y_{1}, \ldots, Y_{k}, \theta_{1}^{*}, \ldots, \theta_{k}^{*}\right)$ has law $\mu_{k}$ for all $k=1, \ldots, T$, and $\theta^{*}=\left(\theta_{1}^{*}, \ldots, \theta_{T}^{*}\right)$ is a trading strategy.

As $\theta_{1}(j)$ are deterministic numbers, weak convergence implies that they converge to some (deterministic) $\theta_{1}^{*}$ which is then $\mathcal{F}_{0}$-measurable. Clearly, $\left(Y_{1}, \theta_{1}^{*}\right)$ has law $\mu_{1}$.

Carrying on, let us assume that we have found $\theta_{i}^{*}, i=1, \ldots, k$ such that $F_{k}$ has law $\mu_{k}$ and $\theta_{j}^{*}$ is $\mathcal{F}_{j-1}$-measurable for $j=1, \ldots, k$.

We now apply Lemma 6.16 with $N_{1}=d, N_{2}=k(d+L), E=U_{k}$ and $Y=F_{k}$ to get $G$ such that $\left(F_{k}, G\left(F_{k}, U_{k}\right)\right)$ has the same law as $\left(M^{1}, \ldots, M^{k L}, M^{T d+1}, M^{T d+(k+1) L}\right)$, we denote this law by $\bar{\mu}_{k}$ henceforth (note that, by Assumption 3.2, $U_{k}$ is independent of $F_{k}$ ). Define $\theta_{k+1}^{*}:=G\left(F_{k}, U_{k}\right)$, this is clearly $\mathcal{F}_{k}$-measurable. It remains to show that $F_{k+1}$ has law $\mu_{k+1}$. As $\mu_{k+1}$ is the weak limit of $\operatorname{Law} I_{j}^{(k+1)}$, it is enough to prove that the latter is $F_{k+1}$. We first decompose the laws of $I_{j}^{(k+1)}$ and $F_{k+1}$ by means of conditioning.

By Assumption 3.2 one can write $Y_{k+1}=f_{k+1}\left(Y_{1}, \ldots, Y_{k}, Z_{k+1}\right)$ with some continuous function $f_{k+1}$. Notice that the law of the $(k+1)(d+L)$-dimensional random variable $F_{k+1}$ is

$$
\mu_{k+1}(d x)=\bar{\mu}_{k}\left(d \sigma_{1}, \ldots, d \sigma_{k}, d \tau_{1}, \ldots, d \tau_{k+1}\right) \rho\left(d \sigma_{k+1} \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right)
$$

where we write

$$
d x=\left(d x_{1}, \ldots, d x_{(k+1)(L+d)}\right)=\left(d \sigma_{1}, \ldots, d \sigma_{k+1}, d \tau_{1}, \ldots, d \tau_{k+1}\right), d \sigma_{j}=\left(d x_{(j-1) L+1}, \ldots, d x_{j L}\right)
$$

for $j=1, \ldots, k+1$ and $d \tau_{i}=\left(d x_{(k+1) L+(i-1) d+1}, \ldots, d x_{(k+1) L+i d}\right)$ for $i=1, \ldots, k+1$. The probabilistic kernel $\rho$ is defined by

$$
\begin{aligned}
& \rho\left(A \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right):=P\left(Y_{k+1} \in A \mid Y_{1}=\sigma_{1}, \ldots, Y_{k}=\sigma_{k}, \theta_{1}^{*}=\tau_{1}, \ldots, \theta_{k+1}^{*}=\tau_{k+1}\right)= \\
& P\left(f_{k+1}\left(\sigma_{1}, \ldots, \sigma_{k}, Z_{k+1}\right) \in A \mid Y_{1}=\sigma_{1}, \ldots, Y_{k}=\sigma_{k}, \theta_{1}^{*}=\tau_{1}, \ldots, \theta_{k+1}^{*}=\tau_{k+1}\right)= \\
& P\left(f_{k+1}\left(\sigma_{1}, \ldots, \sigma_{k}, Z_{k+1}\right) \in A\right)
\end{aligned}
$$

for $A \in \mathcal{B}\left(\mathbb{R}^{d}\right),\left(\sigma_{1}, \ldots, \sigma_{k}\right) \in \mathbb{R}^{k L},\left(\tau_{1}, \ldots, \tau_{k+1}\right) \in \mathbb{R}^{(k+1) d}$, by independence of $Z_{k+1}$ from $\mathcal{F}_{k}$. The crucial observation here is that $\rho$ does not depend on $\left(\tau_{1}, \ldots, \tau_{k+1}\right)$.

It follows in the same way that, for all $j$, the law of $I_{j}^{(k+1)}$ is

$$
\mu_{k+1}(j)(d x)=\bar{\mu}_{k}(j)\left(d \sigma_{1}, \ldots, d \sigma_{k}, d \tau_{1}, \ldots, d \tau_{k+1}\right) \rho\left(d \sigma_{k+1} \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right)
$$

where $\bar{\mu}_{k}(j)$ is the law of $\left(Y_{1}, \ldots, Y_{k}, \theta_{1}(j), \ldots, \theta_{k+1}(j)\right)$.
Clearly, the weak convergence of $\operatorname{Law}\left(I_{j}\right)$ to $\mu$ implies that their marginals $\bar{\mu}_{k}(j)$ converge weakly to $\bar{\mu}_{k}$, for each $k$. To conclude the proof, we have to show that this implies also

$$
\begin{array}{r}
\mu_{k+1}(j)(d x)=\bar{\mu}_{k}(j)\left(d \sigma_{1}, \ldots, d \sigma_{k}, d \tau_{1}, \ldots, d \tau_{k+1}\right) \rho\left(d \sigma_{k+1} \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right) \rightarrow \\
\mu_{k+1}(d x)=\bar{\mu}_{k}\left(d \sigma_{1}, \ldots, d \sigma_{k}, d \tau_{1}, \ldots, d \tau_{k+1}\right) \rho\left(d \sigma_{k+1} \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right) \tag{100}
\end{array}
$$

weakly as $j \rightarrow \infty$.
First notice that, for any sequence $v_{n} \rightarrow v$ in $\mathbb{R}^{k(d+L)+d}, \rho\left(\cdot \mid v_{n}\right)$ tends to $\rho(\cdot \mid v)$ weakly. Indeed, taking any continuous and bounded $h$ on $\mathbb{R}^{L}$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^{L}} h(\sigma) \rho\left(d \sigma \mid v_{n}\right) & =E h\left(f_{k+1}\left(v_{n}^{1}, \ldots, v_{n}^{L k}, Z_{k+1}\right)\right) \rightarrow \\
E h\left(f_{k+1}\left(v^{1}, \ldots, v^{L k}, Z_{k+1}\right)\right) & =\int_{\mathbb{R}^{L}} h(\sigma) \rho(d \sigma \mid v)
\end{aligned}
$$

by continuity of $h, f_{k+1}$, boundedness of $h$ and Lebesgue's theorem.
Now take any uniformly continuous and bounded $g: \mathbb{R}^{(k+1)(d+L)} \rightarrow \mathbb{R}$. Define

$$
\bar{g}(v):=\int_{\mathbb{R}^{L}} g(v, \sigma) \rho(d \sigma \mid v), \quad v \in \mathbb{R}^{k(d+L)+d} .
$$

We claim that $\bar{g}$ is continuous. Indeed, let $v_{n} \rightarrow v$. Then

$$
\begin{aligned}
\left|\bar{g}\left(v_{n}\right)-\bar{g}(v)\right| \leq \mid & \left|\int_{\mathbb{R}^{L}} g\left(v_{n}, \sigma\right) \rho\left(d \sigma \mid v_{n}\right)-\int_{\mathbb{R}^{L}} g(v, \sigma) \rho\left(d \sigma \mid v_{n}\right)\right|+ \\
& \quad+\left|\int_{\mathbb{R}^{L}} g(v, \sigma) \rho\left(d \sigma \mid v_{n}\right)-\int_{\mathbb{R}^{L}} g(v, \sigma) \rho(d \sigma \mid v)\right| .
\end{aligned}
$$

Here the first term tends to zero by uniform continuity, the second term tends to zero by the weak convergence of $\rho\left(\cdot \mid v_{n}\right)$ to $\rho(\cdot \mid v)$. This shows the continuity of $\bar{g}$.

As $\bar{\mu}_{k}(j)$ converge weakly to $\bar{\mu}_{k}$, it follows that

$$
\begin{array}{r}
\int_{\mathbb{R}^{k}(d+L)+d} \bar{g}\left(\sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right) \bar{\mu}_{k}(j)\left(d \sigma_{1}, \ldots, d \sigma_{k}, d \tau_{1}, \ldots, d \tau_{k+1}\right) \rightarrow \\
\int_{\mathbb{R}^{k}(d+L)+d} \bar{g}\left(\sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right) \bar{\mu}_{k}\left(d \sigma_{1}, \ldots, d \sigma_{k}, d \tau_{1}, \ldots, d \tau_{k+1}\right),
\end{array}
$$

as $j \rightarrow \infty$. This implies that

$$
\int_{\mathbb{R}^{(k+1)(d+L)}} g\left(\sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}, \sigma_{k+1}\right) \rho\left(d \sigma_{k+1} \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right) \bar{\mu}_{k}(j)\left(d \sigma_{1}, \ldots\right)
$$

tends to

$$
\int_{\mathbb{R}^{(k+1)(d+L)}} g\left(\sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}, \sigma_{k+1}\right) \rho\left(d \sigma_{k+1} \mid \sigma_{1}, \ldots, \sigma_{k}, \tau_{1}, \ldots, \tau_{k+1}\right) \bar{\mu}_{k}\left(d \sigma_{1}, \ldots\right),
$$

showing that (100) holds (recall that, in order to check weak convergence, it is enough to use uniformly continuous bounded functions, see Theorem 1.1.1 of [100]) and the induction step is completed. We finally arrive at $\left(Y_{1}, \ldots, Y_{T}, \theta_{1}^{*}, \ldots, \theta_{T}^{*}\right)$ with law $\mu_{T}=\mu$.

As $u_{ \pm}, w_{ \pm}$are continuous, $w_{ \pm}\left(P\left(u_{ \pm}\left(\left[X^{\theta(j), z}-B\right]_{ \pm}\right) \geq y\right)\right)$ tend to $w_{ \pm}\left(P\left(u_{ \pm}\left(\left[X^{\theta^{*}, z}-B\right]_{ \pm}\right) \geq\right.\right.$ y)) outside the discontinuity points of the cumulative distribution functions of $u_{ \pm}\left(\left[X^{\theta^{*}, z}-\right.\right.$ $\left.B]_{ \pm}\right)$, in particular, for Lebesgue-a.e. $y$. Note that $w_{+}\left(P\left(u_{+}\left(\left[X_{T}^{z, \phi(j)}-B\right]_{+}\right) \geq y\right) \leq 1_{[0, C]}\right.$ where $C$ is an upper bound for $u_{+}$. Fatou's lemma implies

$$
\limsup _{j \rightarrow \infty} V(\theta(j), z) \leq V\left(\theta^{*}, z\right),
$$

which shows that $\theta^{*}$ satisfies (95).

### 3.3 A surprising example

Take $z=0$. Let us define $\mathcal{P}:=\left\{\operatorname{Law}\left(X_{T}^{0, \phi}\right): \phi \in \Phi\right\}$. The proof of Theorem 3.4 consisted of two steps: first, the relative compactness of the sequence of optimisers (for the weak convergence of probability measures) was shown using Lemma 3.7; second, it was established
that for a tight sequence of strategies $\phi(n), n \in \mathbb{N}$ one may always find a limit point of (a subsequence of) $\operatorname{Law}\left(X_{T}^{0, \phi(n)}\right), n \in \mathbb{N}$ in $\mathcal{P}$, under Assumption 3.2.

In this section we provide an example which shows that the latter property can easily fail and $\mathcal{P}$ is not closed for weak convergence unless additional assumptions (such as Assumption 3.2 above or Assumption 6.1 of [22]) are made. This fact is surprising since the set $\left\{X_{T}^{0, \phi}: \phi \in\right.$ $\Phi\}$ is closed in probability, even without (NA), see Proposition 2 of [99] and Proposition 6.8.1 of [35]. In the sequel Leb refers to the Lebesgue measure on $[0,1]$.

Our example will be a one-step model with one risky asset and a non-trivial initial sigmaalgebra. Let $U$ be uniform on $[0,1]$ and let $Y$ be a $\mathbb{Z}$-valued random variable, independent of $U$, with $P(Y=-1)=1 / 2, P(Y=k)=1 / 2^{k+1}, k \geq 1$. Define $\mathcal{F}_{0}:=\sigma(U), \mathcal{F}_{1}:=\sigma(U, Y)$. Set $S_{0}=0, S_{1}=\Delta S_{1}:=-1$ if $Y=-1$ and $\Delta S_{1}=f_{k}(U)$ if $Y=k, k \geq 1$ where $f_{k}(x):=$ $3^{k}+1 / 2+q_{k}(x), x \in[0,1]$ and $q_{k}$ is a complete orthogonal system in the Hilbert space

$$
\left\{h \in L^{2}([0,1], \mathcal{B}([0,1]), \text { Leb }): \int_{0}^{1} h(x) d x=0\right\}
$$

such that each $q_{k}$ is continuous and $\left|q_{k}(x)\right| \leq 1 / 2, x \in[0,1]$. Such a system can easily be constructed e.g. from the trigonometric system. This model clearly satisfies (NA) but we claim that

$$
\mathcal{P}=\left\{\operatorname{Law}\left(\phi \Delta S_{1}\right): \phi \text { is } \mathcal{F}_{0} \text {-measurable }\right\}
$$

is not closed for weak convergence.
We first construct a certain limit point for a sequence in $\mathcal{P}$. A "creation of more randomness" takes place in the next lemma.

Lemma 3.8. Define $g_{n}(x):=n(x-k / n), k / n \leq x<(k+1) / n, k=0, \ldots, n-1$ and set $g_{n}(1)=1$, for $n \in \mathbb{N}$. We claim that $\mu_{n}:=\operatorname{Law}\left(U, g_{n}(U)\right)$ converges weakly to $\mu_{\infty}:=\operatorname{Law}(U, V), n \rightarrow \infty$, where $V$ is uniform on $[0,1]$ and it is independent of $U$.

Proof. It suffices to prove that, for all $0 \leq a, b \leq 1$, we have $\mu_{n}([0, a] \times[0, b]) \rightarrow \mu_{\infty}([0, a] \times[0, b])$, see Theorem 29.1 in [14]. Fix $a, b$ and define, for all $n, l(n)$ as the largest integer with $l(n) / n \leq$ $a$. By the definition of $g_{n}$, we have that, for all $n \in \mathbb{N} \cup\{\infty\}$,

$$
\mu_{n}([0, l(n) / n] \times[0, b])=b l(n) / n
$$

It is also clear that $\mu_{n}([0, a] \times[0, b])-\mu_{n}([0, l(n) / n] \times[0, b]) \leq 1 / n$ holds for all $n \in \mathbb{N}$ and also $\mu_{\infty}([0, a] \times[0, b])-\mu_{\infty}([0, l(n) / n] \times[0, b]) \leq 1 / n$, hence $\mu_{n}([0, a] \times[0, b]) \rightarrow \mu_{\infty}([0, a] \times[0, b])$, $n \rightarrow \infty$.

Define $\phi_{n}:=g_{n}(U)+1, n \in \mathbb{N}$. It follows that the sequence of triplets $\left(U, Y, \phi_{n}\right), n \in \mathbb{N}$ converges to $(U, Y, W)$ in law, where $W$ is uniform on $[1,2]$ and independent of $(U, Y)$. Define $\bar{\phi}:=W$. Note that equipping $\mathbb{Z}$ with the discrete topology, $\Delta S_{1}$ is a continuous function of ( $U, Y$ ), hence, by the continuous mapping theorem, Law $\left(\phi_{n} \Delta S_{1}\right)$ converges weakly to $\nu:=$ Law $\left(\bar{\phi} \Delta S_{1}\right)$. We claim, however, that $\nu \notin \mathcal{P}$.

Arguing by contradiction, let us suppose the existence of a Borel-function $g$ such that with $\phi:=g(U)$ one has $\nu=\mu:=\operatorname{Law}\left(\phi \Delta S_{1}\right)$. Let $s$ denote the support of the law of $\phi$. If $s \cap(-\infty, 0) \neq \emptyset$ then the support of $\mu$ would be unbounded from below hence it cannot be equal to $\nu$. Hence $\phi \geq 0$ a.s. and then $-s=\operatorname{supp}(\nu) \cap(-\infty, 0]=[-2,-1]$, so $s=[1,2]$.

This implies that the following (a.s.) equalities hold between events:

$$
A_{k}:=\left\{\phi \Delta S_{1} \in\left[3^{k}, 2 \times 3^{k}+2\right]\right\}=\{Y=k\}=\left\{\bar{\phi} \Delta S_{1} \in\left[3^{k}, 2 \times 3^{k}+2\right]\right\}
$$

for all $k \geq 1$. Then, by independence of $U$ from $Y$ and $W$ of $(U, Y)$,

$$
\begin{aligned}
E\left[\phi \Delta S_{1} 1_{A_{k}}\right]=P(Y=k) E\left[g(U) f_{k}(U)\right] & =\left(1 / 2^{k+1}\right) \int_{0}^{1} g(x) f_{k}(x) d x= \\
E\left[\bar{\phi} \Delta S_{1} 1_{A_{k}}\right]=\left(1 / 2^{k+1}\right) E W E f_{k}(U) & =\left(1 / 2^{k+1}\right)(3 / 2)\left(3^{k}+1 / 2\right)
\end{aligned}
$$

It follows that, for all $k$,

$$
c_{k}:=\int_{0}^{1} g(x) q_{k}(x) d x=E\left[g(U)\left(f_{k}(U)-3^{k}-1 / 2\right)\right]=[3 / 2-E g(U)]\left(3^{k}+1 / 2\right) .
$$

Since the $q_{k}$ are orthogonal and uniformly bounded, necessarily $\sum_{k=1}^{\infty} c_{k}^{2}<\infty$, so $c_{k}=0$ for all $k$. This implies $E g(U)=3 / 2$. By the completeness of the sequence $q_{k}$ we also get that $g$ is a.s. constant, so $\phi=3 / 2$. This contradiction with $s=[1,2]$ shows our claim.

### 3.4 A first look at well-posedness

Well-posedness is trivial for $u_{+}$bounded above. We will thus concentrate on the case of unbounded $u_{+}$in this section.

Example 3.9. A typical choice for $u_{ \pm}$, $w_{ \pm}$(going back to [101]) is taking

$$
u_{+}(x)=x^{\alpha}, \quad u_{-}(x)=k x^{\beta}
$$

for some $k>0$ and setting

$$
w_{+}(p)=\frac{p^{\gamma}}{\left(p^{\gamma}+(1-p)^{\gamma}\right)^{1 / \gamma}}, \quad w_{-}(p)=\frac{p^{\delta}}{\left(p^{\delta}+(1-p)^{\delta}\right)^{1 / \delta}},
$$

with constants $0<\alpha, \beta, \gamma, \delta \leq 1$. This is still one of the most commonly used specifications of $u_{ \pm}, w_{ \pm}$in the literature.

Requiring that $0<\alpha, \beta \leq 1$, we ensure that both $u_{+}$and $u_{-}$are concave. Moreover, assuming $\gamma, \delta \leq 1$, we get that the probability distortions are "inverse S -shaped" and that $w_{ \pm}(x) \geq x$ holds for all $x$ close to 0 . This latter property captures the fact that a behavioural agent overweights small probabilities.

We are concerned with maximizing $V(\theta, z)$ over $\theta \in \mathcal{A}(z)$ in the case where $u$ is not necessarily bounded above. We seek to find conditions ensuring well-posedness, i.e.

$$
\begin{equation*}
\sup _{\theta \in \mathcal{A}(z)} V(\theta, z)<\infty \tag{101}
\end{equation*}
$$

and the existence of $\theta^{*} \in \mathcal{A}(z)$ attaining this supremum.
Remark 3.10. One may wonder whether the set $\mathcal{A}(z)$ is rich enough. Assume that $u_{-}(x) \leq$ $c\left(1+x^{\beta}\right)$ for some $c, \beta>0 ; B \in \mathcal{W}$ and $w_{-}(p) \leq C p^{\delta}$ for some $\delta, C>0$. Then Lemma 6.14 below implies that the strategy $\theta_{t}=0, t=1, \ldots, T$ is in $\mathcal{A}(z)$, in particular, the latter set is non-empty. If, furthermore, $\Delta S_{t} \in \mathcal{W}$ for all $t$ then $\theta \in \mathcal{A}(z)$ whenever $\theta_{t} \in \mathcal{W}, t=1, \ldots, T$. This remark applies, in particular, to $u_{-}$and $w_{-}$in Example 3.9 above.

In the rest of this section we will find parameter restrictions that need to hold in order to have a well-posed problem in the setting of e.g. Example 3.9. The discussion below sheds light on the assumptions we will make later in Section 3.5.

For simplicity, we assume that $u_{+}(x)=x^{\alpha}$ and $u_{-}(x)=x^{\beta}$ for some $\alpha, \beta>0$; the distortion functions are $w_{+}(p)=p^{\gamma}, w_{-}(p)=p^{\delta}$ for some $\gamma, \delta>0$. The example given below applies also to $w_{ \pm}$with a power-like behavior near 0 such as those in Example 3.9 above.

Let us consider a two-step market model with $S_{0}=0, \Delta S_{1}$ uniform on [-1,1], $P\left(\Delta S_{2}=\right.$ $\pm 1)=1 / 2$ and $\Delta S_{2}$ is independent of $\Delta S_{1}$. Let $\mathcal{F}_{0}, \mathcal{F}_{1}, \mathcal{F}_{2}$ be the natural filtration of $S_{0}, S_{1}, S_{2}$. It is easy to check that (NA) holds for this model.

Let us choose initial capital $z=0$ and reference point $B=0$. We consider the strategy $\theta \in \Phi$ given by $\theta_{1}=0$ and $\theta_{2}=g\left(\Delta S_{1}\right)$ with $g:[-1,1) \rightarrow[1, \infty)$ defined by $g(x)=\left(\frac{2}{1-x}\right)^{1 / \ell}$, where $\ell>0$ will be chosen later. Then the distribution function of $\theta_{2}$ is given by

$$
F(y)=0, y<1, \quad F(y)=1-\frac{1}{y^{\ell}}, y \geq 1
$$

It follows that

$$
V_{+}(\theta, 0)=\int_{0}^{\infty} P^{\gamma}\left(\left(\theta_{2} \Delta S_{2}\right)_{+}^{\alpha} \geq y\right) d y=\int_{1}^{\infty} \frac{1}{2^{\gamma}} \frac{1}{y^{\ell \gamma / \alpha}} d y,
$$

and

$$
V_{-}(\theta, 0)=\int_{0}^{\infty} P^{\delta}\left(\left(\theta_{2} \Delta S_{2}\right)_{-}^{\beta} \geq y\right) d y=\int_{1}^{\infty} \frac{1}{2^{\delta}} \frac{1}{y^{\delta \delta / \beta}} d y .
$$

If we have $\alpha / \gamma>\beta / \delta$ then there is $\ell>0$ such that

$$
\frac{\ell \gamma}{\alpha}<1<\frac{\ell \delta}{\beta},
$$

which entails $V_{-}(\theta, 0)<\infty$ (so indeed $\left.\theta \in \mathcal{A}(0)\right)$ and $V_{+}(\theta, 0)=\infty, n \rightarrow \infty$ so the optimization problem becomes ill-posed.

One may wonder whether this phenomenon could be ruled out by restricting the set of strategies e.g. to bounded ones. The answer is no. Considering $\theta_{1}(n):=0, \theta_{2}(n):=\min \left\{\theta_{2}, n\right\}$ for $n \in \mathbb{N}$ we obtain easily that $\theta(n) \in \Psi(0)$ and $V_{+}(\theta(n), 0) \rightarrow \infty, V_{-}(\theta(n), 0) \rightarrow V_{-}(\theta, 0)<\infty$ by monotone convergence, which shows that we still have

$$
\sup _{\psi} V(\psi, 0)=\infty
$$

where $\psi$ ranges over the family of bounded strategies in $\mathcal{A}(0)$ only. This shows that the illposedness phenomenon is not just a pathology but it comes from the multi-periodic setting: one may use the information available at time 1 when choosing the investment strategy $\theta_{2}$.

We mention another case of ill-posedness which is present already in one-step models, as noticed in [49] and [12], see also Example 2.10 above. We slightly change the previous setting: we allow general distortions, assuming only that $w_{+}(y)>0$ for $y>0$. The market is defined by $S_{0}=0, \Delta S_{1}= \pm 1$ with probabilities $p, 1-p$ for some $0<p<1$ and $\mathcal{F}_{0}, \mathcal{F}_{1}$ the natural filtration of $S_{0}, S_{1}$. Now the set $\mathcal{A}(z)$ can be identified with $\mathbb{R}$. Take $z=B=0$ and $\theta_{1}(n):=n$, $n \in \mathbb{N}$, then $V_{+}(\theta(n), 0)=w_{+}(p) n^{\alpha}$ and $V_{-}(\theta(n), 0)=w_{-}(1-p) n^{\beta}$. If $\alpha>\beta$ then, whatever $w_{+}, w_{-}$are, we have $V(\theta(n), 0) \rightarrow \infty, n \rightarrow \infty$. Hence, in order to get a well-posed problem one needs to have $\alpha \leq \beta$, as already observed in [12] and [49].

We add a comment on the case $\alpha=\beta$ : whatever $w_{+}, w_{-}$are, we may easily choose $p$ such that the problem becomes ill-posed: indeed, it happens if $w_{+}(p)>w_{-}(1-p)$ (note the assumed continuity of $w_{ \pm}$).

Since it would be difficult to dismiss the simple models of this section based on economic grounds we are led to the conclusion that, in order to get a mathematically meaningful optimization problem for a reasonably wide range of price processes, one needs to assume both

$$
\begin{equation*}
\alpha<\beta \quad \text { and } \quad \alpha / \gamma \leq \beta / \delta . \tag{102}
\end{equation*}
$$

We conjecture that (102) (with < instead of $\leq$ ) is sufficient for well-posedness and for the existence of optimisers but this is still an open problem.

In the following section we propose an easily verifiable sufficient condition. Roughly speaking, what we require is $\alpha / \gamma<\beta$, see (107) below. This is stronger than (102) but it is still reasonably general. If $w_{-}(p)=p$ (i.e. $\delta=1$, no distortion on loss probabilities) then (107) below is essentially sharp, as the present section highlights. See Theorem 4.16 in Section 4 for another partial result.

### 3.5 Unbounded utilities in CPT

Basically, we will require below that $u_{ \pm}$are comparable to power functions at infinity and that $w_{ \pm}$do likewise in the neighborhood of 0 . We stress that no concavity or monotonicity assumptions are made on $u_{ \pm}$, unlike in all the related papers.

Assumption 3.11. We assume that $u_{ \pm}: \mathbb{R}_{+} \rightarrow \mathbb{R}_{+}$and $w_{ \pm}:[0,1] \rightarrow[0,1]$ are continuous functions such that $u_{ \pm}(0)=0, w_{ \pm}(0)=0$ and $w_{ \pm}(1)=1$. They satisfy

$$
\begin{align*}
u_{+}(x) & \leq k_{+}\left(x^{\alpha}+1\right)  \tag{103}\\
k_{-}\left(x^{\beta}-1\right) & \leq u_{-}(x)  \tag{104}\\
w_{+}(p) & \leq g_{+} p^{\gamma}  \tag{105}\\
w_{-}(p) & \geq g_{-} p^{\delta} \tag{106}
\end{align*}
$$

with $0<\alpha, \beta, \gamma, \delta, k_{ \pm}, g_{ \pm}$fixed constants.
Assumption 3.12. Let $\delta \leq 1$ and let either

$$
\begin{equation*}
\gamma \leq 1, \quad \frac{\alpha}{\gamma}<\beta \tag{107}
\end{equation*}
$$

or

$$
\begin{equation*}
\gamma>1, \quad \alpha<\beta \tag{108}
\end{equation*}
$$

hold.
Assumption 3.12 allows us to fix $\lambda>0$ such that $\lambda \gamma>1$ and $\lambda \alpha<\beta$. If $\gamma>1$ then we may and will choose $\lambda=1$. Note that if $\gamma>1$ then $\alpha / \gamma<\beta$ is implied by $\alpha<\beta$. Similarly, for $\gamma \leq 1, \alpha<\beta$ is implied by $\alpha / \gamma<\beta$. In other words, under either (107) or (108), both $\alpha<\beta$ and $\alpha / \gamma<\beta$ hold.
Remark 3.13. As already described at the beginning of the present chapter, according to CPT one should have $\alpha, \beta, \gamma, \delta \leq 1$ (since $u_{ \pm}$should be concave and $w_{ \pm}(p)$ should exceed $p$ for small $p$ ). Our results, however, apply to a more general setting, as reflected by Assumption 3.11.

Condition (107) has already been mentioned in the previous section. It has a rather straightforward interpretation: the investor takes losses more seriously than gains. The distortion function $w_{+}$, being majorized by a power function of order $\gamma$, exaggerates the probabilities of rare events. In particular, the probability of large portfolio returns is exaggerated. In this way, for large portfolio values, the distortion counteracts the risk-aversion expressed by $u_{+}$, which is majorized by a concave power function of order $\alpha$. These observations explain the meaning of the term $\alpha / \gamma$ in (107) as "risk aversion of the agent on large gains modulated by his/her distortion function". Note that the agent will have a maximal risk aversion in the modified sense if either $\alpha$ is high, i.e. close to 1 or $\gamma$ is low i.e. close to 0 (for small value of $\gamma$ the agent distorts a lot the probability of rare events and, in particular, of large gains). Thus in (107) we stipulate that this modulated risk-aversion parameter should still be outbalanced by the loss aversion of the investor (as represented by parameter $\beta$ coming from (104)).

A similar interpretation for the term $\beta / \delta$ in (102) can be given. The case (108) can also easily be explained if the agent is "pessimistic" (or very cautious) and underestimates the probabilities of rare events, say, large gains (i.e. $\gamma>1$ ). We also note that the functions in Example 3.9 satisfy Assumption 3.11.
Assumption 3.14. Let $B \in \mathcal{W}$ with $b \leq B$ for some $b \in \mathbb{R}$.
Remark 3.15. One can weaken this assumption to $B \in \mathcal{W}$ and to the existence of $y \in \mathbb{R}$, $\psi \in \Phi$ such that

$$
X_{T}^{y, \psi} \leq B
$$

just like in the case of expected utility, see Remark 2.24 above.
The main result of the present chapter is the following.
Theorem 3.16. Under Assumptions 2.19, 3.11, 3.12 and 3.14 , for all $z \in \mathbb{R}$,

$$
\sup _{\theta \in \mathcal{A}(z)} V(\theta, z)<\infty
$$

If, furthermore, Assumption 3.2 holds and $V_{-}(z-B)<\infty$ then there exists $\theta^{*} \in \mathcal{A}(z)$ with

$$
-\infty<\sup _{\theta \in \mathcal{A}(z)} V(\theta, z)=V\left(\theta^{*}, z\right)
$$

Lemma 3.17. Let Assumptions 3.11 and 3.12 hold. There exist constants $\tilde{k}_{ \pm}>0$, such that for all $z \in \mathbb{R}$ and $\theta \in \Phi$ :

$$
\begin{aligned}
& V_{+}(\theta, z) \leq \tilde{k}_{+} E\left(1+\left(\left[X_{T}^{z, \theta}-B\right]_{+}\right)^{\lambda \alpha}\right), \\
& V_{-}(\theta, z) \geq \tilde{k}_{-}\left(E\left(\left[X_{T}^{z, \theta}-B\right]_{-}\right)^{\beta}-1\right) .
\end{aligned}
$$

It follows that, for $\theta \in \mathcal{A}(z)$,

$$
\begin{equation*}
V(\theta, z) \leq E u\left(X_{T}^{z, \theta}-B\right), \tag{109}
\end{equation*}
$$

where

$$
\begin{equation*}
u(x):=k+k x^{\alpha \lambda}, x \geq 0, \quad u(x):=k-\tilde{k}_{-}|x|^{\beta}, x<0, \tag{110}
\end{equation*}
$$

and $k:=\max \left\{\tilde{k}_{+}, \tilde{k}_{-}\right\}$.
Proof. First we assume $\gamma \leq 1$. We get, using (105) and Markov's inequality:

$$
\begin{equation*}
V^{+}(\theta, z) \leq 1+g_{+} \int_{1}^{\infty} \frac{E^{\gamma}\left(u_{+}^{\lambda}\left(\left[X_{T}^{z, \theta}-B\right]_{+}\right)\right)}{y^{\lambda \gamma}} d y \tag{111}
\end{equation*}
$$

Evaluating the integral and using (103) we continue the estimation as

$$
\begin{aligned}
V^{+}(\theta, z) & \leq 1+\frac{g_{+}}{\lambda \gamma-1} E^{\gamma}\left(2^{\lambda-1} k_{+}^{\lambda}\left[X_{T}^{z, \theta}-B\right]_{+}^{\lambda \alpha}+2^{\lambda-1} k_{+}^{\lambda}\right) \\
& \leq C_{1}+C_{2} E\left[X_{T}^{z, \theta}-B\right]_{+}^{\lambda \alpha},
\end{aligned}
$$

for some $C_{1}, C_{2}>0$, using the rough estimate $x^{\gamma} \leq x+1, x \geq 0$.
If $\gamma>1$ then $w_{+}(P(\cdot)) \leq g_{+} P(\cdot)$ and hence

$$
V^{+}(\theta, z) \leq g_{+}\left(k_{+}+k_{+} E\left[X_{T}^{z, \theta}-B\right]_{+}^{\alpha}\right),
$$

recall that now $\lambda \alpha=\lambda$. Note that, by (104), (106) and $\delta \leq 1$ (see Assumption 3.12),

$$
\begin{aligned}
V^{-}(\theta, z) & \geq g_{-} \int_{0}^{\infty} P\left(u_{-}\left(\left[X_{T}^{z, \theta}-B\right]_{-}\right) \geq y\right) d y \\
=g_{-} E u_{-}\left(\left[X_{T}^{z, \theta}-B\right]_{-}\right) & \geq g_{-} k_{-} E\left[X_{T}^{z, \theta}-B\right]_{-}^{\beta}-g_{-} k_{-} .
\end{aligned}
$$

Choosing $\tilde{k}_{ \pm}$as these estimates suggest we get $V(\theta, z) \leq E u\left(X_{T}^{z, \theta}-B\right)$, as claimed.
Proof of well-posedness in Theorem 3.16. Note that the $u$ defined in Lemma 3.17 satisfies Assumption 2.13. Clearly, $\Phi(z)$ of Chapter 2 corresponding to $u$ contains $\mathcal{A}(z)$, hence

$$
\sup _{\theta \in \mathcal{A}(z)} V(\theta, z) \leq \sup _{\theta \in \Phi(z)} E u\left(X_{T}^{z, \theta}-B\right)<\infty
$$

since Corollary 2.20 applies to the optimization problem involving $u$ and the given market model.

For the subsequent developments we need to extend and refine the arguments of Corollary 2.20. The next, innocent-looking lemma is the heart of the matter. Let $U_{t}, t=T, \ldots, 0$ be defined as in the proof of Corollary 2.20 for the function $u$ of (110).
Lemma 3.18. Let Assumptions 2.19, 3.11, 3.12 and 3.14 be in force. Fix $c \in \mathbb{R}$ and $\iota, o$ satisfying $\lambda \alpha<\iota<o<\beta$. Let $H_{t} \in \Xi_{t}^{1}$ with $E\left|H_{t}\right|^{\circ}<\infty$ and let $\theta_{t+1} \in \widehat{\Xi}_{t}^{d}$ (recall the definition of $\widehat{\Xi}_{t}^{d}$ from Section 1.2) such that

$$
\begin{equation*}
E U_{t+1}\left(H_{t}+\theta_{t+1} \Delta S_{t+1}\right) \geq c \tag{112}
\end{equation*}
$$

holds. Then there exists a constant $K_{t}$ such that

$$
E\left|\theta_{t+1}\right|^{L} \leq K_{t}\left[E\left|H_{t}\right|^{o}+1\right],
$$

where $K_{t}$ does not depend either on $H_{t}$ or on $\theta_{t+1}$.

Proof. Fix $\chi$ satisfying $\lambda \alpha<\chi<\iota$. By Assumption 2.19, the event

$$
A:=\left\{\theta_{t+1} \Delta S_{t+1} \leq-\nu_{t}\left|\theta_{t+1}\right|\right\}
$$

satisfies $P\left(A \mid \mathcal{F}_{t}\right) \geq \kappa_{t}$ with $1 / \kappa_{t} \in \mathcal{W}$. Define also

$$
\begin{equation*}
F:=\left\{\frac{\left|\theta_{t+1}\right| \nu_{t}}{2} \geq\left|H_{t}\right|+N_{t}\right\} \in \mathcal{F}_{t} \tag{113}
\end{equation*}
$$

recall $\nu_{t}$ from Proposition 1.6 and $N_{t}$ from the proof of Corollary 2.20.
By estimations of Lemma 2.31 and Corollary 2.20,

$$
\begin{align*}
E\left[U_{t+1}\left(H_{t}+\theta_{t+1} \Delta S_{t+1}\right) \mid \mathcal{F}_{t}\right] \leq & E\left[C_{t+1}\left(1+\left|H_{t}\right|^{\lambda \alpha}+\left|\theta_{t+1}\right|^{\lambda \alpha}\right) 1_{[A \cap F]} \mid \mathcal{F}_{t}\right]+ \\
& E\left[\left.U_{t+1}\left(-N_{t}-\frac{\nu_{t}\left|\theta_{t+1}\right|}{2}\right) 1_{A \cap F} \right\rvert\, \mathcal{F}_{t}\right], \tag{114}
\end{align*}
$$

with $C_{t+1} \in \mathcal{W}$. Using (71) and (72), the last term can be estimated as

$$
\begin{align*}
& E\left[\left.\left(\frac{N_{t}+\nu_{t}\left|\theta_{t+1}\right| / 2}{N_{t}}\right)^{\beta} U_{t+1}\left(-N_{t}\right) 1_{A}+C\left(\frac{N_{t}+\nu_{t}\left|\theta_{t+1}\right| / 2}{N_{t}}\right)^{\beta} 1_{A} \right\rvert\, \mathcal{F}_{t}\right] 1_{F} \leq \\
& -\left(\frac{N_{t}+\nu_{t}\left|\theta_{t+1}\right| / 2}{N_{t}}\right)^{\beta} \frac{\kappa_{t}}{2} 1_{F} \leq-\left(\frac{\nu_{t}\left|\theta_{t+1}\right|}{2}\right)^{\beta} \tilde{\kappa}_{t} 1_{F}, \tag{115}
\end{align*}
$$

where we set $\tilde{\kappa}_{t}:=\kappa_{t} / 2 N_{t}^{\beta}$. We thus get

$$
\begin{align*}
c \leq & E U_{t+1}\left(H_{t}+\theta_{t+1} \Delta S_{t+1}\right) \leq E C_{t+1}\left(1+\left|H_{t}\right|^{\lambda \alpha}+\left|\theta_{t+1}\right|^{\lambda \alpha}\right)- \\
& E\left(1_{F}\left(\frac{\left|\theta_{t+1}\right| \nu_{t}}{2}\right)^{\beta} \tilde{\kappa}_{t}\right) . \tag{116}
\end{align*}
$$

We now push further the latter estimation.
We estimate, using the Hölder inequality for $p=\beta / o$ and its conjugate $q$,

$$
E\left(1_{F}\left(\frac{\left|\theta_{t+1}\right| \nu_{t}}{2}\right)^{\beta} \tilde{\kappa}_{t}\right) \geq \frac{E^{p}\left(1_{F}\left(\left|\theta_{t+1}\right| \nu_{t} / 2\right)^{o} \tilde{\kappa}_{t}^{1 / p} \tilde{\kappa}_{t}^{-1 / p}\right)}{E^{p / q}\left(\tilde{\kappa}_{t}^{-q / p}\right)} .
$$

Now let us note the trivial fact that for random variables $X, Y \geq 0$ such that $E Y^{\circ} \geq 2 E X^{o}$ one has $E\left[1_{\{Y \geq X\}} Y^{o}\right] \geq \frac{1}{2} E Y^{o}$.

It follows that if

$$
\begin{equation*}
E\left(\left|\theta_{t+1}\right| \nu_{t} / 2\right)^{o} \geq 2 E\left(\left|H_{t}\right|+N_{t}\right)^{o} \tag{117}
\end{equation*}
$$

holds true then, applying the trivial $x \leq x^{p}+1, x \geq 0$,

$$
\begin{aligned}
\frac{E^{p}\left(1_{F}\left(\left|\theta_{t+1}\right| \nu_{t} / 2\right)^{o}\right)}{E^{p / q}\left(\tilde{\kappa}_{t}^{-q / p}\right)} & \geq \frac{E^{p}\left(\left(\left|\theta_{t+1}\right| \nu_{t} / 2\right)^{o}\right)}{2^{p} E^{p / q}\left(\tilde{\kappa}_{t}^{-q / p}\right)} \\
& \geq \frac{E\left(\left|\theta_{t+1}\right| \nu_{t} / 2\right)^{o}-1}{2^{p} E^{p / q}\left(\tilde{\kappa}_{t}^{-q / p}\right)}=c_{1} E\left(\left|\theta_{t+1}\right| \nu_{t}\right)^{o}-c_{2}
\end{aligned}
$$

with suitable $c_{1}, c_{2}>0$. Using again Hölder's inequality with $p=o / \iota$ and its conjugate $q$,

$$
\begin{equation*}
E\left(\left|\theta_{t+1}\right| \nu_{t}\right)^{o} \geq \frac{E^{p}\left|\theta_{t+1}\right|^{c}}{E^{p / q}\left(\nu_{t}^{-L q}\right)} \geq \frac{E\left|\theta_{t+1}\right|^{\iota}-1}{E^{p / q}\left(\nu_{t}^{-\iota q}\right)} . \tag{118}
\end{equation*}
$$

With suitable $c_{1}^{\prime}, c_{2}^{\prime}>0$, we get, whenever (117) holds, that

$$
\begin{equation*}
E\left(1_{F}\left(\frac{\left|\theta_{t+1}\right| \nu_{t}}{2}\right)^{\beta} \tilde{\kappa}_{t}\right) \geq c_{1}^{\prime} E\left|\theta_{t+1}\right|^{\iota}-c_{2}^{\prime} \tag{119}
\end{equation*}
$$

Estimate also, with $p:=\chi /(\lambda \alpha)$ and its conjugate $q$,

$$
\begin{align*}
E\left(C_{t+1}\left(1+\left|H_{t}\right|^{\lambda \alpha}+\left|\theta_{t+1}\right|^{\lambda \alpha}\right)\right) & \leq E^{1 / q}\left[C_{t+1}^{q}\right]\left[1+E^{1 / p}\left|H_{t}\right|^{\chi}+E^{1 / p}\left|\theta_{t+1}\right|^{\chi}\right] \\
& \leq E^{1 / q}\left[C_{t+1}^{q}\right]\left[3+E\left|H_{t}\right|^{\chi}+E\left|\theta_{t+1}\right|^{\chi}\right] \\
& \leq \tilde{c}\left[1+E\left|H_{t}\right|^{o}+E\left|\theta_{t+1}\right|^{\chi}\right], \tag{120}
\end{align*}
$$

with some $\tilde{c}>0$, using that $x^{1 / p} \leq x+1, x^{\chi} \leq x^{o}+1$. Furthermore, Jensen's inequality gives

$$
E\left|\theta_{t+1}\right|^{\chi} \leq E^{\chi / \iota}\left|\theta_{t+1}\right|^{\chi} .
$$

Clearly, whenever

$$
\begin{equation*}
\left(E\left|\theta_{t+1}\right|^{\iota}\right)^{1-\chi / \iota} \geq \frac{2 \tilde{c}}{c_{1}^{\prime}} \tag{121}
\end{equation*}
$$

one also has

$$
\begin{equation*}
\tilde{c} E\left|\theta_{t+1}\right|^{\chi} \leq \frac{c_{1}^{\prime}}{2} E\left|\theta_{t+1}\right|^{\nu} . \tag{122}
\end{equation*}
$$

Finally, consider the condition

$$
\begin{equation*}
\frac{c_{1}^{\prime}}{2} E\left|\theta_{t+1}\right|^{\iota} \geq \tilde{c}\left[1+E\left|H_{t}\right|^{o}\right]+\left(c_{2}^{\prime}-c+1\right) \tag{123}
\end{equation*}
$$

It is easy to see that we can find some $K_{t}$, large enough, such that

$$
\begin{equation*}
E\left|\theta_{t+1}\right|^{\iota} \geq K_{t}\left[E\left|H_{t}\right|^{o}+1\right] \tag{124}
\end{equation*}
$$

implies (117) (recall (118)), (121) and (123). In this case we have, from (116), (120), (122), (119) and (123),

$$
\begin{aligned}
E U_{t+1}\left(H_{t}+\theta_{t+1} \Delta S_{t+1}\right) & \leq \tilde{c}\left[1+E\left|H_{t}\right|^{o}\right]+\frac{c_{1}^{\prime}}{2} E\left|\theta_{t+1}\right|^{l} \\
-c_{1}^{\prime} E\left|\theta_{t+1}\right|^{\iota}+c_{2}^{\prime} & \leq c-1<c .
\end{aligned}
$$

This shows that (112) cannot hold when (124) does so the statement of this lemma follows.
Lemma 3.19. Let Assumptions 2.19, 3.11, 3.12 and 3.14 be in force. Fix $c \in \mathbb{R}$ and $\tau$ with $\lambda \alpha<\tau<\beta$. Then there exist constants $G_{t}, t=0, \ldots, T-1$ such that

$$
E\left|\theta_{t+1}\right|^{\tau} \leq G_{t}\left[E|z|^{\beta}+1\right] \text { for } t=0, \ldots, T-1
$$

for any $\theta \in \mathcal{A}(z) \cap \widehat{\Phi}$ satisfying

$$
E u\left(X_{T}^{z, \theta}-B\right) \geq c,
$$

where the constants $G_{t}, t=0, \ldots, T-1$ do not depend either on $z$ or on $\theta$.
Proof. Take $\tau=: \alpha_{T}<\alpha_{T-1}<\ldots<\alpha_{1}<\alpha_{0}:=\beta$. We first prove, by induction on $t$, that $H_{t}:=z+\sum_{j=1}^{t} \theta_{j} \Delta S_{j}, t \geq 0$ satisfy

$$
E\left|H_{t}\right|^{\alpha_{t}} \leq O_{t}\left[|z|^{\beta}+1\right],
$$

for suitable constants $O_{t}>0$. For $t=0$ this is trivial. Assuming it for $t$ we will show it for $t+1$. We first remark that

$$
c \leq E u\left(X_{T}^{z, \theta}-B\right) \leq E U_{t+1}\left(X_{t}^{z, \theta}+\theta_{t+1} \Delta S_{t+1}\right)
$$

and that, by the induction hypothesis, $E\left|H_{t}\right|^{\alpha_{t}}<\infty$ holds.

Thus Lemma 3.18 applies with the choice $\iota:=\left(\alpha_{t+1}+\alpha_{t}\right) / 2$ and $o:=\alpha_{t}$, and we can estimate, using Hölder's inequality with $p:=\iota / \alpha_{t+1}$ (and its conjugate number $q$ ), plugging in the induction hypothesis:

$$
\begin{aligned}
E\left|H_{t+1}\right|^{\alpha_{t+1}} & =E\left|H_{t}+\theta_{t+1} \Delta S_{t+1}\right|^{\alpha_{t+1}} \\
& \leq 2^{\alpha_{t+1}}\left[E\left|H_{t}\right|^{\alpha_{t+1}}+E\left|\theta_{t+1} \Delta S_{t+1}\right|^{\alpha_{t+1}}\right] \\
& \leq 2^{\alpha_{t+1}}\left[E\left|H_{t}\right|^{\alpha_{t}}+1+E^{1 / p}\left|\theta_{t+1}\right|^{\kappa} E^{1 / q}\left|\Delta S_{t+1}\right|^{q \alpha_{t+1}}\right] \\
& \leq \tilde{C}\left[E\left|H_{t}\right|^{\alpha_{t}}+1\right]+\tilde{C}\left(E\left|\theta_{t+1}\right|^{\kappa}+1\right) \\
& \leq \tilde{C}\left[E\left|H_{t}\right|^{\alpha_{t}}+1\right]+\tilde{C}\left(K_{t}\left(E\left|H_{t}\right|^{\alpha_{t}}+1\right)+1\right) \\
& \leq\left(\tilde{C}+\tilde{C} K_{t}\right) O_{t}\left(|z|^{\beta}+1\right)+\tilde{C}+\tilde{C} K_{t}+\tilde{C},
\end{aligned}
$$

with $\tilde{C}:=2^{\alpha_{t+1}}\left[1+E^{1 / q}\left|\Delta S_{t+1}\right|^{q \alpha_{t+1}}\right]$, this proves the induction hypothesis for $t+1$.
Now let us observe that, by Lemma 3.18 (with $\iota=\alpha_{t+1}, o=\alpha_{t}$ ), with some constants $K_{t}^{\prime}$,

$$
\begin{aligned}
E\left|\theta_{t+1}\right|^{\tau} & \leq E\left|\theta_{t+1}\right|^{\alpha_{t+1}}+1 \\
& \leq K_{t}^{\prime}\left[E\left|H_{t}\right|^{\alpha_{t}}+1\right]+1 \leq K_{t}^{\prime}\left[O_{t}\left(|z|^{\beta}+1\right)+1\right]+1
\end{aligned}
$$

completing the proof by setting $G_{t}:=K_{t}^{\prime} O_{t}+K_{t}^{\prime}+1$.
Proof of Theorem 3.16. Take $\theta(j) \in \mathcal{A}(z) \cap \widehat{\Phi}, j \in \mathbb{N}$ such that

$$
\lim _{j \rightarrow \infty} V(\theta(j), z)=\sup _{\theta \in \mathcal{A}(z)} V(\theta, z)
$$

We can fix $c$ such that $-\infty<c<\inf _{j} V(\theta(j), z)$. By Lemma 3.17 this implies that, for all $j$,

$$
E u\left(X_{T}^{z, \theta(j)}-B\right)>c
$$

Apply Lemma 3.19 for some $\tau$ such that $\lambda \alpha<\tau<\beta$ to get

$$
\sup _{j, t} E\left|\theta_{t}(j)\right|^{\tau}<\infty
$$

It follows that the sequence of $T(d+L)$-dimensional random variables

$$
I_{j}:=\left(Y_{1}, \ldots, Y_{T}, \theta_{1}(j), \ldots, \theta_{T}(j)\right)
$$

are bounded in $L^{\tau}$ so the sequence of the laws of $I_{j}$ is tight, admitting a subsequence (which we continue to denote by $j$ ) weakly convergent to some probability law $\mu$ on $\mathcal{B}\left(\mathbb{R}^{T(d+L)}\right)$. Now, following verbatim the proof of Theorem 3.4, we get $\theta_{i}^{*}, i=1, \ldots, T$ such that $\mu=$ $\operatorname{Law}\left(Y_{1}, \ldots, Y_{T}, \theta_{1}^{*}, \ldots, \theta_{T}^{*}\right)$.

We will now show that

$$
\begin{equation*}
V\left(\theta^{*}, z\right) \geq \lim \sup _{j \rightarrow \infty} V(\theta(j), z) \tag{125}
\end{equation*}
$$

which will complete the proof.
Indeed, $H_{j}:=z+\sum_{t=1}^{T^{P}} \theta_{t}(j) \Delta S_{t}-B$ clearly converges in law to $H:=z+\sum_{t=1}^{T} \theta_{t}^{*} \Delta S_{t}-B$, $j \rightarrow \infty$ (note that $B$ is a continuous function of $Y_{1}, \ldots, Y_{T}$ ). By continuity of $u_{+}, u_{-}$also $u_{ \pm}\left(\left[H_{j}\right]_{ \pm}\right)$tend to $u_{ \pm}\left([H]_{ \pm}\right)$in law which entails that $P\left(u_{ \pm}\left(\left[H_{j}\right]_{ \pm}\right) \geq y\right) \rightarrow P\left(u_{ \pm}\left([H]_{ \pm}\right) \geq y\right)$ for all $y$ outside a countable set (the points of discontinuities of the cumulative distribution functions of $u_{ \pm}\left([H]_{ \pm}\right)$).

It suffices thus to find a measurable function $h(y)$ with $w_{+}\left(P\left(u_{+}\left[H_{j}\right]_{+} \geq y\right)\right) \leq h(y), j \geq 1$ and $\int_{0}^{\infty} h(y) d y<\infty$ and then Fatou's lemma will imply (125). If $\gamma \leq 1$ we get, just like in Lemma 3.17, using Markov's inequality, Assumption 3.14, (103) and (105), for $y \geq 1$ :

$$
\begin{aligned}
w_{+}\left(P\left(u_{+}\left[H_{j}\right]_{+} \geq y\right)\right) & \leq C \frac{1+|z|^{\lambda \alpha}+\sum_{t=1}^{T} E\left(\left|\theta_{t}(j)\right|^{\lambda \alpha}\left|\Delta S_{t}\right|^{\lambda \alpha}\right)}{y^{\lambda \gamma}} \\
& \leq \frac{C}{y^{\lambda \gamma}}\left(1+|z|^{\lambda \alpha}+\sum_{t=1}^{T} E^{1 / p}\left|\theta_{t}(j)\right|^{\tau} E^{1 / q}\left|\Delta S_{t}\right|^{\lambda \alpha q}\right)
\end{aligned}
$$

for some constant $C>0$, using Hölder's inequality with $p:=\tau /(\lambda \alpha)$ and its conjugate $q$ (recall that $\Delta S_{t} \in \mathcal{W}$ ). We know from the construction that $\sup _{j, t} E\left|\theta_{t}(j)\right|^{\tau}<\infty$. Thus we can find some constant $C^{\prime}>0$ such that $w_{+}\left(P\left(u_{+}\left[H_{j}\right]_{+} \geq y\right)\right) \leq C^{\prime} / y^{\lambda \gamma}$, for all $j$. Now trivially $w_{+}\left(P\left(u_{+}\left[H_{j}\right]_{+} \geq y\right)\right) \leq w_{+}(1)=1$ for $0 \leq y \leq 1$. Setting $h(y):=1$ for $0 \leq y \leq 1$ and $h(y):=C^{\prime} / y^{\lambda \gamma}$ for $y>1$, we conclude since $\lambda \gamma>1$ and thus $1 / y^{\lambda \gamma}$ is integrable on $[1, \infty)$. The case of $\gamma>1$ follows similarly.

### 3.6 Examples

In this section we present some classical market models where Assumptions 2.19 and 3.2 hold true and hence Theorems 3.4 and 3.7 apply.

Example 3.20. Fix $d \leq L \leq N$. Take $Y_{0} \in \mathbb{R}^{L}$ constant and define $Y_{t}$ by the difference equation

$$
Y_{t+1}-Y_{t}=\mu\left(Y_{t}\right)+\rho\left(Y_{t}\right) Z_{t+1}
$$

where $\mu: \mathbb{R}^{L} \rightarrow \mathbb{R}^{L}$ and $\rho: \mathbb{R}^{L} \rightarrow \mathbb{R}^{L \times N}$ are bounded and measurable. We assume that there is $h>0$ such that

$$
\begin{equation*}
v^{T} \rho(x) \rho^{T}(x) v \geq h v^{T} v, \quad v \in \mathbb{R}^{L} \tag{126}
\end{equation*}
$$

for all $x \in \mathbb{R}^{L} ; Z_{t} \in \mathcal{W}, t=1, \ldots, T$ are independent with $\operatorname{supp}\left(\operatorname{Law}\left(Z_{t}\right)\right)=\mathbb{R}^{N}$.
Thus $Y_{t}$ can be chosen to be e.g. the Euler approximation of a non-degenerate diffusion process. We may think that $Y_{t}$ represent the evolution of $L$ economic factors. Take $\mathcal{F}_{0}$ trivial and $\mathcal{F}_{t}:=\sigma\left(Z_{j}, j \leq t\right), t \geq 1$.

We claim that $Y_{t}$ satisfies Assumption 2.19 with respect to $\mathcal{F}_{t}$. Indeed, $Y_{t} \in \mathcal{W}$ is trivial and we will show that (27) holds with $\kappa_{t}, \nu_{t}$ constants.

Take $v \in \mathbb{R}^{L}$. By the Markov property of $Y$ w.r.t. $\mathcal{F}$.,

$$
P\left(v\left(Y_{t+1}-Y_{t}\right) \leq-|v| \mid \mathcal{F}_{t}\right)=P\left(v\left(Y_{t+1}-Y_{t}\right) \leq-|v| \mid Y_{t}\right) .
$$

It is thus enough to show for each $t=1, \ldots, T$ that there is $c>0$ such that for each unit vector $v \in \mathbb{R}^{L}$ and for each $x \in \mathbb{R}^{L}$

$$
P\left(v\left(\mu(x)+\rho(x) Z_{t}\right) \leq-1\right) \geq c
$$

Denoting by $m$ an upper bound for $|\mu(x)|, x \in \mathbb{R}^{L}$, we may write

$$
P\left(v\left(\mu(x)+\rho(x) Z_{t}\right) \leq-1\right) \geq P\left(v\left(\rho(x) Z_{t}\right) \leq-(m+1)\right)
$$

Here $y=v^{T} \rho(x)$ is a vector of length at least $\sqrt{h}$, hence the absolute value of one of its components is at least $\sqrt{h / N}$. Thus we have

$$
\begin{align*}
P\left(v^{T} \rho(x) Z_{t} \leq-(m+1)\right) \geq & \min \left(\min _{i, k_{i}} P\left(\sqrt{h / N} Z_{t}^{i} \leq-(m+1), k_{i}(j) Z_{t}^{j} \leq 0, j \neq i\right),\right. \\
& \left.\min _{i, k_{i}} P\left(\sqrt{h / N} Z_{t}^{i} \geq(m+1), k_{i}(j) Z_{t}^{j} \leq 0, j \neq i\right)\right) \tag{127}
\end{align*}
$$

where $i$ ranges over $1, \ldots, N$ and $k_{i}$ ranges over the (finite) set of all functions from $\{1,2, \ldots, i-$ $1, i+1, \ldots, N\}$ to $\{1,-1\}$ (representing all the possible configurations for the signs of $y^{j}, j \neq i$ ). This minimum is positive by our assumption on the support of $Z_{t}$.

Now we can take $S_{t}^{i}:=Y_{t}^{i}, i=1, \ldots, d$ for some $d \leq L$. When $L>d$, we may think that the $Y_{j}, d<j \leq L$ are not prices of some traded assets but other relevant economic variables that influence the market. It is trivial to check that Assumption 2.19 holds for $S_{t}$, too, with respect to $\mathcal{F}_{t}$.

Example 3.21. Take $\tilde{Y}_{t}:=\exp \left(Y_{t}\right)$ where $Y_{t}$ is as in the above example. Let $Z_{t}, t=1, \ldots, T$ be such that for all $\zeta>0$,

$$
E e^{\zeta\left|Z_{t}\right|}<\infty
$$

Set $S_{t}^{i}:=\tilde{Y}_{t}^{i}, i=1, \ldots, d$. We claim that Assumption 2.19 holds true for $S_{t}$ with respect to the filtration $\mathcal{F}_{t}$.

We prove this only for the case $N=L=d=1$, for simplicity. We choose $\kappa_{t}:=S_{t} / 2$. Clearly, $1 / \kappa_{t} \in \mathcal{W}$ and $\Delta S_{t} \in \mathcal{W}, t \geq 1$. It suffices to prove that $1 / P\left(S_{t+1}-S_{t} \leq-S_{t} / 2 \mid \mathcal{F}_{t}\right)$ and $1 / P\left(S_{t+1}-S_{t} \geq S_{t} / 2 \mid \mathcal{F}_{t}\right)$ belong to $\mathcal{W}$. We shall show only the second containment, the first one being similar. This amounts to checking

$$
1 / P\left(\exp \left\{Y_{t+1}-Y_{t}\right\} \geq 3 / 2 \mid Y_{t}\right) \in \mathcal{W}
$$

We may and will assume $\rho(x) \geq h>0, x \in \mathbb{R}$. Let us notice that

$$
\begin{aligned}
P\left(\exp \left\{Y_{t+1}-Y_{t}\right\} \geq 3 / 2 \mid Y_{t}\right) & =P\left(\mu\left(Y_{t}\right)+\rho\left(Y_{t}\right) Z_{t+1} \geq \ln (3 / 2) \mid Y_{t}\right) \\
& =P\left(\left.Z_{t+1} \geq \frac{\ln (3 / 2)-\mu\left(Y_{t}\right)}{\rho\left(Y_{t}\right)} \right\rvert\, Y_{t}\right) \\
& \geq P\left(Z_{t+1} \geq \frac{\ln (3 / 2)+m}{\sqrt{h}}\right)
\end{aligned}
$$

which is a deterministic positive constant, by the assumption on the support of $Z_{t+1}$. Examples 3.20 and 3.21 are pertinent, in particular, when the $Z_{t}$ are Gaussian.

We now show that, if $d \leq L<N$ then Assumption 3.2 holds for both Examples above and hence Theorems 3.4 and 3.7 apply to them.

Example 3.22. Let us consider the setting of Example 3.20 with $d \leq L<N$. This corresponds to the case when an incomplete diffusion market model has been discretized (the number of driving processes, $N$, exceeds the number $L$ of economic variables).

Let us furthermore assume that for all $t$, the law of $Z_{t}$ has a density w.r.t. the $N$ dimensional Lebesgue measure (when we say "density" from now on we will always mean density w.r.t. a Lebesgue measure of appropriate dimension) and that $\mu, \rho$ are continuous.

It is clear that in this case $Y_{t+1}=f_{t+1}\left(Y_{1}, \ldots, Y_{t}, Z_{t+1}\right)$ for some continuous function $f_{t+1}$. It remains to construct $U_{t+1}$ as required in Assumption 3.2.

We will denote by $\rho_{i}(x)$ the $i$ th row of $\rho(x), i=1, \ldots, d$. First let us notice that (126) implies that $\rho(x)$ has full rank for all $x$ and hence the $\rho_{i}(x), i=1, \ldots, d$ are linearly independent for all $x$.

It follows that the set $\left\{(\omega, w) \in \Omega \times \mathbb{R}^{N}: \rho_{i}\left(Y_{t}\right) w=0, i=1, \ldots, d,|w|=1\right\}$ has full projection on $\Omega$ and it is easily seen to be in $\mathcal{F}_{t} \otimes \mathcal{B}\left(\mathbb{R}^{N}\right)$. It follows by measurable selection (see e.g. Proposition III. 44 of [36]) that there is a $\mathcal{F}_{t}$-measurable $N$-dimensional random variable $\xi_{d+1}$ such that $\xi_{d+1}$ has unit length and it is a.s. orthogonal to $\rho_{i}\left(Y_{t}\right), i=1, \ldots, d$. Continuing in a similar way we get $\xi_{d+1}, \ldots, \xi_{N}$ such that they have unit length, they are a.s. orthogonal to each other as well as to the $\rho_{i}\left(Y_{t}\right)$. Let $\Sigma$ denote the $\mathbb{R}^{N \times N}$-valued $\mathcal{F}_{t}$-measurable random variable whose rows are $\rho_{1}\left(Y_{t}\right), \ldots, \rho_{d}\left(Y_{t}\right), \xi_{d+1}, \ldots, \xi_{N}$. Note that $\Sigma$ is a.s. nonsingular (by (126) and by construction).
$\Sigma$ is $\mathcal{F}_{t}$-measurable, so $\Sigma=\Psi\left(Z_{1}, \ldots, Z_{t}\right)$ with some (measurable) $\Psi$. For any $\left(z_{1}, \ldots, z_{t}\right) \in$ $\mathbb{R}^{t N}$, the conditional law of $\Sigma Z_{t+1}$ knowing $\left\{Z_{1}=z_{1}, \ldots, Z_{t}=z_{t}\right\}$ equals the law of the random variable $\Psi\left(z_{1}, \ldots, z_{t}\right) Z_{t+1}$. Recall that $Z_{t+1}$ has a density w.r.t. the $N$-dimensional Lebesgue measure thus $\Psi\left(z_{1}, \ldots, z_{t}\right) Z_{t+1}$, and (a.s.) the conditional law of $\Sigma Z_{t+1}$ knowing $\mathcal{F}_{t}$, has a density.

As $\left(\rho\left(Y_{t}\right) Z_{t+1}, \xi_{d+1} Z_{t+1}\right)$ is the first $d+1$ coordinates of $\Sigma Z_{t+1}$, using Fubini's theorem, the conditional law of $\left(\rho\left(Y_{t}\right) Z_{t+1}, \xi_{d+1} Z_{t+1}\right)$ knowing $\mathcal{F}_{t}$ also has a density. It follows that the random variable $\left(Y_{t+1}, \xi_{d+1} Z_{t+1}\right)$ has a $\mathcal{F}_{t}$-conditional density. This implies that $\xi_{d+1} Z_{t+1}$ has an $\mathcal{F}_{t} \vee \sigma\left(Y_{t+1}\right)$-conditional density and, a fortiori, its conditional law is atomless.

Lemma 6.17 with the choice $X:=\xi_{d+1} Z_{t+1}$ and $W:=\left(Z_{1}, \ldots, Z_{t}, Y_{t+1}\right)$ provides a uniform $U_{t+1}=G\left(\xi_{d+1} Z_{t+1}, Z_{1}, \ldots, Z_{t}, Y_{t+1}\right)$ independent of $\sigma\left(Z_{1}, \ldots, Z_{t}, Y_{t+1}\right)=\mathcal{F}_{t} \vee \sigma\left(Y_{t}\right)$ but $\mathcal{F}_{t+1^{-}}$ measurable. Clearly, the same considerations apply to Example 3.21 as well.

Example 3.23. Let $Z_{1}^{\prime}, \ldots, Z_{T}^{\prime}$ be independent $N$-dimensional random variables and let the random variables $\left(\varepsilon_{1}, \ldots, \varepsilon_{T}\right)$ be independent of the $Z^{\prime}$ with uniform law on $[0,1]^{T}$. Let $Y_{0}=$ $S_{0} \in \mathbb{R}^{d}$ and $Y_{t+1}=S_{t+1}:=f_{t+1}\left(S_{0}, \ldots, S_{t}, Z_{t+1}^{\prime}\right)$ with some continuous $f_{t+1}: \mathbb{R}^{(t+1) d+N} \rightarrow \mathbb{R}^{d}$.

$$
\begin{aligned}
& \\
& \text { Define } Z_{t}:=\left(Z_{t}^{\prime}, \varepsilon_{t}\right), t=1, \ldots, T \text {. This market model clearly satisfies Assumption } 3.2 \text { with } \\
& U_{t}:=\varepsilon_{t} \text { and with } \mathcal{F}_{0} \text { trivial, } \mathcal{F}_{t}:=\sigma\left(Z_{1}, \ldots, Z_{t}\right), t \geq 1 \text {. } \\
& \text { The interpretation of this example is that the investor randomizes his/her strategy at each } \\
& \text { time } t \text { using } \varepsilon_{t} \text { ("throwing a dice"), which is independent of the assets' driving noise } Z^{\prime} \text {. In the } \\
& \text { case of EUT such a randomization cannot increase satisfaction but when distortions appear } \\
& \text { it may indeed be advantageous to gamble, see Section } 6 \text { of [22] for a detailed discussion. }
\end{aligned}
$$


#### Abstract




$\qquad$

## 4 Continuous-time models in CPT

In the present chapter we study investors whose preferences are as in Assumption 3.11 above but this time trading is assumed continuous. The results presented here pioneer in finding an explicit necessary and sufficient condition for well-posedness on the parameters that applies to the class of distortions proposed by [101], see Example 3.9 and Assumption 3.11.

In continuous time only a very narrow class of models have been tractable up to now (complete markets and some incomplete markets of a very particular structure, see [74] and Chapter 4 of [80]). Results of the present chapter also provide the first ingredient for eventual extensions to incomplete models: the tightness estimates of Section 4.3.

Under rather stringent conditions (almost market completeness, see Assumption 4.4) we will prove the existence of an optimal strategy as well. All these results apply, in particular, to the well-known Black-Scholes model. They could be extended to other complete financial markets using techniques of [80].

The problem of optimal investment assuming a complete continuous-time market arose also in [52]. Existence results in [52], however, are provided under conditions that are not easily verifiable and whose economic interpretation is unclear. Note also that concavity of $u_{ \pm}$ is essential in [52] while we do not need this property. Some related investigations have been carried out in [17], but they use the risk-neutral (instead of the physical) probability in the definition of the objective function, which leads to a problem that is entirely different from ours.

The more realistic case of incomplete markets is yet unexplored territory. Our results on well-posedness and tightness carry over to this case without any modification but for the existence we need Assumption 4.4 below which is only slightly less than completeness. See also [74] and Chapter 4 of [80] for some other ad hoc methods which, however, cover only few models. Results covering a new class of continuous-time incomplete models appear in [77] but we will not review them in the present dissertation due to volume constraints. See also [75] for the case where $u_{+}$is bounded above.

This chapter is based on [74, 76].

### 4.1 Model description

We stay in the setting of Section 1.3 and Assumption 3.11 above. We fix a scalar-valued $\mathcal{F}_{T}$-measurable random variable $B$ which will serve as our reference point. Let us introduce the following technical assumptions.

Assumption 4.1. Let $\mathcal{M} \neq \emptyset$ and fix $Q \in \mathcal{M}$ with $\rho:=d Q / d P$.
Assumption 4.2. The cumulative distribution function (CDF) of $\rho$ under $Q$, denoted by $F_{\rho}^{Q}$, is continuous.

Assumption 4.3. Both $\rho$ and $1 / \rho$ belong to $\mathcal{W}$.
Assumption 4.4. There exists an $\mathcal{F}_{T}$-measurable random variable $U_{*}$ such that, under $P, U_{*}$ has uniform distribution on $(0,1)$ and it is independent of $\rho$. We have $B \in L^{1}(Q)$. Furthermore, $B$ and all $\sigma\left(\rho, U_{*}\right)$-measurable random variables in $L^{1}(Q)$ are replicable, i.e. they are equal to $X_{T}^{z, \phi}$ for some $z$ and $\phi \in \Phi_{a}(Q)$.

Just as in Assumption 3.2 above, the existence of $U_{*}$ means that there is enough "noise" in the market model. Such an assumption seems valid in practice. The condition of being replicable is a kind of completeness hypothesis, although for a certain type of claims only. In complete markets every $X \in L^{1}(Q)$ is replicable, by Lemma 1.11 above.

In the present chapter it is more convenient to work with a slightly different form of the functionals $V_{+}, V_{-}, V$. Define, for all random variables $X \geq 0$,

$$
V_{+}(X):=\int_{0}^{\infty} w_{+}(P(X \geq y)) d y
$$

and

$$
V_{-}(X):=\int_{0}^{\infty} w_{-}(P(X \geq y)) d y
$$

For an arbitrary random variable $X$ we set $V(X):=V_{+}\left(X^{+}\right)-V_{-}\left(X^{-}\right)$whenever $V_{-}\left(X^{-}\right)<$ $\infty$.

Under Assumption 4.1, we define $\mathcal{A}(z)$, the set of feasible strategies from initial capital $z$ as

$$
\begin{equation*}
\mathcal{A}(z):=\left\{\phi \in \Phi_{a}: V_{-}\left(\left[X_{T}^{z, \phi}-B\right]^{-}\right)<\infty\right\} \tag{128}
\end{equation*}
$$

where $\Phi_{a}=\Phi_{a}(Q)$. Note that, unlike in Chapter 3 above, this definition requires the martingale property for the process $X^{z, \phi}$. The continuous-time portfolio choice problem for an investor with CPT preferences then consists in maximising the expected distorted payoff functional $V\left(X_{T}^{z, \phi}-B\right)$ over $\mathcal{A}(z)$, that is, finding $\phi^{*} \in \mathcal{A}(z)$ satisfying

$$
\begin{equation*}
\sup _{\phi \in \mathcal{A}(z)} V\left(X_{T}^{z, \phi}-B\right)=V\left(X_{T}^{z, \phi^{*}}-B\right) \tag{129}
\end{equation*}
$$

If $V(z-B)>-\infty$ then $\mathcal{A}(z)$ is nonempty: it contains the identically zero strategy.
We end this short discussion by fixing the convention that, whenever $X$ is a random variable admitting a replicating portfolio that belongs to the set $\mathcal{A}(z)$, by abuse of language we may write " $X$ is in $\mathcal{A}(z)$ ".

### 4.2 Well-posedness

We are concerned with seeking conditions on the parameters under which the portfolio problem is a well-posed one. We fix $u_{+}(x)=x^{\alpha}, u_{-}(x)=x^{\beta}, x \in \mathbb{R}_{+}, w_{+}(p)=p^{\gamma}$ and $w_{-}(p)=p^{\delta}, p \in[0,1]$, for some $\alpha, \beta, \gamma, \delta>0$. Our results apply, with trivial modifications, to $u_{ \pm}, w_{ \pm}$as in Example 3.9. Inspired by Chapter 3, we start by proving that, as in the incomplete discrete-time multiperiod case, we need to assume $\alpha<\beta$ in order to obtain a well-posed optimisation problem.

Proposition 4.5. Under Assumptions 4.1, 4.2 and 4.4, if $\alpha>\beta$, then the problem (129) is ill-posed for any initial capital $z$ and $B=0$.

Proof. Suppose that $\alpha>\beta$ and let $U$ be the random variable given by $U:=F_{\rho}^{Q}(\rho)$. By Lemma $6.15, U$ has uniform distribution on $(0,1)$ under $Q$.

For each $n \in \mathbb{N}$, we define $Y_{n}:=n 1_{A}$, with $A:=\left\{\omega \in \Omega: U(\omega) \geq \frac{1}{2}\right\}$. Then $E_{Q}\left[Y_{n}\right]=$ $n Q(A)=\frac{n}{2}$ and

$$
V_{+}\left(Y_{n}\right)=\int_{0}^{+\infty} P\left(Y_{n}^{\alpha}>y\right)^{\gamma} d y=\int_{0}^{n^{\alpha}} P\left(Y_{n}^{\alpha}>y\right)^{\gamma} d y=n^{\alpha} P(A)^{\gamma}
$$

Now set $Z_{n}:=(n-2 z) 1_{A^{c}}$, for every $n \in \mathbb{N}$, where $A^{c}$ denotes the complement of $A$ (in $\Omega$ ). Clearly, we have that $E_{Q}\left[Z_{n}\right]=(n-2 z) Q\left(A^{c}\right)=\frac{n}{2}-z$, so $E_{Q}\left[Y_{n}\right]-E_{Q}\left[Z_{n}\right]=z$. Furthermore,

$$
V_{-}\left(Z_{n}^{+}\right)=\int_{0}^{+\infty} P\left(\left(Z_{n}^{+}\right)^{\beta}>y\right)^{\delta} d y=\left([n-2 z]^{+}\right)^{\beta} P\left(A^{c}\right)^{\delta}
$$

Finally, since $2 z \leq n_{0}$ for some $n_{0} \in \mathbb{N}$, let us define for each $n \in \mathbb{N}$ the random variable $X_{n}:=Y_{n_{0}+n}-Z_{n_{0}+n}$, which is clearly $\sigma(\rho)$-measurable and bounded from below by $2 z-n_{0}-n$. Also $X_{n}^{+}=Y_{n_{0}+n}$ and $X_{n}^{-}=Z_{n_{0}+n}$, so $E_{Q}\left[X_{n}\right]=z$. Therefore, for each $n$ the r.v. $X_{n}$ is in $\mathcal{A}(z)$, however, the sequence

$$
V\left(X_{n}\right)=\left(n_{0}+n\right)^{\alpha} P(A)^{\gamma}-\left(n_{0}+n-2 z\right)^{\beta} P\left(A^{c}\right)^{\delta}
$$

goes to infinity as $n \rightarrow+\infty$ (we recall that $P(A)>0$ because $P$ and $Q$ are equivalent measures), hence $\sup _{\phi \in \mathcal{A}(z)} V\left(X_{T}^{z, \phi}\right)=+\infty$.

Remark 4.6. Suppose that there exists an event $A \in \sigma(\rho)$, with $Q(A)=1 / 2$, for which $P(A)^{\gamma}>[1-P(A)]^{\delta}$ also holds true. Then even in the case where $\alpha=\beta$,

$$
V\left(X_{n}\right)=n^{\alpha}\left[\left(1+\frac{n_{0}}{n}\right)^{\alpha} P(A)^{\gamma}-\left(1+\frac{n_{0}-2 z}{n}\right)^{\alpha} P\left(A^{c}\right)^{\delta}\right] \rightarrow \infty, n \rightarrow \infty
$$

shows us that the optimisation problem (129) is ill-posed.
We shall now provide a very simple example of a financial market model in which such an event can be found. First, let us define the function $f(p) \triangleq p^{\gamma}-(1-p)^{\delta}$ for $p \in[0,1]$. Clearly, there exists some $\epsilon>0$ such that $f(x)>0$ for all $x \in(1-\epsilon, 1]$. On the other hand, choosing $\mu>0$ to be sufficiently large, we have that

$$
\begin{equation*}
\int_{-\infty}^{-\mu} \frac{1}{\sqrt{2 \pi}} e^{-x^{2} / 2} d x<\epsilon \tag{130}
\end{equation*}
$$

Set $T:=1$, let $W$ be a one-dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ with its natural filtration $\mathcal{F}_{t}, t \geq 0$ (augmented by $P$-zero sets). Let the price process of the risky asset be given by

$$
d S_{t}=\mu S_{t} d t+S_{t} d W_{t}, \quad S_{0}=s>0
$$

for all $t \in[0,1]$. This is a paticular example of the standard Black-Scholes model, see [15]. Thus, setting $\rho:=\exp \left\{-\mu W_{1}-\frac{\mu^{2}}{2}\right\}$, it is well-known that the probability measure $Q$ given by $d Q / d P=\rho$ is the unique element of $\mathcal{M}$, Assumptions 4.2 and 4.4 also hold true and the process $\tilde{W}=\left\{\tilde{W}_{t} ; 0 \leq t \leq 1\right\}$ defined by

$$
\widetilde{W}_{t}:=W_{t}+\mu t
$$

is a $Q$-Wiener process. Now take $A:=\left\{\widetilde{W}_{1}>0\right\}$. Clearly $Q(A)=1 / 2$ and

$$
P(A)=P\left(W_{1}+\mu>0\right)=1-P\left(W_{1} \leq-\mu\right)>1-\epsilon,
$$

which then guarantees that $P(A)^{\gamma}-[1-P(A)]^{\delta}>0$, as intended, and the problem is ill-posed provided that $\mu$ is large enough to satisfy (130).

Let us now mention the following auxiliary lemma, which will be used later. The proof is easy.
Lemma 4.7. Let $X$ be a random variable such that $X \geq 0$ a.s. and $E[X]=+\infty$. Then for each nonnegative real number $b$, there exists some $a=a(b) \in[b,+\infty)$ such that $b=E[X \wedge a]$.

In view of Proposition 4.5 and Remark 4.6 above, it is now evident that we must impose $\alpha<\beta$ as a necessary condition if we wish to have well-posedness for a reasonably large class of models. However, this is not enough to rule out ill-posedness, as shown by the next two propositions.
Proposition 4.8. Under Assumptions 4.1, 4.2, 4.3 and 4.4, if $\beta / \delta<1$, then problem (129) is ill-posed.

Proof. There exists some $\chi$ such that $\frac{\beta}{\delta}<\chi<1$ so we can choose $p \in\left(1, \frac{\delta \chi}{\beta}\right)$. Also, fix $0<\xi<\frac{\alpha}{\gamma}$. In particular, this implies that $q:=\frac{\alpha}{\gamma \xi}>1$. We define the nonnegative random variables

$$
Y:=\left\{\begin{array}{ll}
\frac{1}{U^{1 / \xi}}, & \text { if } U<\frac{1}{2}, \\
0, & \text { if } U \geq \frac{1}{2},
\end{array} \quad \text { and } \quad Z:= \begin{cases}0, & \text { if } U<\frac{1}{2} \\
\frac{1}{(1-U)^{1 / x}}, & \text { if } U \geq \frac{1}{2}\end{cases}\right.
$$

with $U$ given in the proof of the Proposition 4.5. Then, since $\frac{1}{\chi}>1$, we obtain

$$
E_{Q}[Z]=\int_{0}^{1} \frac{1}{(1-u)^{1 / \chi}} 1_{\left[\frac{1}{2}, 1\right]}(u) d u=\int_{0}^{\frac{1}{2}} \frac{1}{u^{1 / \chi}} d u=+\infty
$$

In addition, using the Hölder inequality we conclude that

$$
\begin{align*}
\int_{0}^{+\infty} P\left(Y^{\alpha}>y\right)^{\gamma} d y & =\int_{0}^{+\infty} E_{Q}^{\gamma}\left[\frac{1}{\rho} 1_{\left\{Y^{\alpha}>y\right\}}\right] d y \\
& \geq C_{1} \int_{0}^{+\infty} Q\left(Y^{\alpha}>y\right)^{\gamma q} d y \geq C_{1} \int_{2^{\frac{\alpha}{\xi}}}^{+\infty} \frac{1}{y^{\frac{\gamma q \xi}{\alpha}}} d y=+\infty \tag{131}
\end{align*}
$$

where $C_{1}:=1 / E_{Q}^{\gamma(q-1)}\left[\rho^{1 /(q-1)}\right]>0$, and the last inequality follows from the fact that

$$
Q\left(Y^{\alpha}>y\right)=Q\left(U<\frac{1}{y^{\xi / \alpha}}\right)
$$

for all $y \geq 2^{\alpha / \xi}$. Analogously,

$$
\begin{align*}
\int_{0}^{+\infty} P\left(Z^{\beta}>y\right)^{\delta} d y & \leq C_{2} \int_{0}^{+\infty} Q\left(Z^{\beta}>y\right)^{\frac{\delta}{p}} d y \\
& \leq 2^{\frac{\beta}{\chi}} C_{2}+C_{2} \int_{2^{\frac{\beta}{\chi}}}^{+\infty} \frac{1}{y^{\frac{\delta \chi}{\beta p}}} d y<+\infty \tag{132}
\end{align*}
$$

with $C_{2}:=E_{Q}^{\delta \frac{p-1}{p}}\left[\frac{1}{\rho^{1 /(p-1)}}\right]<\infty$ since $Q\left(Z^{\beta}>y\right)=Q\left(U>1-\frac{1}{y^{\chi / \beta}}\right)$ for all $y \geq 2^{\beta / \chi}$, and $\frac{\delta \chi}{\beta p}>1$.

Now we set $Y_{n}:=Y \wedge n$ for each $n \in \mathbb{N}$. It follows from Lemma 4.7 that $E_{Q}\left[Y_{n}\right]=E_{Q}\left[Z \wedge a_{n}\right]$ for some $a_{n}$. Let us define $Z_{n}:=Z \wedge a_{n}$ for every $n$. Then each $Z_{n}$ is a nonnegative random variable satisfying $V_{-}\left(Z_{n}\right) \leq V_{-}(Z)<+\infty$.

Finally, we consider the sequence of $\sigma(\rho)$-measurable random variables $X_{n}, n \in \mathbb{N}$, with $X_{n}:=Y_{n}-Z_{n}$. It is clear, by the way it was constructed, that $E_{Q}\left[X_{n}\right]=0, X_{n}^{+}=Y_{n}, X_{n}^{-}=Z_{n}$, and $X_{n} \geq-a_{n}$, for all $n$. Also, $V_{-}\left(X_{n}^{-}\right)=V_{-}\left(Z_{n}\right)<+\infty$, so $X_{n}$ is in $\mathcal{A}(z)$. Consequently, the problem is ill-posed because we get

$$
V\left(X_{n}\right)=V_{+}\left(Y_{n}\right)-V_{-}\left(Z_{n}\right) \rightarrow V_{+}(Y)-V_{-}(Z)=+\infty, n \rightarrow \infty
$$

using monotone convergence, (131) and (132).
Proposition 4.9. Under Assumptions 4.1, 4.2, 4.3 and 4.4, if $\alpha / \gamma>1$, then the problem (129) is ill-posed.

Proof. Let $U$ be as in Proposition 4.8. Denoting by $Y$ the nonnegative random variable given by

$$
Y:= \begin{cases}\frac{1}{U^{1 / \xi}}, & \text { if } U<\frac{1}{2} \\ 0, & \text { if } U \geq \frac{1}{2}\end{cases}
$$

where $\xi$ is chosen in such a way that $1<\xi<\frac{\alpha}{\gamma}$, we see that

$$
\begin{equation*}
E_{Q}[Y]=\int_{0}^{\frac{1}{2}} \frac{1}{u^{1 / \xi}} d u<+\infty \tag{133}
\end{equation*}
$$

Moreover, applying Hölder's inequality as in the proof of Proposition 4.8 with $q:=\frac{\alpha}{\gamma \xi}>1$,

$$
\begin{equation*}
V_{+}(Y)=\int_{0}^{+\infty} P\left(Y^{\alpha}>y\right)^{\gamma} d y \geq C \int_{2^{\frac{\alpha}{\xi}}}^{+\infty} \frac{1}{y^{\xi q \gamma / \alpha}} d y=+\infty \tag{134}
\end{equation*}
$$

follows, where $C=1 / E_{Q}^{\gamma(q-1)}\left[\rho^{1 /(q-1)}\right]$. Finally, for each $n \in \mathbb{N}$, we define $Y_{n}:=Y \wedge n$ and set $C_{n}:=E_{Q}\left[Y_{n}\right]$. Then, for $Z_{n}=2 C_{n} 1_{\left\{U \geq \frac{1}{2}\right\}}$, we have that $V_{-}\left(Z_{n}\right)=\left(2 C_{n}\right)^{\beta} P\left(U \geq \frac{1}{2}\right)^{\delta}$, for all $n$. We note further that $C_{n} \rightarrow E_{Q}[Y], n \rightarrow \infty$.

Now take $X_{n}:=Y_{n}-Z_{n}$. It is clear that $X_{n}^{+}=Y_{n}$ and $X_{n}^{-}=Z_{n}$. Additionally, being $\sigma(\rho)-$ measurable, bounded from below by $-2 C_{n}$, and satisfying $E_{Q}\left[X_{n}\right]=E_{Q}\left[Y_{n}\right]-2 C_{n} Q\left(U \geq \frac{1}{2}\right)=$ 0 , as well as $V_{-}\left(X_{n}^{-}\right)<+\infty$, each $X_{n}$ is in $\mathcal{A}(z)$. However, because of (133) and (134),

$$
\begin{aligned}
V\left(X_{n}\right) & =V_{+}\left(Y_{n}\right)-\left(2 C_{n}\right)^{\beta} P\left(U \geq \frac{1}{2}\right)^{\delta} \\
& \rightarrow V_{+}(Y)-\left(2 E_{Q}[Y]\right)^{\beta} P\left(U \geq \frac{1}{2}\right)^{\delta}=+\infty
\end{aligned}
$$

as $n \rightarrow+\infty$, which completes the proof.
Remark 4.10. It was proved in [82], using a different argument, that ill-posedness happens in any of the cases $\alpha=\gamma$ and $\beta=\delta$ as well. Hence the problem (129) is well-posed (in a reasonably large class of models) only if

$$
\begin{equation*}
\alpha<\beta \quad \text { and } \quad \frac{\alpha}{\gamma}<1<\frac{\beta}{\delta} \tag{135}
\end{equation*}
$$

Remark 4.11. In the particular case where $\delta=1$ (no distortion on the negative side), it follows from Proposition 4.8 that the problem is ill-posed for all $\beta \in(0,1]$. Hence, a probability distortion on losses is a necessary condition for the well-posedness of (129), which is a phenomenon in line with Theorem 3.2 of [52]. We stress, however, that under the assumptions of Theorem 2.18, and regardless of the fact that there is a probability distortion on losses or not, the optimal portfolio problem in a multiperiod discrete-time financial market model can be well-posed. Ill-posedness in our continuous-time model is due to the richness of attainable payoffs, see Remark 2.44 above.

The rest of this chapter is devoted to the proof of well-posedness and existence under (135). This requires that we present some auxiliary lemmata first.

Lemma 4.12. If $a, b, s>0$ satisfy $\frac{b}{s a}>1$ then there exists $D \geq 0$ (depending on $a, b$ and $s$ ) such that

$$
\begin{equation*}
E_{P}\left[X^{s}\right] \leq 1+D\left(\int_{0}^{\infty} P\left(X^{b}>y\right)^{a} d y\right)^{\frac{1}{a}} \tag{136}
\end{equation*}
$$

for all random variables $X \geq 0$.
Proof. Let $t>0$ be arbitrary. Then

$$
\begin{aligned}
\int_{0}^{+\infty} P\left(X^{b}>y\right)^{a} d y & =\int_{0}^{+\infty} P\left(\left(X^{s}\right)^{\frac{b}{s}}>y\right)^{a} d y \geq \int_{0}^{t^{\frac{b}{s}}} P\left(\left(X^{s}\right)^{\frac{b}{s}}>y\right)^{a} d y \\
& \geq \int_{0}^{t^{\frac{b}{s}}} P\left(\left(X^{s}\right)^{\frac{b}{s}}>t^{\frac{b}{s}}\right)^{a} d y=t^{\frac{b}{s}} P\left(X^{s}>t\right)^{a}
\end{aligned}
$$

where the last inequality follows from the inclusion $\left\{\left(X^{s}\right)^{\frac{b}{s}}>t^{\frac{b}{s}}\right\} \subseteq\left\{\left(X^{s}\right)^{\frac{b}{s}}>y\right\}$ for all $0 \leq y \leq t^{\frac{b}{3}}$. Hence

$$
\begin{gathered}
P\left(X^{s}>t\right) \leq \frac{1}{t^{\frac{b}{s a}}}\left(\int_{0}^{+\infty} P\left(X^{b}>y\right)^{a} d y\right)^{\frac{1}{a}} \\
E_{P}\left[X^{s}\right]=\int_{0}^{\infty} P\left(X^{s}>t\right) d t \leq 1+\left(\int_{0}^{\infty} P\left(X^{b}>y\right)^{a} d y\right)^{\frac{1}{a}} \int_{1}^{+\infty} \frac{1}{t^{\frac{b}{s a}}} d t
\end{gathered}
$$

and we can conclude recalling that $\frac{b}{s a}>1$ by hypothesis.

Lemma 4.13. Fix $m \in \mathbb{R}$. Let Assumptions 4.1, 4.3 be in force. Let $0<\alpha<\beta, 0<\gamma, \delta$ and $\frac{\alpha}{\gamma}<1<\frac{\beta}{\delta}$. Then there is some $\eta>0$ satisfying $\alpha<\eta<\beta$, and there exists a constant $L$ such that

$$
\begin{equation*}
\int_{0}^{+\infty} P\left(\left(X^{+}\right)^{\alpha}>y\right)^{\gamma} d y \leq L+L \int_{0}^{+\infty} P\left(\left(X^{-}\right)^{\eta}>y\right)^{\delta} d y \tag{137}
\end{equation*}
$$

for all random variables $X$ with $E_{Q}[X]=m$.
Proof. We start by noticing that the hypothesis $\alpha<\gamma$ implies that $\frac{1}{\gamma}<\frac{1}{\alpha}$. Moreover, since $\alpha<\beta$ and $\delta<\beta$, there exists $\eta$ such that $\max \{\alpha, \delta\}<\eta<\beta$. In particular, we deduce that $\frac{\eta}{\alpha}>1$, and thus $\frac{\eta}{\alpha \gamma}>\frac{1}{\gamma}$. We choose $\lambda$ such that $\frac{1}{\gamma}<\lambda<\min \left\{\frac{1}{\alpha}, \frac{\eta}{\alpha \gamma}\right\}$. Then, given that $\frac{1}{\lambda \alpha}>1$, there exists some $p$ satisfying $1<p<\frac{1}{\lambda \alpha}$. Finally, we note that $1<\frac{\eta}{\delta}$ and $\frac{\alpha \lambda \gamma}{\delta}<\frac{\eta}{\delta}$ (because $\lambda<\frac{\eta}{\alpha \gamma}$, that is, $\alpha \gamma \lambda<\eta$ ), so we can take $q$ such that $\max \left\{1, \frac{\alpha \lambda \gamma}{\delta}\right\}<q<\frac{\eta}{\delta}$.

Since for all $y \geq 1$ we have $P\left(\left(X^{+}\right)^{\alpha}>y\right) \leq \frac{E_{P}\left[\left(X^{+}\right)^{\alpha \lambda}\right]}{y^{\lambda}}$ by Markov's inequality,

$$
\int_{0}^{+\infty} P\left(\left(X^{+}\right)^{\alpha}>y\right)^{\gamma} d y \leq 1+C_{1} E_{P}^{\gamma}\left[\left(X^{+}\right)^{\alpha \lambda}\right]
$$

with $C_{1}:=\int_{1}^{+\infty} 1 / y^{\lambda \gamma} d y<\infty$ (we recall that $\lambda \gamma>1$ ). Applying Hölder's inequality yields

$$
E_{P}\left[\left(X^{+}\right)^{\alpha \lambda}\right]=E_{P}\left[\frac{1}{\rho^{1 / p}} \rho^{1 / p}\left(X^{+}\right)^{\alpha \lambda}\right] \leq C_{2} E_{P}^{\frac{1}{p}}\left[\rho\left(X^{+}\right)^{\alpha \lambda p}\right]=C_{2} E_{Q}^{\frac{1}{p}}\left[\left(X^{+}\right)^{\alpha \lambda p}\right]
$$

where $C_{2}:=E_{P}^{\frac{p-1}{p}}\left[\frac{1}{\rho^{1 /(p-1)}}\right]<\infty$ Thus, from Jensen's inequality,

$$
\begin{align*}
E_{P}^{\gamma}\left[\left(X^{+}\right)^{\alpha \lambda}\right] & \leq C_{3} E_{Q}^{\frac{\gamma}{p}}\left[\left(X^{+}\right)^{\alpha \lambda p}\right] \leq C_{3} E_{Q}^{\alpha \lambda \gamma}\left[X^{+}\right] \\
& =C_{3}\left(m+E_{Q}\left[X^{-}\right]\right)^{\alpha \lambda \gamma} \leq C_{4}+C_{4} E_{Q}^{\alpha \lambda \gamma}\left[X^{-}\right] \tag{138}
\end{align*}
$$

Now, we again use Hölder's inequality to see that $E_{Q}^{\alpha \lambda \gamma}\left[X^{-}\right] \leq C_{5} E_{P}^{\frac{\alpha \lambda \gamma}{q}}\left[\left(X^{-}\right)^{q}\right]$ (here $C_{5}:=$ $\left.E_{P}^{\alpha \lambda \gamma(q-1) / q}\left[\rho^{q /(q-1)}\right]\right)$. Moreover, since $\frac{\alpha \lambda \gamma}{q}<\delta$, we have $E_{P}^{\frac{\alpha \lambda \gamma}{q}}\left[\left(X^{-}\right)^{q}\right] \leq 1+E_{P}^{\delta}\left[\left(X^{-}\right)^{q}\right]$. Therefore, these inequalities combined with (138) yield

$$
\begin{align*}
E_{P}^{\gamma}\left[\left(X^{+}\right)^{\alpha \lambda}\right] & \leq C_{6}+C_{6} E_{P}^{\delta}\left[\left(X^{-}\right)^{q}\right] \\
& \leq C_{7}+C_{7}\left[1+D\left(\int_{0}^{+\infty} P\left(\left(X^{-}\right)^{\eta}>y\right)^{\delta} d y\right)^{\frac{1}{\delta}}\right]^{\delta} \\
& \leq C_{8}+C_{8} \int_{0}^{+\infty} P\left(\left(X^{-}\right)^{\eta}>y\right)^{\delta} d y \tag{139}
\end{align*}
$$

where we apply Lemma 4.12 above with $s=q, b=\eta, a=\delta$ (note that $\frac{\eta}{\delta q}>1$ ). Hence,

$$
\int_{0}^{+\infty} P\left(\left(X^{+}\right)^{\alpha}>y\right)^{\gamma} d y \leq L+L \int_{0}^{+\infty} P\left(\left(X^{-}\right)^{\eta}>y\right)^{\delta} d y
$$

with $L$ that does not depend on $X$ (only on the parameters), as intended.
Lemma 4.14. Let $a, b, s>0$ such that $s<a<b$ and $s \leq 1$. Then there exist $0<\zeta<1$ and $a$ constant $R \geq 0$ such that

$$
\begin{equation*}
\int_{0}^{+\infty} P\left(X^{a}>y\right)^{s} d y \leq R+R\left(\int_{0}^{+\infty} P\left(X^{b}>y\right)^{s} d y\right)^{\zeta} \tag{140}
\end{equation*}
$$

for all random variables $X \geq 0$. In particular, this implies that we have

$$
\int_{0}^{+\infty} P\left(X_{n}^{b}>y\right)^{s} d y \rightarrow+\infty, n \rightarrow \infty
$$

whenever

$$
\int_{0}^{+\infty} P\left(X_{n}^{a}>y\right)^{s} d y \rightarrow+\infty, n \rightarrow \infty
$$

for any sequence $X_{n}, n \in \mathbb{N}$ of nonnegative random variables.
Proof. We start by fixing some $\chi$ satisfying $\frac{1}{s}<\chi<\frac{b}{s a}$. We also note that, because $\chi a<\frac{b}{s}$, we can choose $\xi$ so that $\chi a<\xi<\frac{b}{s}$.

Since $\frac{b}{s \xi}>1$, we know from Lemma 4.12 that $E_{P}\left[X^{\xi}\right] \leq 1+D\left(\int_{0}^{+\infty} P\left(X^{b}>y\right)^{s} d y\right)^{1 / s}$ for some $D$. Therefore, recalling that $s \leq 1$, it follows that

$$
\begin{equation*}
E_{P}^{s}\left[X^{\xi}\right] \leq 1+C_{1} \int_{0}^{+\infty} P\left(X^{b}>y\right)^{s} d y \tag{141}
\end{equation*}
$$

with $C_{1}:=D^{s}$. Now, by Jensen's inequality (note that $\frac{a \chi}{\xi}<1$ ), we obtain

$$
\begin{equation*}
E_{P}\left[X^{a \chi}\right]=E_{P}\left[\left(X^{\xi}\right)^{\frac{a x}{\xi}}\right] \leq E_{P}^{\frac{a x}{\xi}}\left[X^{\xi}\right] . \tag{142}
\end{equation*}
$$

Moreover, using Markov's inequality, we get

$$
\begin{equation*}
\int_{0}^{+\infty} P\left(X^{a}>y\right)^{s} d y \leq 1+C_{2} E_{P}^{s}\left[X^{a \chi}\right] \tag{143}
\end{equation*}
$$

with $C_{2}:=\int_{1}^{+\infty} \frac{1}{y^{s} \chi} d y$ (note that $s \chi>1$ ).
Thus, combining inequalities (141), (142) and (143) yields

$$
\begin{aligned}
\int_{0}^{+\infty} P\left(X^{a}>y\right)^{s} d y & \leq 1+C_{2}\left(E_{P}^{s}\left[X^{\xi}\right]\right)^{\frac{a \chi}{\xi}} \\
& \leq 1+C_{2}\left(1+C_{1} \int_{0}^{+\infty} P\left(X^{b}>y\right)^{s} d y\right)^{\frac{a x}{\xi}} \\
& \leq R+R\left(\int_{0}^{+\infty} P\left(X^{b}>y\right)^{s} d y\right)^{\frac{a \chi}{\xi}}
\end{aligned}
$$

where $R$ depends only on the parameters. Setting $\zeta=\frac{a \chi}{\xi}$ completes the proof.

### 4.3 Tightness

Fix $z \in \mathbb{R}$. Define

$$
\begin{equation*}
V^{*}:=\sup _{\phi \in \mathcal{A}(z)} V\left(X_{T}^{z, \phi}-B\right) . \tag{144}
\end{equation*}
$$

The following result shows that (135) is sufficient for well-posedness; it even provides a crucial compactness property.

Theorem 4.15. Suppose that Assumptions 3.11, 4.1 and 4.3 hold, $V_{-}(z-B)<\infty$,

$$
\begin{equation*}
\alpha<\beta \quad \text { and } \quad \frac{\alpha}{\gamma}<1<\frac{\beta}{\gamma} . \tag{145}
\end{equation*}
$$

Then the optimisation problem (129) is well-posed, i.e. $V^{*}<\infty$. Furthermore, if $\phi_{n}$ is a sequence of feasible strategies with $V\left(X_{T}^{z, \phi_{n}}-B\right) \rightarrow V^{*}$ then $\sup _{n} E\left|X_{T}^{z, \phi_{n}}\right|^{\tau}<\infty$ for some $\tau>0$, a fortiori, the sequence $\operatorname{Law}\left(X_{T}^{z, \phi_{n}}\right), n \in \mathbb{N}$ is tight.

Proof. For the sake of convenience we shall henceforth set $X_{n}:=X_{T}^{z, \phi_{n}}$. By contradiction, let us suppose that the optimisation problem is ill-posed. Then we have $V_{+}\left(\left[X_{n}-B\right]^{+}\right) \rightarrow+\infty$ as $n \rightarrow \infty$. Note that, for any $X \geq 0$,

$$
V_{+}(X) \leqslant \int_{0}^{\infty} g_{+} P\left(X^{\alpha}>\left(y / k_{+}\right)-1\right)^{\gamma} d y \leq \int_{0}^{\infty} g_{+} k_{+} P\left(X^{\alpha}>t\right)^{\gamma} d t+g_{+} k_{+}
$$

using the change of variable $t:=y / k_{+}-1$. Thus it follows from Lemma 4.13 (with the choice $\left.m:=z-E_{Q}[B]\right)$ that

$$
\lim _{n \rightarrow+\infty} \int_{0}^{+\infty} P\left(\left(\left[X_{n}-B\right]^{-}\right)^{\eta}>y\right)^{\delta} d y=+\infty
$$

for some $\eta$ satisfying $\alpha<\eta<\beta$. Notice that

$$
\begin{equation*}
V_{-}(X) \geqslant \int_{0}^{\infty} g_{-} P\left(k_{-} X^{\beta}-k_{-}>y\right)^{\delta} d y \geq \int_{1}^{\infty} g_{-} k_{-} P\left(X^{\beta}>t\right)^{\delta} d t \tag{146}
\end{equation*}
$$

Consequently, we can apply Lemma 4.14 to conclude that also

$$
\lim _{n \rightarrow+\infty} V_{-}\left(\left[X_{n}-B\right]^{-}\right)=+\infty
$$

Therefore, using Lemmata 4.13, 4.14 and (146) (recalling $0<\zeta<1$ ),

$$
\begin{align*}
& V\left(X_{n}-B\right) \leq g_{+} k_{+}\left[1+L+L \int_{0}^{+\infty} P\left(\left(\left[X_{n}-B\right]^{-}\right)^{\eta}>y\right)^{\delta} d y\right]-V_{-}\left(\left[X_{n}-B\right]^{-}\right) \leq \\
& g_{+} k_{+}\left(1+L+L R+L R\left[\frac{V_{-}\left(\left[X_{n}-B\right]^{-}\right)}{g_{-} k_{-}}+1\right]^{\zeta}\right)-V_{-}\left(\left[X_{n}-B\right]^{-}\right) \rightarrow-\infty \tag{147}
\end{align*}
$$

as $n \rightarrow \infty$, which is absurd. Hence, as claimed, the problem is well-posed.
Let $\lambda>0$ be as in the proof of Lemma 4.13. We first show that

$$
\begin{equation*}
\sup _{n \in \mathbb{N}} E_{P}\left[\left(\left[X_{n}-B\right]^{+}\right)^{\alpha \lambda}\right]<+\infty \tag{148}
\end{equation*}
$$

Assume by contradiction that this supremum is $\infty$. Then we can take a subsequence $n_{l}$ such that $E_{P}\left[\left(\left[X_{n_{l}}-B\right]^{+}\right)^{\alpha \lambda}\right] \rightarrow+\infty$ as $l \rightarrow+\infty$. By (139) in the proof of Lemma 4.13, we conclude that $\int_{0}^{+\infty} P\left(\left(\left[X_{n_{l}}-B\right]^{-}\right)^{\eta}>y\right)^{\delta} d y \rightarrow+\infty, l \rightarrow \infty$. Therefore, using Lemma 4.14 we also obtain that

$$
V_{-}\left(\left[X_{n_{l}}-B\right]^{-}\right)=\int_{0}^{+\infty} P\left(\left(\left[X_{n_{l}}-B\right]^{-}\right)^{\beta}>y\right)^{\delta} d y \rightarrow \infty
$$

and hence $V\left(X_{n_{l}}-B\right) \rightarrow-\infty$ as in (147) above, which is nonsense.
Clearly, (147) implies

$$
\sup _{n \in \mathbb{N}} V_{-}\left(\left[X_{n}-B\right]^{-}\right)<+\infty
$$

as well.
Recalling that $\frac{\beta}{\delta}>1$, we can choose $\xi \in\left(1, \frac{\beta}{\delta}\right)$. Therefore $\frac{\beta}{\delta \xi}>1$, and it follows from Lemma 4.12 that there exists $D \geq 0$ such that

$$
E_{P}\left[\left(\left[X_{n}-B\right]^{-}\right)^{\xi}\right] \leq 1+D\left(V_{-}\left(\left[X_{n}-B\right]^{-}\right)\right)^{\frac{1}{\delta}}
$$

for all $n \in \mathbb{N}$, which implies that

$$
\begin{equation*}
\sup _{n \in \mathbb{N}} E_{P}\left[\left(\left[X_{n}-B\right]^{-}\right)^{\xi}\right]<+\infty \tag{149}
\end{equation*}
$$

We set $\tau=\alpha \lambda \in(0,1)$. A straightforward application of Jensen's inequality gives

$$
\begin{aligned}
E_{P}\left[\left|X_{n}-B\right|^{\tau}\right] & \leq E_{P}\left[\left(\left[X_{n}-B\right]^{+}\right)^{\tau}\right]+E_{P}\left[\left(\left[X_{n}-B\right]^{-}\right)^{\tau}\right] \\
& \leq E_{P}\left[\left(\left[X_{n}-B\right]^{+}\right)^{\tau}\right]+E_{P}^{\frac{\tau}{\xi}}\left[\left(\left[X_{n}-B\right]^{-}\right)^{\xi}\right],
\end{aligned}
$$

hence $\sup _{n \in \mathbb{N}} E_{P}\left[\left|X_{n}\right|^{\tau}\right]<+\infty$ follows from (148), (149) and $E_{P}|B|^{\tau} \leq C^{\prime} E_{Q}|B|<\infty$ (with some $C^{\prime}$, this is a consequence of $\left.d Q / d P \in \mathcal{W}\right)$.

### 4.4 A digression - back to discrete time

Using the arguments for continuous-time markets it is possible to prove a complement to Theorem 3.16: we can replace Assumption 3.12 by (135). During this brief section we get back to the discrete-time setting of Chapter 3.

Theorem 4.16. Let Assumptions 2.19, 3.2 and 3.11 be in force. Let furthermore $B \in L^{1}(P)$, $V_{-}(z-B)<\infty, u_{-}, w_{-}$non-decreasing and

$$
\alpha<\beta, \quad \frac{\alpha}{\gamma}<1<\frac{\beta}{\delta}
$$

hold. Then

$$
\sup _{\theta \in \mathcal{A}(z)} V(\theta, z)<\infty
$$

and there exists $\theta^{*} \in \mathcal{A}(z)$ with

$$
\sup _{\theta \in \mathcal{A}(z)} V(\theta, z)=V\left(\theta^{*}, z\right)
$$

Proof. Corollary 2.47 provides $Q \in \mathcal{M}$ with $d Q / d P \in L^{\infty}, d P / d Q \in \mathcal{W}$. Clearly, $B \in L^{1}(Q)$. We claim that for each $\theta \in \mathcal{A}(z):=\left\{\theta \in \Phi: V_{-}(\theta, z)<\infty\right\}$ we also have $X_{T}^{z, \theta} \in L^{1}(Q)$, i.e. the strategy $\theta$ is also in $\mathcal{A}(z)$ as defined in (128) of the present chapter (this explains why we did not seek new notation). Indeed, $V_{-}\left(\left[X_{T}^{z, \theta}-B\right]^{-}\right)<\infty$ and Lemma 4.12 with $s:=1<\beta / \delta$ imply that $E_{P}\left[\left[X_{T}^{z, \theta}-B\right]^{-}\right]<\infty$ hence also $E_{P}\left(\left[X_{T}^{z, \theta}\right]^{-}\right)<\infty$. By $d Q / d P \in L^{\infty}$ we also have $E_{Q}\left(\left[X_{T}^{z, \theta}\right]^{-}\right)<\infty$. Proposition 5.3 .2 of [54] (see also its proof) entails that $X_{t}^{z, \theta}, t=0, \ldots, T$ is a $Q$-martingale, in particular, $X_{T}^{z, \theta} \in L^{1}(Q)$.

Then the argument for proving Theorem 4.15 implies that $\sup _{n \rightarrow \infty} V_{-}\left(\left[X_{n}-B\right]^{-}\right)<\infty$ for any maximising sequence $X_{n}=X_{T}^{z, \phi(n)}$. Notice that $w_{-}(x)>0, x>0$ by Assumption 3.11 hence, by the proof of Lemma 3.7, we get that $\left(\phi_{1}(n), \ldots, \phi_{T}(n)\right)$ is a tight sequence. Now we can conclude just like in the proofs of Theorems 3.4 and 3.16 above, using Assumption 3.2.

Remark 4.17. It is interesting to note that the proofs of tightness in Theorems 3.16 and 4.16 follow entirely different ideas. In Theorem 3.16 we manage to find an EUT optimal investment problem whose value function is above that of the CPT problem. In Theorem 4.16 we use $Q$ to find estimates for $X_{T}^{z, \theta}$ which then translate into estimates for $\left(\phi_{1}(n), \ldots, \phi_{T}(n)\right)$.

### 4.5 Existence

We return to the setting of Theorem 4.15. Let $\nu_{n}$ denote the joint law of the random vector $\left(\rho, X_{n}\right)$. As a consequence of Theorem 4.15, the sequence $\left\{\nu_{n} ; n \in \mathbb{N}\right\}$ is also tight. and we can extract a weakly convergent subsequence $\left\{\nu_{n_{k}} ; k \in \mathbb{N}\right\}$ with limit $\pi$ for some probability measure $\pi$ on $\mathbb{R}^{2}$.

We recall that a mapping $K$ from $\mathbb{R} \times \mathcal{B}(\mathbb{R})$ into $[0,+\infty)$ is called a transition probability kernel on a probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ if the mapping $x \mapsto K(x, B)$ is measurable for every set $B \in \mathcal{B}(\mathbb{R})$, and the mapping $B \mapsto K(x, B)$ is a probability measure for $\mu$-a.e. $x \in \mathbb{R}$.

By the disintegration theorem (see e.g. [36]) there exists a probability measure $\lambda$ on $\mathbb{R}$ and a transition probability kernel $K$ on $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ such that $\pi\left(A_{1} \times A_{2}\right)=\int_{A_{1}} K\left(x, A_{2}\right) d \lambda(x)$ for all $A_{1}, A_{2} \in \mathcal{B}(\mathbb{R})$. Clearly, $\lambda(A)=P(\rho \in A)$ for all Borel sets $A \subseteq \mathbb{R}$.

Theorem 4.18. Under Assumptions 3.11, 4.1, 4.2, 4.3, 4.4 and $V_{-}(z-B)<\infty$ there exists an optimal trading strategy $\phi^{*}=\phi^{*}(z)$ for (129).
Proof. As $B$ is replicable by Assumption 4.4 with a replicating portfolio, say, $\psi$, we may replace $\phi_{n}$ by $\phi_{n}-\psi$ and assume $B=0$.

Let us set $X_{*}:=G\left(\rho, U_{*}\right)$, where $G$ is the measurable function given by Lemma 6.16 applied with $\delta:=\lambda, \nu:=K$ and $Y:=\rho$. Clearly, the random variable $X_{*}$ is $\sigma\left(\rho, U_{*}\right)$-measurable.

Moreover, the subsequence of random variables $X_{n_{k}}, k \in \mathbb{N}$ converges in law to $X_{*}$ as $k \rightarrow+\infty$. So $\left\{u_{+}\left(X_{n_{k}}^{+}\right)\right\}_{k \in \mathbb{N}}$ also converges in law to $u_{+}\left(X_{*}^{+}\right)$. Hence, $\lim _{k} P\left(u_{+}\left(X_{n_{k}}^{+}\right)>y\right)=$ $P\left(u_{+}\left(X_{*}^{+}\right)>y\right)$ for every $y \in \mathbb{R}$ at which the cumulative distribution function of $u_{+}\left(X_{*}^{+}\right)$is continuous (i.e. outside a countable set). Analogously, we conclude that $P\left(u_{-}\left(X_{n_{k}}^{-}\right)>y\right) \rightarrow$ $P\left(u_{-}\left(X_{*}^{-}\right)>y\right)$ as $k \rightarrow+\infty$ for all $y$ outside a countable set.

We start by showing that $V_{ \pm}\left(X_{*}^{ \pm}\right)<+\infty$. The distortion functions being continuous, it is obvious that $w_{ \pm}\left(P\left(u_{ \pm}\left(X_{n_{k}}^{ \pm}\right)>y\right)\right) \rightarrow w_{ \pm}\left(P\left(u_{ \pm}\left(X_{*}^{ \pm}\right)>y\right)\right)$ for Lebesgue a.e. $y, n \rightarrow \infty$. Thus, applying Fatou's lemma we get

$$
\begin{align*}
V_{ \pm}\left(X_{*}^{ \pm}\right) & =\int_{0}^{+\infty} w_{ \pm}\left(P\left(u_{ \pm}\left(X_{*}^{ \pm}\right)>y\right)\right) d y \\
& \leq \lim _{k}^{\inf } \int_{0}^{+\infty} w_{ \pm}\left(P\left(u_{ \pm}\left(X_{n_{k}}^{ \pm}\right)>y\right)\right) d y=\lim _{k} \inf ^{\prime}\left(X_{ \pm}^{ \pm}\right) . \tag{150}
\end{align*}
$$

But $\liminf _{k} V_{ \pm}\left(X_{n_{k}}^{ \pm}\right) \leq \sup _{k \in \mathbb{N}} V_{ \pm}\left(X_{n_{k}}^{ \pm}\right) \leq \sup _{n \in \mathbb{N}} V_{ \pm}\left(X_{n}^{ \pm}\right)$, and we know from the proof of Theorem 4.15 that $\sup _{n \in \mathbb{N}} V_{ \pm}\left(X_{n}^{ \pm}\right)<+\infty$, so we have the intended result.

Secondly, we prove that the inequality $V\left(X_{*}\right) \geq V^{*}$ holds. We already know, from the previous step, that $V_{-}\left(X_{*}^{-}\right) \leq \liminf _{k} V_{-}\left(X_{n_{k}}^{-}\right)$. We note further that, by the proof of Theorem 4.15, $\sup _{k \in \mathbb{N}} E_{P}\left[\left(X_{n_{k}}^{+}\right)^{\alpha \lambda}\right]<\infty$, for some $\lambda>0$ such that $\alpha \lambda<1<\gamma \lambda$. Therefore, defining $g(y):=1$ for $y \in[0,1]$ and

$$
g(y):=g_{+} \frac{\left(\sup _{n \in \mathbb{N}} k_{+} E_{P}\left[\left(X_{n}^{+}\right)^{\alpha \lambda}\right]+k_{+}\right)^{\gamma}}{y^{\gamma \lambda}}
$$

for $y>1$, we see that $g$ is an integrable function on $[0,+\infty)$. It follows from Markov's inequality that $w_{+}\left(P\left(u_{+}\left(X_{n_{k}}^{+}\right)>y\right)\right) \leq g(y)$ for all $y \geq 0$ and for all $k \in \mathbb{N}$. Hence Fatou's lemma gives

$$
V_{+}\left(X_{*}^{+}\right) \geq \underset{k}{\lim \sup } \int_{0}^{+\infty} w_{+}\left(P\left(u_{+}\left(X_{n_{k}}^{+}\right)>y\right)\right) d y=\limsup _{k} V_{+}\left(X_{n_{k}}^{+}\right) .
$$

Combining the previous inequalities then yields

$$
\begin{aligned}
V\left(X_{*}\right) & =V_{+}\left(X_{*}^{+}\right)-V_{-}\left(X_{*}^{-}\right) \\
& \geq \underset{k}{\lim \sup ^{2}}\left\{V_{+}\left(X_{n_{k}}^{+}\right)-V_{-}\left(X_{n_{k}}^{-}\right)\right\}=V^{*} .
\end{aligned}
$$

Lastly, we check that $E_{Q}\left[X_{*}\right] \leq z$. To see this, we start by noting that, since ( $\rho, X_{n_{k}}$ ) tends to ( $\rho, X_{*}$ ) in law, we have $\rho X_{n_{k}} \rightarrow \rho X_{*}$ in law. Thus, we can use Skorohod's theorem to find real-valued random variables $Y$ and $Y_{k}$, with $k \in \mathbb{N}$, on some auxiliary probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{Q})$, such that each $Y_{k}$ has the same law as $\rho X_{n_{k}}, Y$ has the same law as $\rho X_{*}$, and $Y_{k} \rightarrow Y \hat{Q}$-a.s., as $n \rightarrow \infty$. It is then clear that $E_{Q}\left[X_{n_{k}}\right]=E_{P}\left[\rho X_{n_{k}}\right]=E_{\hat{Q}}\left[Y_{k}\right]$ for every $k \in \mathbb{N}$. We know from the proof of Theorem 4.15 that $\sup _{k \in \mathbb{N}} E_{P}\left[\left(X_{n_{k}}^{-}\right)^{\xi}\right]<+\infty$, for some $1<\xi<\frac{\beta}{\delta}$. Consequently, we can choose $\vartheta>1$ such that $\vartheta<\xi$, and using Hölder's inequality we obtain

$$
E_{\hat{Q}}\left[\left(Y_{k}^{-}\right)^{\vartheta}\right]=E_{P}\left[\left(\rho X_{n_{k}}^{-}\right)^{\vartheta}\right] \leq E_{P}\left[\rho^{\frac{\partial \xi}{\xi-\vartheta}}\right]^{\frac{\xi-\vartheta}{\xi}} E_{P}\left[\left(X_{n_{k}}^{-}\right)^{\xi}\right]^{\frac{\vartheta}{\xi}},
$$

for every $k \in \mathbb{N}$, which implies that $\sup _{k \in \mathbb{N}} E_{\hat{Q}}\left[\left(Y_{k}^{-}\right)^{\vartheta}\right]<+\infty$. Hence, by the de la ValléePoussin criterion, the family $Y_{k}^{-}, k \in \mathbb{N}$ is uniformly integrable and thus

$$
\begin{equation*}
\lim _{k \in \mathbb{N}} E_{\hat{Q}}\left[Y_{k}^{-}\right]=E_{\hat{Q}}\left[Y^{-}\right]<+\infty . \tag{151}
\end{equation*}
$$

Furthermore, using Fatou's lemma we get the inequality $E_{\hat{Q}}\left[Y^{+}\right] \leq \lim \inf _{k} E_{\hat{Q}}\left[Y_{k}^{+}\right]$. Combining these observations yields

$$
E_{Q}\left[X_{*}\right]=E_{\hat{Q}}[Y] \leq \underset{k}{\liminf } E_{\hat{Q}}\left[Y_{k}\right]=\liminf _{k} E_{Q}\left[X_{n_{k}}\right]=z
$$

and (151) yields $E_{Q}\left|X_{*}\right|<\infty$, that is, $X_{*} \in L^{1}(Q)$. Hence, by Assumption 4.4, $X_{*}$ admits a replicating portfolio $\phi^{*} \in \Phi_{a}(Q)$ from initial capital $E_{Q} X_{*}$.

Let us define the $\mathcal{F}_{T}$-measurable random variable $Z_{*}:=X_{*}+c$, where the constant $c$ is given by $c:=z-E_{Q}\left[X_{*}\right] \geq 0$. Then it is trivial that $Z_{*}$ is replicable by $\phi^{*} \in \mathcal{A}(z)$ since $V_{-}\left(Z_{*}\right) \leq V_{-}\left(X_{*}\right)<\infty$ by (150) above. Besides, $V^{*} \leq V\left(X_{*}\right) \leq V\left(Z_{*}\right)$ so necessarily $V^{*}=$ $V\left(Z_{*}\right)$ must hold, by the definition of $V^{*}$. The proof is complete.

### 4.6 Examples

Let $W_{t}, t \geq 0$ be a standard $k$-dimensional Brownian motion with its natural filtration ( $P$-zero sets added) $\mathcal{F}_{t}, t \geq 0$. The dynamics of the price process of the $i$ th stock $S^{i}=\left\{S_{t}^{i} ; 0 \leq t \leq T\right\}$ is described, under the measure $P$, by

$$
\begin{equation*}
d S_{t}^{i}=\mu^{i}(t) S_{t}^{i} d t+\sum_{j=1}^{k} \sigma^{i j}(t) S_{t}^{i} d W_{t}^{j}, \quad S_{0}^{i}=s_{i}>0 \tag{152}
\end{equation*}
$$

for any $i \in\{1, \ldots, d\}$, with $\mu^{i}, \sigma^{i j}$ deterministic measurable functions on $[0, T]$ satisfying $\int_{0}^{T}\left|\mu^{i}(t)\right| d t+\int_{0}^{T} \sum_{j=1}^{k}\left|\sigma^{i j}(t)\right|^{2} d t<+\infty$. We assume that $\sigma(t) \sigma(t)^{T}$ is non-singular for Leb-a.e. $t \in[0, T]$ (this implies, in particular, $k \geq d$ ).

Let us suppose that there are as many risky assets as sources of randomness, that is, $k=d$. Then it is trivial that there exists a uniquely determined $d$-dimensional, deterministic process $\theta=\left\{\theta(t)=\left(\theta^{1}(t), \ldots, \theta^{d}(t)\right)^{\top} ; 0 \leq t \leq T\right\}$ such that

$$
-\mu^{i}(t)=\sum_{j=1}^{d} \sigma^{i j}(t) \theta^{j}(t), \quad \text { for Lebesgue a.e. } t \in[0, T]
$$

holds simultaneously for all $i \in\{1, \ldots, d\}$. If we assume, in addition, that the condition $0<\int_{0}^{T} \sum_{i=1}^{d}\left|\theta^{i}(t)\right|^{2} d t<+\infty$ is satisfied, then

$$
\begin{equation*}
d Q / d P=\exp \left\{\sum_{i=1}^{d} \int_{0}^{T} \theta^{i}(s) d W_{s}^{i}-\frac{1}{2} \int_{0}^{T} \sum_{i=1}^{d}\left|\theta^{i}(s)\right|^{2} d s\right\} \tag{153}
\end{equation*}
$$

defines the unique element $Q \in \mathcal{M}$. It is straightforward to check that $\rho_{T}$ is lognormally distributed both under $P$ and under $Q$. In particular, $\rho_{T}, 1 / \rho_{T} \in \mathcal{W}$, so Assumptions 4.1, 4.2 and 4.3 hold.

Any contingent claim in $L^{1}(Q)$ is replicable by the martingale representation theorem. It is trivial to see that there must be some $0<\hat{t}<T$ for which

$$
0<\int_{0}^{\hat{t}} \sum_{i=1}^{d}\left|\theta^{i}(s)\right|^{2} d s<\int_{0}^{T} \sum_{i=1}^{d}\left|\theta^{i}(s)\right|^{2} d s
$$

holds true, so the vector

$$
\left(\sum_{i=1}^{d} \int_{0}^{\hat{t}} \theta^{i}(s) d W_{s}^{i}, \sum_{i=1}^{d} \int_{0}^{T} \theta^{i}(s) d W_{s}^{i}\right)
$$

has a non-degenerate joint normal distribution. It is easy to see that

$$
G_{*}:=\frac{\int_{0}^{T} \sum_{i=1}^{d}\left|\theta^{i}(s)\right|^{2} d s}{\int_{0}^{\hat{t}} \sum_{i=1}^{d}\left|\theta^{i}(s)\right|^{2} d s} \sum_{i=1}^{d} \int_{0}^{\hat{t}} \theta^{i}(s) d W_{s}^{i}-\sum_{i=1}^{d} \int_{0}^{T} \theta^{i}(s) d W_{s}^{i}
$$

is an $\mathcal{F}_{T}$-measurable and non-degenerate Gaussian random variable which is independent of $\sum_{i=1}^{d} \int_{0}^{T} \theta^{i}(s) d W_{s}^{i}$ and hence of $\rho_{T}$. Lemma 6.15 provides a uniform $U_{*}$ independent of $\rho_{T}$, that is, satisfying Assumption 4.4. When $d=1, \sigma \neq 0, \mu \in \mathbb{R}$ constants then we get the Black-Scholes model, see e.g. [15].

Remark 4.19. In [80] a new method for constructing $X_{*}$ in Theorem 4.18 was found which applies to all complete markets (see Section 1.3) and it allows to construct $X_{*}$ which is a function of $\rho$ only. Hence, in the case of complete markets, the existence requirement of $U_{*}$ can be dropped in Assumption 4.4.

There are also examples of incomplete markets satisfying Assumption 4.4 where Theorem 4.18 applies. However, the class of such models is rather narrow. We refer to [74] and [80].

Extending results of the present chapter to larger classes of incomplete models is a challenge. Some progress in this direction has been made in [77].

## 5 Illiquid markets

In financial practice, trading moves prices against the trader: buying faster increases execution prices, and selling faster decreases them. This aspect of liquidity, known as market depth [16] or price-impact, is widely documented empirically [40, 30], and has received increasing attention, see [62, 10, 2, 97, 84, 45]. These models depart from the literature on frictionless markets, where prices are the same for any amount traded.

The growing interest in price-impact has also highlighted a shortage of effective theoretical tools. In discrete time, several researchers have studied these fundamental questions, [ $5,70,38,69$ ], but extensions to continuous time have proved challenging. In this chapter we shall prove an existence theorem for optimal strategies in a very general continuous-time model under the assumption that trading costs are superlinear functions of the trading speed. This assumption is consistent with empirical data, see [30].

Superlinear frictions in the sense of the present dissertation entail that execution prices become arbitrarily unfavorable as traded quantities per unit of time grow: buying or selling too fast becomes impossible. As a result, trading is feasible only at finite rates - the number of shares $\varphi_{t}$ will be assumed absolutely continuous. This feature sets apart superlinear frictions from frictionless markets, in which the number of shares is merely predictable, see Sections 1.3 and 5.1.

This chapter is based on [47].

### 5.1 Model

For a finite time horizon $T>0$, consider a continuous-time filtered probability space $\left(\Omega, \mathcal{F},\left(\mathcal{F}_{t}\right)_{t \in[0, T]}, P\right)$ where the filtration is right-continuous and $\mathcal{F}_{0}$ coincides with the family of $P$-zero sets. $\mathcal{O}$ denotes the optional sigma-field on $\Omega \times[0, T]$, that is, the sigma-field generated by the family of càdlàg adapted processes. The market includes a riskless and perfectly liquid asset $S^{0}$ with $S_{t}^{0} \equiv 1, t \in[0, T]$, and $d$ risky assets, described by càdlàg, adapted processes $\left(S_{t}^{i}\right)_{t \in[0, T]}^{1 \leq i \leq d}$. Henceforth $S$ denotes the $d$-dimensional process with components $S^{i}$, $1 \leq i \leq d$. The components of a $(d+1)$-dimensional vector $x$ are denoted by $x^{0}, \ldots, x^{d}$.

The next definition identifies those strategies for which the number of shares changes over time at some finite rate.
Definition 5.1. A feasible strategy is a process $\phi$ in the class

$$
\begin{equation*}
\mathcal{A}:=\left\{\phi: \phi \text { is an } \mathbb{R}^{d} \text {-valued, } \mathcal{O} \text {-measurable process, } \int_{0}^{T}\left|\phi_{u}\right| d u<\infty \text { a.s. }\right\} \tag{154}
\end{equation*}
$$

In this definition, the process $\phi$ represents the trading rate, that is, the speed at which the number of shares in each asset changes over time, and the condition $\int_{0}^{T}\left|\phi_{u}\right| d u<\infty$ means that absolute turnover (the cumulative number of shares bought or sold) remains finite in finite time. Define, for each $\phi \in \mathcal{A}$,

$$
\varphi_{t}:=\int_{0}^{t} \phi_{u} d u, t \in[0, T]
$$

the number of stock in the portfolio at time $t$ in the respective assets (integration is meant componentwise).

The above definition significantly differs from the one of admissible strategies in frictionless markets in Section 1.3: this definition restricts the number of shares to be (absolutely) continuous, while usual admissible strategies have an arbitrarily irregular number of shares. Note also that the definition of feasibility does not involve the asset price at all.

Assume $S$ to be a semimartingale and recall Section 1.3. Note that $\varphi$ above is a predictable, locally bounded process, hence it is $S$-integrable.

In the absence of frictions the value of a self-financing portfolio at time $T$ is

$$
z^{0}+\int_{0}^{T} \varphi_{t} d S_{t}
$$

where $z^{0}$ represents the initial capital, see (6). Note that the investor holds $\varphi_{T}$ units of stock at the terminal date which is worth $\varphi_{T} S_{T}$. Hence the value of his/her cash (bank account) position at the terminal date is

$$
\begin{equation*}
z^{0}+\int_{0}^{T} \varphi_{t} d S_{t}-\varphi_{T} S_{T}=z^{0}-\int_{0}^{T} S_{t} \phi_{t} d t, \tag{155}
\end{equation*}
$$

where we performed, formally, an integration by parts and recalled $\varphi_{0}=0$ as well. Notice, however, that the right-hand side of (155) makes sense for any càdlàg $S$ and not only for semimartingales. Indeed, by the càdlàg property the function $S_{t}(\omega), t \in[0, T]$ is bounded for almost every $\omega \in \Omega$, hence the integral in question is finite a.s. for each $\phi$ satisfying $\int_{0}^{T}\left|\phi_{t}\right| d t<\infty$ a.s.

Now we look at how (155) changes in the presence of illiquidity. For a given trading strategy $\phi$, frictions reduce the cash position, by making purchases more expensive, and sales less profitable. We model this effect by introducing a function $G$, which summarizes the impact of frictions on the execution price at different trading rates:

Assumption 5.2. Let $G: \Omega \times[0, T] \times \mathbb{R}^{d} \rightarrow \mathbb{R}_{+}$be a $\mathcal{O} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$-measurable function, such that $G(\omega, t, \cdot)$ is convex with $G(\omega, t, x) \geq G(\omega, t, 0)$ for all $\omega, t, x$. Henceforth, set $G_{t}(x):=G(\omega, t, x)$, i.e. the dependence on $\omega$ is omitted, and $t$ is used as a subscript.

Taking price-impact into accout, for a given strategy $\phi \in \mathcal{A}$ and an initial asset position $z \in \mathbb{R}^{d+1}$, the resulting positions at time $t \in[0, T]$ in the risky and safe assets are defined as:

$$
\begin{align*}
X_{t}^{i}=X_{t}^{i}(z, \phi) & :=z^{i}+\int_{0}^{t} \phi_{u}^{i} d u \quad 1 \leq i \leq d  \tag{156}\\
X_{t}^{0}=X_{t}^{0}(z, \phi) & :=z^{0}-\int_{0}^{t} \phi_{u} S_{u} d u-\int_{0}^{t} G_{u}\left(\phi_{u}\right) d u \tag{157}
\end{align*}
$$

The first equation merely says that the cumulative number of shares $X_{t}^{i}$ in the $i$-th asset is given by the initial number of shares, plus subsequent flows. The second equation, compared to (155), contains a new term involving the friction $G$, which summarizes the impact of trading on execution prices. The condition $G(\omega, t, x) \geq G(\omega, t, 0)$ means that inactivity is always cheaper than any trading activity. Most models in the literature assume $G(\omega, t, 0)=0$, but the above definition allows for $G(\omega, t, 0)>0$, which is interpreted as a cost of participation in the market, such as the fees charged by exchanges to trading firms. The convexity of $x \mapsto G_{t}(x)$ implies that trading twice as fast for half the time locally increases execution costs - speed is expensive. Indeed, let $g(x)=G(\omega, t, x)$, i.e. focus on a local effect. Then, by convexity, $g(x) \leq(1-1 / k) g(0)+(1 / k) g(k x)$ for $k>1$, and therefore $(g(k x)-g(0)) T / k \geq(g(x)-g(0)) T$, which means that increasing trading speed by a factor of $k$ and reducing trading time by the same factor implies higher trading costs, excluding the participation cost captured by $g(0)$. Finally note that, in general, $X_{t}^{0}$ may take the value $-\infty$ for some (unwise) strategies.

With a single risky asset and with $G(\omega, t, 0)=0$, the above specification is equivalent to assuming that a trading rate of $\phi_{t} \neq 0$ implies an instantaneous execution price equal to

$$
\begin{equation*}
\tilde{S}_{t}=S_{t}+G_{t}\left(\phi_{t}\right) / \phi_{t} \tag{158}
\end{equation*}
$$

which is (by positivity of $G$ ) higher than $S_{t}$ when buying, and lower when selling. Thus, $G \equiv 0$ boils down to a frictionless market, while proportional transaction costs correspond to $G_{t}(x)=\varepsilon S_{t}|x|$ with some $\varepsilon>0$. Yet, we focus on neither of these settings, which entail either zero or linear costs, but rather on superlinear frictions, defined as those that satisfy the following conditions. Note that we require a strong form of superlinearity here (i.e. the cost functional grows at least as a superlinear power of the traded volume).

Assumption 5.3. There is $\alpha>1$ and a càdlàg process $H$ such that

$$
\begin{align*}
\inf _{t \in[0, T]} H_{t} & >0 \quad \text { a.s. }  \tag{159}\\
G_{t}(x) & \geq H_{t}|x|^{\alpha}, \quad \text { for all } t, x ; \text { a.s. }  \tag{160}\\
\int_{0}^{T}\left(\sup _{|x| \leq N} G_{t}(x)\right) d t & <\infty \quad \text { a.s. for all } N>0 \tag{161}
\end{align*}
$$

Condition (160) is the central superlinearity assumption. Condition (159) requires that frictions never disappear, and (161) says that they remain finite in finite time for uniformly bounded trading rates. In summary, these conditions characterize nontrivial, finite, superlinear frictions. Note that (160) implies that $\tilde{S}_{t}$ in (158) becomes arbitrarily negative as $\phi_{t}$ becomes negative enough, i.e. when selling too fast.

Remark 5.4. Although we can treat a general $S$, the most important case is where $S$ has non-negative components, and therefore a positive number of units of risky positions has positive value. Otherwise, if $S$ can take negative values, a larger number of units does not imply a position with higher value, but only a larger exposure to default.

Assume in the rest of this remark that $S$ is non-negative and one-dimensional (for simplicity). Take $\phi \in \mathcal{A}$ and consider the (optional) set $A:=\left\{(\omega, t): \phi_{t}(\omega)<0, S_{t}(\omega)+\right.$ $\left.G\left(\omega, t, \phi_{t}(\omega)\right) / \phi_{t}(\omega) \geq 0\right\}$, which identifies the times at which execution prices are positive. Clearly, $X_{T}^{i}\left(z, \phi^{\prime}\right) \geq X_{T}^{i}(z, \phi), i=1,2$ for $\phi_{t}^{\prime}(\omega):=\phi_{t}(\omega) 1_{A}$. Hence one may always replace the set of strategies $\mathcal{A}$ by

$$
\mathcal{A}_{+}:=\left\{\phi \in \mathcal{A}: S_{t}(\omega)+G\left(\omega, t, \phi_{t}(\omega)\right) / \phi_{t}(\omega) \geq 0 \text { when } \phi_{t}(\omega)<0\right\}
$$

without losing any "good" investment. In other words, we may restrict ourselves to trading strategies with positive execution prices at all times, because any other strategy is dominated pointwise by a strategy that trades at the same rate when the execution price is positive, and otherwise does not trade. The class $\mathcal{A}_{+}$may be economically more appealing as it excludes the unintended consequence of (160) that $S_{t}(\omega)+G\left(\omega, t, \phi_{t}(\omega)\right) / \phi_{t}(\omega) \rightarrow-\infty$ whenever $\phi_{t}(\omega) \rightarrow$ $-\infty$.

The most common example in the literature is, with one risky asset, the friction

$$
G_{t}(x):=\Lambda|x|^{\alpha} \text { for some } \Lambda>0, \alpha>1
$$

(see e.g. [38]). Another possibility is $G_{t}(x):=\Lambda S_{t}|x|^{\alpha}$. In multiasset models the friction $G_{t}(x):=x^{T} \Lambda x$ for some symmetric, positive-definite, $d \times d$ matrix $\Lambda$ has been suggested in [45].

Remark 5.5. Our results remain valid assuming that (160) holds for $|x| \geq M$ only, with some $M>0$. Such an extension requires only minor modifications of the proofs, and may accommodate models for which a low trading rate incurs, for instance, either zero or linear costs.

### 5.2 Bounds for the market and for the trading volume

Superlinear frictions in the sense of Assumption 5.3 lead to a striking boundedness property: for a fixed initial position, all payoffs of feasible strategies are bounded above by a single random variable $W<\infty$, the market bound, which depends on the friction $G$ and on the price $S$, but not on the strategy. This property clearly fails in frictionless markets, where any payoff with zero initial capital can be scaled arbitrarily, and therefore admits no uniform bound. In such markets, a much weaker boundedness property holds: Corollary 9.3. of [35] shows that the set of payoff of $x$-admissible strategies is bounded in $L^{0}$ if the market is arbitrage-free in a certain strong sense.

A central tool in this analysis is the function $G^{*}$, the Fenchel-Legendre conjugate of $G$. Its importance was first recognized by [38]. $G^{*}$ is defined as

$$
G_{t}^{*}(y):=\sup _{x \in \mathbb{R}^{d}}\left(x y-G_{t}(x)\right), y \in \mathbb{R}^{d}, t \in[0, T]
$$

Note that the supremum can be taken over $\mathbb{Q}^{d}$, hence $G^{*}$ is $\mathcal{O} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$-measurable. Note also that, under Assumption 5.3, $G_{t}^{*}(\cdot)$ is a finite (see the proof of Lemma 5.7), convex function.

The typical case $d=1, G_{t}(x)=\Lambda|x|^{\alpha}$ leads to $G_{t}^{*}(y)=\frac{\alpha-1}{\alpha} \alpha^{\frac{1}{1-\alpha}} \Lambda^{\frac{1}{1-\alpha}}|y|^{\frac{\alpha}{\alpha-1}}$ (in particular, $G_{t}^{*}(y)=y^{2} /(4 \Lambda)$ for $\alpha=2$ ). The key observation is the following:

Lemma 5.6. Under Assumption 5.3, any $\phi \in \mathcal{A}$ satisfies

$$
X_{T}^{0}(z, \phi) \leq z^{0}+\int_{0}^{T} G_{t}^{*}\left(-S_{t}\right) d t<\infty \quad \text { a.s. }
$$

Proof. Indeed, this follows from (157), the definition of $G_{t}^{*}$, and Lemma 5.7 below.
Lemma 5.7. Under Assumption 5.3, the random variable $W:=\int_{0}^{T} G_{t}^{*}\left(-S_{t}\right) d t$ is finite almost surely.

Proof. Consider first the case $d=1$. Then, by direct calculation,

$$
G_{t}^{*}(y) \leq \sup _{r \in \mathbb{R}}\left(r y-H_{t}|r|^{\alpha}\right)=\frac{\alpha-1}{\alpha} \alpha^{\frac{1}{1-\alpha}} H_{t}^{\frac{1}{1-\alpha}}|y|^{\frac{\alpha}{\alpha-1}} .
$$

Noting that $\sup _{t \in[0, T]}\left|S_{t}\right|$ is finite a.s. by the càdlàg property of $S$, and $\inf _{t \in[0, T]} H_{t}$ is a positive random variable, it follows that

$$
\sup _{t \in[0, T]} G_{t}^{*}\left(-S_{t}\right)<\infty \text { a.s. }
$$

which clearly implies the statement. If $d>1$ then note that
$G_{t}^{*}(y) \leq \sup _{r \in \mathbb{R}^{d}}\left(\sum_{i=1}^{d} r^{i} y^{i}-H_{t}|r|^{\alpha}\right) \leq \sum_{i=1}^{d} \sup _{r \in \mathbb{R}^{d}}\left(r^{i} y^{i}-\left(H_{t} / d\right)|r|^{\alpha}\right) \leq \sum_{i=1}^{d} \sup _{x \in \mathbb{R}}\left(x y^{i}-\left(H_{t} / d\right)|x|^{\alpha}\right)$
and the conclusion follows from the scalar case.
Since $W<\infty$ a.s, it is impossible to achieve a scalable arbitrage: though a trading strategy may realize an a.s. positive terminal value, one cannot get an arbitrarily large profit by scaling the trading strategy (i.e. by multiplying it with large positive constants) since bigger trading values also enlarge costs. Even if an arbitrage exists, amplifying it too much backfires, because the superlinear friction eventually overrides profits. Yet, arbitrage opportunities can exist in limited size.

For $Q \sim P$, denote by $L^{1}(Q)$ the Banach space of $(d+1)$-dimensional, $Q$-integrable random variables; given a subset $A$ of a Euclidean space, $L^{0}(A)$ denotes the set of ( $P$-a.s. equivalence classes of) $A$-valued random variables, equipped with the topology of convergence in probability. Fix $1<\beta<\alpha$, where $\alpha$ is as in Assumption 5.3. Let $\gamma$ be the conjugate number of $\beta$, defined by

$$
\frac{1}{\beta}+\frac{1}{\gamma}=1
$$

The next definition identifies a class of reference probability measures with integrability properties that fit the friction $G$ and the price process $S$ well.

Definition 5.8. $\mathcal{P}$ denotes the set of probabilities $Q \sim P$ such that

$$
E_{Q} \int_{0}^{T} H_{t}^{\beta /(\beta-\alpha)}\left(1+\left|S_{t}\right|\right)^{\beta \alpha /(\alpha-\beta)} d t<\infty
$$

$\tilde{\mathcal{P}}$ denotes the set of probability measures $Q \in \mathcal{P}$ such that

$$
E_{Q} \int_{0}^{T}\left|S_{t}\right| d t<\infty
$$

Under Assumption 5.3, note that $\tilde{\mathcal{P}} \neq \emptyset$ by Lemma 6.9. The next lemma shows that, if a payoff has a finite negative part under some probability in $\mathcal{P}$, then its trading rate must also be (suitably) integrable. There is no analogue to such a result in frictionless markets. The intuition is that, with frictions, excessive trading causes unbounded losses. Hence, a bound on losses translates into one for trading volume. Lemma 5.9 is crucial to establish the closedness of the set of attainable payoffs (Proposition 5.10 below).

We recall that $x_{-}$denotes the negative part of $x \in \mathbb{R}$.
Lemma 5.9. Let $Q \in \mathcal{P}$ and $\phi \in \mathcal{A}$ be such that $E_{Q} \xi_{-}<\infty$, where

$$
\xi:=-\int_{0}^{T} S_{t} \phi_{t} d t-\int_{0}^{T} G_{t}\left(\phi_{t}\right) d t
$$

Then

$$
E_{Q} \int_{0}^{T}\left|\phi_{t}\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t<\infty
$$

Proof. For simplicity, set $T:=1$. Define $\phi_{t}(n):=\phi_{t} 1_{\left\{\left|\phi_{t}\right| \leq n\right\}} \in \mathcal{A}, n \in \mathbb{N}$. As $n \rightarrow \infty$, clearly $\phi_{t}(n) \rightarrow \phi_{t}$ for all $t$ and $\omega \in \Omega$, and the random variables

$$
\begin{aligned}
\xi_{n} & :=-\int_{0}^{1} S_{t} \phi_{t}(n) d t-\int_{0}^{1} G_{t}\left(\phi_{t}(n)\right) d t= \\
& -\sum_{i=1}^{d} \int_{0}^{1} S_{t}^{i} \phi_{t}^{i}(n)\left[1_{\left\{S_{t}^{i} \leq 0, \phi_{t}^{i} \leq 0\right\}}+1_{\left\{S_{t}^{i}>0, \phi_{t}^{i} \leq 0\right\}}+1_{\left\{S_{t}^{i} \leq 0, \phi_{t}^{i}>0\right\}}+1_{\left\{S_{t}^{i}>0, \phi_{t}^{i}>0\right\}}\right] d t \\
& -\int_{0}^{1} G_{t}\left(\phi_{t}(n)\right) d t
\end{aligned}
$$

converge to $\xi$ a.s. by monotone convergence. (Note that each of the terms with an indicator converges monotonically, and that $G_{t}(0) \leq G_{t}(x)$ for all $x$.) Hölder's inequality yields

$$
\begin{gather*}
\int_{0}^{1}\left|\phi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t=\int_{0}^{1}\left|\phi_{t}(n)\right|^{\beta} H_{t}^{\beta / \alpha} \frac{1}{H_{t}^{\beta / \alpha}}\left(1+\left|S_{t}\right|\right)^{\beta} d t \leq  \tag{162}\\
{\left[\int_{0}^{1}\left|\phi_{t}(n)\right|^{\alpha} H_{t} d t\right]^{\beta / \alpha}\left[\int_{0}^{1}\left(\frac{1}{H_{t}^{\beta / \alpha}}\left(1+\left|S_{t}\right|\right)^{\beta}\right)^{\alpha /(\alpha-\beta)} d t\right]^{(\alpha-\beta) / \alpha} \leq} \\
{\left[\int_{0}^{1} G_{t}\left(\phi_{t}(n)\right) d t\right]^{\beta / \alpha}\left[\int_{0}^{1}\left(\frac{1}{H_{t}^{\beta / \alpha}}\left(1+\left|S_{t}\right|\right)^{\beta}\right)^{\alpha /(\alpha-\beta)} d t\right]^{(\alpha-\beta) / \alpha}}
\end{gather*}
$$

All these integrals are finite by Assumption 5.3 and the càdlàg property of $S$. Now, set

$$
m:=\left[\int_{0}^{1}\left(\frac{1}{H_{t}^{\beta / \alpha}}\left(1+\left|S_{t}\right|\right)^{\beta}\right)^{\alpha /(\alpha-\beta)} d t\right]^{(\alpha-\beta) / \alpha}
$$

and note that, by Jensen's inequality,

$$
\left|\int_{0}^{1} S_{t} \phi_{t}(n) d t\right| \leq \int_{0}^{1}\left|\phi_{t}(n)\right|\left(1+\left|S_{t}\right|\right) d t \leq\left[\int_{0}^{1}\left|\phi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t\right]^{1 / \beta}
$$

Note also that if $x \geq 1$ and $x \geq 2^{\frac{\beta}{\alpha-\beta}} m^{\frac{\alpha}{\alpha-\beta}}$ then $x^{1 / \beta}-(x / m)^{\alpha / \beta} \leq x-2 x=-x$. This observation, applied to

$$
x:=\int_{0}^{1}\left|\phi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t
$$

implies that $\xi_{n} \leq-x$ on the event $\left\{x \geq 2^{\frac{\beta}{\alpha-\beta}} m^{\frac{\alpha}{(\alpha-\beta)}}+1\right\}$. Thus,

$$
\int_{0}^{1}\left|\phi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t \leq\left(\xi_{n}\right)_{-}+2^{\frac{\beta}{\alpha-\beta}} m^{\frac{\alpha}{(\alpha-\beta)}}+1, \text { a.s. }
$$

Letting $n$ tend to $\infty$, it follows that

$$
\begin{equation*}
\int_{0}^{1}\left|\phi_{t}\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t \leq \xi_{-}+2^{\frac{\beta}{\alpha-\beta}} m^{\frac{\alpha}{(\alpha-\beta)}}+1 \tag{163}
\end{equation*}
$$

which implies the claim, since $E_{Q} \xi_{-}<\infty$ by assumption, and $E_{Q} m^{\frac{\alpha}{\alpha-\beta}}<\infty$ from $Q \in \mathcal{P}$.

### 5.3 Closed payoff space

The central implication of the previous result is that the class of multivariate payoffs that are dominated by the terminal value of a feasible portfolio, defined as $C:=\left[\left\{X_{T}(0, \phi): \phi \in\right.\right.$ $\left.\mathcal{A}\}-L^{0}\left(\mathbb{R}_{+}^{d+1}\right)\right] \cap L^{0}\left(\mathbb{R}^{d+1}\right)$, is closed in a rather strong sense; recall the componentwise definition of the $(d+1)$-dimensional random variable $X_{T}(0, \phi)$ in (156) and (157). Note also that $X_{T}^{0}$ may well be $-\infty$ hence the intersection with $L^{0}\left(\mathbb{R}^{d+1}\right)$ in the definition of $C$ is reasonable. Closedness is the key property for establishing Theorem 5.12 below.

Proposition 5.10. Under Assumption 5.3, the set $C \cap L^{1}(Q)$ is closed in $L^{1}(Q)$ for all $Q \in \tilde{\mathcal{P}}$.
Proof. Take $T=1$ for simplicity, and assume that $\rho_{n}:=\xi_{n}-\eta_{n} \rightarrow \rho$ in $L^{1}(Q)$ where $\eta_{n} \in$ $L^{0}\left(\mathbb{R}_{+}^{d+1}\right)$ and $\xi_{n}=X_{1}(0, \psi(n))$ for some $\psi(n) \in \mathcal{A}$ with $\rho_{n} \in L^{1}(Q)$. Up to passing to a subsequence, this convergence takes place a.s. as well.

Lemma 5.9 implies that $E_{Q} \int_{0}^{1}\left|\psi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t$ must be finite for all $n$ since $\left(\xi_{n}\right)_{-} \leq$ $\left(\rho_{n}\right)_{-}$and the latter is in $L^{1}(Q)$. Applying (163) with the choice $\phi:=\psi(n)$ yields

$$
\int_{0}^{1}\left|\psi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t \leq\left(\rho_{n}\right)_{-}+2^{\frac{\beta}{\alpha-\beta}} m^{\frac{\alpha}{(\alpha-\beta)}}+1 .
$$

Now, since $Q \in \mathcal{P}$, and the sequence $\rho_{n}$ is bounded in $L^{1}(Q)$, it follows that

$$
\begin{equation*}
\sup _{n \geq 1} E_{Q} \int_{0}^{1}\left|\psi_{t}(n)\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t<\infty \tag{164}
\end{equation*}
$$

Consider $\mathbb{L}:=L^{1}(\Omega, \mathcal{F}, Q ; \mathbb{B})$, the Banach space of $\mathbb{B}$-valued integrable functions, see Section 6.2, where $\mathbb{B}:=L^{\beta}([0,1], \mathcal{B}([0,1]), L e b)$ is a separable and reflexive Banach space. The functions $\psi \cdot(n): \Omega \rightarrow \mathbb{B}$ are easily seen to be measurable in the sense of Section 6.2. By (164), the sequence $\psi .(n)$ is bounded in $\mathbb{L}$, so Lemma 6.11 yields convex combinations

$$
\tilde{\psi} \cdot(n)=\sum_{j=n}^{M(n)} \alpha_{j}(n) \psi \cdot(n)
$$

which converge to some $\tilde{\psi} . \in \mathbb{L}$ a.s. in $\mathbb{B}$-norm.
At this point, however, it is not yet clear that $\tilde{\psi}$ has an $\mathcal{O}$-measurable version. By the bound in (164), $\sup _{n} E_{Q} \int_{0}^{1}\left|\phi_{t}(n)\right|\left(1+\left|S_{t}\right|\right) d t<\infty$. Now apply Lemma 6.10 to the sequence $\tilde{\psi} .(n)$ in the space of $d$-dimensional integrable functions $L^{1}(\Omega \times[0,1], \mathcal{O}, \nu)$, where $\nu$ is the measure defined by

$$
\nu(A):=\int_{\Omega \times[0,1]} 1_{A}(\omega, t)\left(1+\left|S_{t}\right|\right) d t d Q(\omega)
$$

for $A \in \mathcal{O}$ ( $\nu$ is finite by $Q \in \tilde{P}$ ). This lemma yields convex combinations $\widehat{\psi} .(n)$ of the $\tilde{\psi} \cdot(n)$ such that $\widehat{\psi}$. $n$ ) converges to some $\psi . \in L^{1}(\Omega \times[0,1], \mathcal{O}, \nu) \nu$-a.e., and hence $P \times L e b$-a.e. This shows that $\psi$. is $\mathcal{O}$-measurable.

Since $\tilde{\psi} .(n)$ converge a.s. in $\mathbb{B}$-norm, also $\widehat{\psi} .(n) \rightarrow \tilde{\psi}$ a.s. in $\mathbb{B}$-norm, so $\psi=\tilde{\psi}, P \times$ Leb-a.e. and $\tilde{\psi} .(n)$ tends to $\psi$. in $\mathbb{B}$-norm a.e.

Define $\tilde{\xi}_{n}:=\sum_{j=n}^{M(n)} \alpha_{j}(n) \xi_{j}$ and $\tilde{\eta}_{n}:=\sum_{j=n}^{M(n)} \alpha_{j}(n) \eta_{j}$. Convergence in $\mathbb{B}$-norm implies that $\lim _{n \rightarrow \infty} \int_{0}^{1} \tilde{\psi}_{t}(n) S_{t} d t=\int_{0}^{1} \psi_{t} S_{t} d t$ almost surely, and also

$$
\lim _{n \rightarrow \infty} \tilde{\xi}_{n}^{i}=\lim _{n \rightarrow \infty} \int_{0}^{1} \tilde{\psi}_{t}^{i}(n) d t=\int_{0}^{1} \psi_{t}^{i} d t \text { a.s. for } i=1, \ldots, d
$$

Hence, $\tilde{\eta}_{n}^{i} \rightarrow \eta^{i}$ a.s. with $\eta^{i}:=\int_{0}^{T} \psi_{t}^{i} d t-\rho^{i} \in L^{0}\left(\mathbb{R}_{+}\right), i=1, \ldots, d$. By the convexity of $G_{t}$,

$$
\begin{aligned}
\rho^{0} & =\lim _{n \rightarrow \infty}\left(\tilde{\xi}_{n}^{0}-\tilde{\eta}_{n}^{0}\right)=\lim _{n \rightarrow \infty}-\left(\sum_{j=n}^{M(n)} \alpha_{j}(n) \int_{0}^{1}\left[\psi_{t}(j) S_{t}+G_{t}\left(\psi_{t}(j)\right)\right] d t\right)-\tilde{\eta}_{n}^{0} \\
& \leq \limsup _{n \rightarrow \infty}\left[-\int_{0}^{1} \tilde{\psi}_{t}(n) S_{t} d t-\int_{0}^{1} G_{t}\left(\tilde{\psi}_{t}(n)\right) d t-\tilde{\eta}_{n}^{0}\right] \\
& =\limsup _{n \rightarrow \infty}\left[-\int_{0}^{1} \tilde{\psi}_{t}(n) S_{t} d t-\int_{0}^{1} G_{t}\left(\psi_{t}\right) d t-\int_{0}^{1} G_{t}\left(\tilde{\psi}_{t}(n)\right) d t+\int_{0}^{1} G_{t}\left(\psi_{t}\right) d t-\tilde{\eta}_{n}^{0}\right] \\
& =-\int_{0}^{1} \psi_{t} S_{t} d t-\int_{0}^{1} G_{t}\left(\psi_{t}\right) d t+\limsup _{n \rightarrow \infty}^{1}\left[-\int_{0}^{1} G_{t}\left(\tilde{\psi}_{t}(n)\right) d t+\int_{0}^{1} G_{t}\left(\psi_{t}\right) d t-\tilde{\eta}_{n}^{0}\right] .
\end{aligned}
$$

Now Fatou's lemma, $G_{t}(x) \geq 0$ and $\tilde{\eta}_{n} \in L^{0}\left(\mathbb{R}_{+}^{d+1}\right)$ imply that the limit superior is in $-L^{0}\left(\mathbb{R}_{+}\right)$ (note that $G_{t}(\cdot)$ is continuous by convexity), hence there is $\eta^{0} \in L^{0}\left(\mathbb{R}_{+}\right)$such that

$$
\rho^{0}=-\int_{0}^{1} \psi_{t} S_{t} d t-\int_{0}^{1} G_{t}\left(\psi_{t}\right) d t-\eta^{0}
$$

which proves the proposition.
Corollary 5.11. Under Assumption 5.3, the set $C$ is closed in probability.
Proof. Let $\rho_{n} \in C$ tend to $\rho$ in probability. Up to a subsequence, convergence also holds almost surely. There exists $Q \in \tilde{\mathcal{P}}$ such that $\rho, \sup _{n}\left|\rho-\rho_{n}\right|$ are $Q$-integrable, see Lemma 6.9. Then $\rho_{n} \rightarrow \rho$ in $L^{1}(Q)$ as well and Proposition 5.10 implies $\rho \in C$.

### 5.4 Utility maximisation

This section discusses utility maximization in the model of Section 5.1. The first result (Theorem 5.12 below) shows that optimal strategies exist under a simple integrability assumption, which is easy to check. In particular, optimal strategies exist regardless of arbitrage (compare to Proposition 1.5 above), since such opportunities are necessarily limited. Put differently, the budget equation is nonlinear. Therefore one cannot add to an optimal strategy an arbitrage opportunity, and expect the resulting wealth to be the sum.

Importantly, these results consider only utilities defined on the real line. This setting is consistent with the definition of feasible strategies, which do not constrain wealth to remain positive. Since the focus is on utility functions defined on a single variable, and with price impact there is no scalar notion of portfolio value, the results below assume for simplicity that all strategies begin and end with cash only.

Let $B$ be an arbitrary real-valued random variable (representing a random payoff) and $c \in \mathbb{R}$ the investor's initial capital. For any $x \in \mathbb{R}$ let $\check{x} \in \mathbb{R}^{d+1}$ be defined as $\check{x}^{0}=x$ and $\check{x}^{i}=0$, $i=1, \ldots, d$.
Theorem 5.12. Let $u: \mathbb{R} \rightarrow \mathbb{R}$ be concave and nondecreasing, and let

$$
\begin{equation*}
E u(c-B+W)<\infty \tag{165}
\end{equation*}
$$

hold for the market bound $W$ of Lemma 5.7. Under Assumption 5.3, there is $\phi^{*} \in \mathcal{A}^{\prime}(u, c)$ such that

$$
E u\left(X_{T}^{0}\left(\check{c}, \phi^{*}\right)-B\right)=\sup _{\phi \in \mathcal{A}^{\prime}(u, c)} E u\left(X_{T}^{0}(\check{c}, \phi)-B\right),
$$

provided that $\mathcal{A}^{\prime}(u, c):=\left\{\phi \in \mathcal{A}: X_{T}^{i}(\check{c}, \phi)=0, i=1, \ldots, d, E u_{-}\left(X_{T}^{0}(\check{c}, \phi)-B\right)<\infty\right\}$ is not empty.

Remark 5.13. If $G_{t}(0) \equiv 0$ then for $u$ and $B$ bounded above condition (165) holds and $\mathcal{A}^{\prime}(u, c) \neq \emptyset$ hence Theorem 5.12 applies.

Proof. Corollary 5.11 implies that

$$
C^{\prime}:=\check{c}+\left(C \cap\left\{X: X^{i}=0 \text { a.s. }, i=1, \ldots, d\right\}\right) \subset L^{0}\left(\mathbb{R}^{d+1}\right)
$$

is closed in probability. Let $\phi(n)$ be a sequence in $\mathcal{A}^{\prime}(u, c)$ with

$$
\lim _{n \rightarrow \infty} E u\left(X_{T}^{0}(\check{c}, \phi(n))-B\right)=\sup _{\phi \in \mathcal{A}^{\prime}(u, c)} E u\left(X_{T}^{0}(\check{c}, \phi)-B\right)
$$

Since $X_{T}^{0}(\check{c}, \phi(n)) \leq c+W$ a.s. for all $n$, by Lemma 6.10 there are convex combinations such that $\sum_{j=n}^{M(n)} \alpha_{j}(n) X_{T}^{0}(\check{c}, \phi(j)) \rightarrow X$ a.s. for some $[-\infty, c+W]$-valued random variable $X$. By convexity of $G$, we have that for $\tilde{\phi}(n):=\sum_{j=n}^{M(n)} \alpha_{j}(n) \phi(j)$,

$$
X_{T}^{0}(\check{c}, \tilde{\phi}(n)) \geq \sum_{j=n}^{M(n)} \alpha_{j}(n) X_{T}^{0}(\check{c}, \phi(j))
$$

so $\sum_{j=n}^{M(n)} \alpha_{j}(n) X_{T}(\check{c}, \phi(j)) \in C^{\prime}$ for each $n$. By concavity of $u$,

$$
E u\left(\sum_{j=n}^{M(n)} \alpha_{j}(n) X_{T}^{0}(\check{c}, \phi(j))-B\right) \geq \sum_{j=n}^{M(n)} \alpha_{j}(n) E u\left(X_{T}^{0}(\check{c}, \phi(j))-B\right)
$$

Fatou's lemma applies because of (165) and it implies that

$$
E u(X-B) \geq \sup _{\phi \in \mathcal{A}^{\prime}(u, c)} E u\left(X_{T}^{0}(\check{c}, \phi)-B\right)
$$

in particular, $X$ is finite-valued a.s. and hence $\check{X} \in C^{\prime}$ by the closedness of $C^{\prime}$. It follows that $X=X_{T}^{0}\left(\check{c}, \phi^{*}\right)-Y^{0}$ for some $\phi^{*} \in \mathcal{A}^{\prime}(u, c)$ and $Y \in L^{0}\left(\mathbb{R}_{+}^{d+1}\right)$. Clearly,

$$
E u\left(X_{T}^{0}\left(\check{c}, \phi^{*}\right)-B-Y^{0}\right)=\sup _{\phi \in \mathcal{A}^{\prime}(u, c)} E u\left(X_{T}^{0}(\check{c}, \phi)-B\right)
$$

and necessarily $E u\left(X_{T}^{0}\left(\check{c}, \phi^{*}\right)-B\right)=\sup _{\phi \in \mathcal{A}^{\prime}(u, c)} E u\left(X_{T}^{0}(\check{c}, \phi)-B\right)$ as well. This completes the proof. Note that $u$ may be constant on an interval hence $Y^{0} \neq 0$ is possible.

Remark 5.14. Theorem 5.12 is the first existence result of its kind in a general, continuoustime setting. In the discrete time case similar results were obtained in [27, 69].
Remark 5.15. Theorem 5.12 can also be proved with

$$
\mathcal{A}^{\prime \prime}(u, c)=\left\{\phi \in \mathcal{A}: X_{T}^{i}(\check{c}, \phi) \geq 0, i=1, \ldots, d, E u_{-}\left(X_{T}^{0}(\check{c}, \phi)-B\right)<\infty\right\}
$$

in lieu of $\mathcal{A}^{\prime}(u, c)$. Note that the two optimization problems are not equivalent, due to illiquidity.

Remark 5.16. Let us assume that $S$ is non-negative and one-dimensional. We may equivalently use either $\mathcal{A}^{\prime \prime}(u, c)$ or

$$
\begin{aligned}
\mathcal{A}_{+}^{\prime \prime}(u, c):= & \left\{\phi \in \mathcal{A}: S_{t}(\omega)+G\left(\omega, t, \phi_{t}(\omega)\right) / \phi_{t}(\omega) \geq 0 \text { when } \phi_{t}(\omega)<0\right. \\
& \left.X_{T}^{1}(\check{c}, \phi) \geq 0, E u_{-}\left(X_{T}^{0}(\check{c}, \phi)-B\right)<\infty\right\}
\end{aligned}
$$

that is, we may restrict our class of strategies to those for which the instantaneous execution price is non-negative, as in Remark 5.4 above.

Remark 5.17. The proofs of Theorem 5.12 and Proposition 5.10 use Lemmata 6.11 and 6.10. They could be replaced, at the price of minor modifications, with Komlós's theorem [60] and its extensions [6, 103].

While the previous result asserts the existence of optimal strategies, the next theorem provides a kind of verification theorem for a strategy's optimality, compare to Remark 2.48.

Theorem 5.18. Set $d:=1$. Let Assumption 5.3 hold,
a) let $u$ be concave, continuously differentiable, with $u^{\prime}$ strictly decreasing, and

$$
\begin{equation*}
u(x) \leq-C|x|^{\delta}, x \leq 0 \tag{166}
\end{equation*}
$$

for some $C>0$ and $\delta>1$;
b) denoting by $\tilde{u}$ the convex conjugate function of $u$, i.e.

$$
\tilde{u}(y):=\sup _{x \in \mathbb{R}}(u(x)-x y), y>0
$$

assume that $\tilde{u}^{\prime}$ exists and it is strictly increasing;
c) let $B$ be a bounded random variable;
d) let $Q \in \mathcal{P}$ be such that

$$
\begin{equation*}
d Q / d P \in L^{\eta} \tag{167}
\end{equation*}
$$

where $(1 / \eta)+(1 / \delta)=1$;
e) let $G_{t}(\cdot)$ be $P \times$ Leb-a.s. continuously differentiable in $x$ and let $G_{t}^{\prime}(\cdot)$ be strictly increasing;
f) let $Z$ be a càdlàg process with $Z_{T} \in L^{\gamma^{\prime}}$ for some $\gamma^{\prime}>\gamma$ and let $\phi^{*}$ be a feasible strategy such that, for some $y^{*}>0$, the following conditions hold:
i) $Z$ is a $Q$-martingale;
ii) $u^{\prime}\left(X_{T}^{0}\left(x, \phi^{*}\right)-B\right)=y^{*}(d Q / d P)$ a.s.;
iii) $Z_{t}=S_{t}+G_{t}^{\prime}\left(\phi_{t}^{*}\right), P \times$ Leb-a.s.

Then the strategy $\phi^{*}$ is optimal for the problem

$$
\begin{equation*}
\max _{\phi \in \mathcal{A}^{\prime}(u, x)} \operatorname{Eu}\left(X_{T}^{0}(x, \phi)-B\right) . \tag{168}
\end{equation*}
$$

Proof. Let $Z_{t}$ be as in the statement of the Theorem, and rewrite the final payoff for some $\phi \in \mathcal{A}^{\prime}(u, x)$ as:

$$
X_{T}^{0}(x, \phi)=x-\int_{0}^{T} Z_{t} \phi_{t} d t+\int_{0}^{T}\left(Z_{t}-S_{t}\right) \phi_{t} d t-\int_{0}^{T} G_{t}\left(\phi_{t}\right) d t
$$

By definition of $G_{t}^{*}$ it follows that:

$$
\begin{equation*}
X_{T}^{0}(x, \phi) \leq x-\int_{0}^{T} Z_{t} \phi_{t} d t+\int_{0}^{T} G_{t}^{*}\left(Z_{t}-S_{t}\right) d t \tag{169}
\end{equation*}
$$

and equality holds if $Z_{t}-S_{t}=G_{t}^{\prime}\left(\phi_{t}\right), P \times L e b-$ a.s., that is, when $\left.i i i\right)$ holds.
It follows from Lemma 5.19 that

$$
\begin{equation*}
0 \leq E_{Q}\left[\left(x-X_{T}^{0}(x, \phi)+\int_{0}^{T} G_{t}^{*}\left(Z_{t}-S_{t}\right) d t\right)\right] \tag{170}
\end{equation*}
$$

Thus, for any $\phi \in \mathcal{A}^{\prime}(u, \phi)$ and for any $y>0$ the following holds:

$$
\begin{align*}
E u\left(X_{T}^{0}(x, \phi)-B\right) & \leq E u\left(X_{T}^{0}(x, \phi)-B\right)+E y(d Q / d P)\left(x-X_{T}^{0}(x, \phi)+\int_{0}^{T} G_{t}^{*}\left(Z_{t}-S_{t}\right) d t\right) \\
& \leq E \tilde{u}(y(d Q / d P))+E y(d Q / d P)\left(\int_{0}^{T} G_{t}^{*}\left(Z_{t}-S_{t}\right) d t-B\right)+y x \tag{171}
\end{align*}
$$

Notice that, by $b), \tilde{u}(y)=u\left(\left[u^{\prime}\right]^{-1}(y)\right)-y\left[u^{\prime}\right]^{-1}(y)$. If $\left.i i i\right)$ is satisfied then there is equality in (169) above for $\phi=\phi^{*}$. If, in addition, $i$, $i i$ ) are satisfied then both inequalities in (171) are equalities for $y=y^{*}$. Thus, if conditions $i$ ), $i i$ ) and $i i i$ ) hold for $\phi^{*}$ then, by (171),

$$
E u\left(X_{T}^{0}\left(x, \phi^{*}\right)-B\right)=E \tilde{u}\left(y^{*}(d Q / d P)\right)+E y^{*}(d Q / d P)\left(\int_{0}^{T} G_{t}^{*}\left(Z_{t}-S_{t}\right) d t-B\right)+y^{*} x
$$

For all $\phi \in \mathcal{A}^{\prime}(u, c)$,

$$
E u\left(X_{T}^{0}(x, \phi)-B\right) \leq E \tilde{u}\left(y^{*}(d Q / d P)\right)+E y^{*}(d Q / d P)\left(\int_{0}^{T} G_{t}^{*}\left(Z_{t}-S_{t}\right) d t-B\right)+y^{*} x
$$

by (171). Hence the strategy $\phi^{*}$ is indeed optimal.
Lemma 5.19. Under the assumptions of Theorem 5.18, any $\phi \in \mathcal{A}^{\prime}(u, x)$ satisfies

$$
E_{Q} \int_{0}^{T} \phi_{t} Z_{t} d t=0
$$

Proof. Assume $T=1$. Define

$$
\Phi_{t}:=\int_{0}^{t} \phi_{s}^{+} d s, \quad \Psi_{t}:=\int_{0}^{t} \phi_{s}^{-} d s, t \in[0,1] .
$$

We will show $E_{Q} \int_{0}^{1} Z_{t} d \Phi_{t}-E_{Q} \int_{0}^{1} Z_{t} d \Psi_{t}=0$.
Since $\phi \in \mathcal{A}^{\prime}(U, x)$, (166), (167) and Hölder's inequality imply that $E_{Q}\left[X_{1}^{0}(x, \phi)\right]_{-}<\infty$, hence Lemma 5.9 shows

$$
E_{Q} \int_{0}^{1}\left|\phi_{t}\right|^{\beta}\left(1+\left|S_{t}\right|\right)^{\beta} d t<\infty
$$

a fortiori,

$$
\begin{equation*}
E_{Q}\left(\Phi_{1}\right)^{\beta}=E_{Q}\left(\int_{0}^{1} \phi_{t}^{+} d t\right)^{\beta}<\infty \tag{172}
\end{equation*}
$$

Define $\Phi_{t}(n):=\Phi_{k_{n}(t)}$ where

$$
k_{n}(t):=\max \left\{i \in \mathbb{N}: \frac{i}{n} \leq t\right\} / n
$$

and observe that $d \Phi_{t}(n) \rightarrow d \Phi_{t}$ a.s., $n \rightarrow \infty$ in the sense of the weak convergence of measures on $\mathcal{B}([0,1])$. As $Z_{t}$ is a.s. càdlàg, its trajectories have at most countably many points of discontinuity (a.s.). By $d \Phi \ll$ Leb, this implies

$$
Y_{n}^{+}:=\int_{0}^{1} Z_{t} d \Phi_{t}(n) \rightarrow \int_{0}^{1} Z_{t} d \Phi_{t}=: Y^{+}
$$

almost surely. Furthermore,

$$
\begin{equation*}
\left|\int_{0}^{1} Z_{t} d \Phi_{t}(n)\right|=\left|\sum_{k=1}^{n} Z_{k / n}\left[\Phi_{k / n}(n)-\Phi_{(k-1) / n}(n)\right]\right| \leq \sup _{t}\left|Z_{t}\right| \Phi_{1} \tag{173}
\end{equation*}
$$

where $\sup _{t \in[0, T]}\left|Z_{t}\right| \in L^{\gamma^{\prime}}(Q)$ by assumption and $\Phi_{1} \in L^{\beta}(Q)$ by (172). It follows by Hölder's inequality that the sequence $Y_{n}^{+}$is $Q$-uniformly integrable, so $E_{Q} Y_{n}^{+} \rightarrow E_{Q} Y^{+}, n \rightarrow \infty$. From (173) we get, noting that $\Phi_{0}(n)=0$,

$$
\begin{equation*}
E_{Q} Y_{n}^{+}=E_{Q}\left[\sum_{l=0}^{n-1}\left(Z_{l / n}-Z_{(l+1) / n}\right) \Phi_{l / n}(n)\right]+E_{Q} Z_{1} \Phi_{1}(n)=E_{Q} Z_{1} \Phi_{1}(n) \tag{174}
\end{equation*}
$$

by the $Q$-martingale property of $Z$. Analogously, as $n \rightarrow \infty$,

$$
E_{Q} Y_{n}^{-}=E_{Q} Z_{1} \Psi_{1}(n) \rightarrow E_{Q} Y^{-}
$$

where $Y_{n}^{-}, \Psi .(n)$ are defined analogously to $Y_{n}^{+}, \Phi .(n)$ using $\Psi$ instead of $\Phi$ and

$$
Y^{-}:=\int_{0}^{1} Z_{t} d \Psi_{t}
$$

Since $\Phi_{1}(n)-\Psi_{1}(n)=\Phi_{1}-\Psi_{1}=0$, (174) implies that $E_{Q}\left(Y_{n}^{+}-Y_{n}^{-}\right)=0$ for all $n$, whence also

$$
E_{Q}\left(Y^{+}-Y^{-}\right)=E_{Q} \int_{0}^{T} \phi_{t} Z_{t} d t=0
$$

completing the proof.
Remark 5.20. The paper [47] did not only deal with utility maximisation but also with theorems on superreplication and on the characterisation of arbitrage. As these latter topics are only loosely connected to the main themes of this dissertation, we confined ourselves to optimal investment here.

## 6 Appendix

We collect results of an auxiliary nature which have been relegated here for reasons of a smooth presentation.

### 6.1 Generalized conditional expectation

Let $W$ be a non-negative random variable on the probability space $(\Omega, \mathcal{F}, P)$. Let $\mathcal{H} \subset \mathcal{F}$ be a sigma-algebra. Define (as in e.g. [36]), the generalized conditional expectation by

$$
E(W \mid \mathcal{H}):=\lim _{n \rightarrow \infty} E(W \wedge n \mid \mathcal{H})
$$

where the limit a.s. exists by monotonicity (but may be $+\infty$ ). In particular, $E W$ is defined (finite or infinite). Note that if $E W<+\infty$, then the generalized and the usual conditional expectations of $W$ coincide.

Lemma 6.1. For all $A \in \mathcal{H}$ and all non-negative random variables $W$, the following equalities hold a.s.:

$$
\begin{align*}
E\left(1_{A} E(W \mid \mathcal{H})\right) & =E\left(W 1_{A}\right)  \tag{175}\\
E\left(W 1_{A} \mid \mathcal{H}\right) & =E(W \mid \mathcal{H}) 1_{A} \tag{176}
\end{align*}
$$

Furthermore, $E(W \mid \mathcal{H})<+\infty$ a.s. if and only if there is a sequence $A_{m} \in \mathcal{H}, m \in \mathbb{N}$ such that $E\left(W 1_{A_{m}}\right)<\infty$ for all $m$ and $\cup_{m} A_{m}=\Omega$. In this case, $E(W \mid \mathcal{H})$ is the Radon-Nykodim derivative of the sigma-finite measure $\mu(A):=E\left(W 1_{A}\right), A \in \mathcal{H}$ with respect to $P$ on $(\Omega, \mathcal{H})$.

Proof. Most of these facts are stated in section II. 39 on page 33 of [36]. We nevertheless give a quick proof. Let $A \in \mathcal{H}$ arbitrary. Then

$$
\begin{aligned}
E\left(1_{A} E(W \mid \mathcal{H})\right) & =\lim _{n \rightarrow \infty} E\left(1_{A} E(W \wedge n \mid \mathcal{H})\right) \\
& =\lim _{n \rightarrow \infty} E\left((W \wedge n) 1_{A}\right)=E\left(W 1_{A}\right)
\end{aligned}
$$

by monotone convergence and by the properties of ordinary conditional expectations. Similarly, (176) is satisfied by monotone convergence and by the properties of ordinary conditional expectations.

Now, if $A_{m}$ is a sequence as in the statement of Lemma 6.1, then $\mu$ is indeed sigma-finite and (175) implies that $E(W \mid \mathcal{H})$ is the Radon-Nykodim derivative of $\mu$ with respect to $P$ on $(\Omega, \mathcal{H})$ and as such, it is a.s. finite.

Conversely, if $E(W \mid \mathcal{H})<+\infty$ a.s. then define $A_{m}:=\{E(W \mid \mathcal{H}) \leq m\}$. We have, by (175),

$$
E\left(W 1_{A_{m}}\right)=E\left(1_{A_{m}} E(W \mid \mathcal{H})\right) \leq m<\infty
$$

showing the existence of a suitable sequence $A_{m}$.
For a real-valued random variable $Z$ we may define, if either $E\left(Z^{+} \mid \mathcal{H}\right)<\infty$ a.s. or $E\left(Z^{-} \mid \mathcal{H}\right)<\infty$ a.s.,

$$
E(Z \mid \mathcal{H}):=E\left(Z^{+} \mid \mathcal{H}\right)-E\left(Z^{-} \mid \mathcal{H}\right)
$$

In particular, $E(Z)$ is defined if either $E\left(Z^{+}\right)<+\infty$ or $E\left(Z^{-}\right)<+\infty$.
Lemma 6.2. If $E(Z)$ is defined then so is $E(Z \mid \mathcal{H})$ a.s. and $E(Z)=E(E(Z \mid \mathcal{H}))$.
Proof. We may suppose that e.g. $E\left(Z^{+}\right)<\infty$. Then $E\left(Z^{+} \mid \mathcal{H}\right)$ exists (in the ordinary sense as well) and is finite, so $E(Z \mid \mathcal{H})$ exists a.s. Then, by (175), we have $E\left(Z^{ \pm}\right)=E\left(E\left(Z^{ \pm} \mid \mathcal{H}\right)\right)$.
Corollary 6.3. Let $Z$ be a random variable and let $W$ be an $\mathcal{H}$-measurable random variable. Assume that there is a sequence $A_{m} \in \mathcal{H}, m \in \mathbb{N}$ such that $\cup_{m} A_{m}=\Omega$ and $E\left(Z 1_{A_{m}} \mid \mathcal{H}\right)$ exists and it is finite a.s. for all $m$. Then
(i) $E(Z \mid \mathcal{H})$ exists and it is finite a.s.
(ii) If $W 1_{A_{m}} \leq E\left(Z 1_{A_{m}} \mid \mathcal{H}\right)$ a.s. for all $m$ then $W \leq E(Z \mid \mathcal{H})$ a.s.
(iii) If $W 1_{A_{m}}=E\left(Z 1_{A_{m}} \mid \mathcal{H}\right)$ a.s. for all $m$ then $W=E(Z \mid \mathcal{H})$ a.s.

Proof. Fix some $m$ such that $E\left(Z 1_{A_{m}} \mid \mathcal{H}\right)$ exists and it is finite a.s., then $E\left(|Z| 1_{A_{m}} \mid \mathcal{H}\right)$ is also finite a.s. and by Lemma 6.1 there exists a sequence $\left(B_{j}^{m}\right)_{j}$ such that $\cup_{j} B_{j}^{m}=\Omega$ and $E\left(|Z| 1_{A_{m}} 1_{B_{j}^{m}}\right)<\infty$ for all $j$.

Then the sets $C(m, j):=A_{m} \cap B_{j}^{m}$ are such that $\cup_{m, j} C(m, j)=\Omega$. Let $C_{n}, n \in \mathbb{N}$ be the enumeration of all the sets $C(m, j)$. We clearly have $E\left(|Z| 1_{C_{n}}\right)<\infty$ for all $n$. Hence, by Lemma 6.1, $E(|Z| \mid \mathcal{H})<\infty$ and thus $E(Z \mid \mathcal{H})$ exists and is finite a.s.

Suppose that, e.g., $\{W>E(Z \mid \mathcal{H})\}$ on a set of positive measure. Then there is $n$ such that $G:=C_{n} \cap\{W>E(Z \mid \mathcal{H})\}$ has positive measure. There is also $m$ such that $C_{n} \subset A_{m}$. Then $E\left(|Z| 1_{G}\right) \leq E\left(|Z| 1_{C_{n}}\right)<\infty$ and

$$
E\left(E(Z \mid \mathcal{H}) 1_{G}\right)=E\left(E\left(Z 1_{A_{m}} \mid \mathcal{H}\right) 1_{G}\right) \geq E\left(W 1_{A_{m}} 1_{G}\right)=E\left(W 1_{G}\right)
$$

but this contradicts the choice of $G$, showing $W \leq E(Z \mid \mathcal{H})$ a.s. Arguing similarly for $\{W<$ $E(Z \mid \mathcal{H})\}$ we can get (iii) as well.

Lemma 6.4. Let $Z_{n}$ be a sequence of random variables with $\left|Z_{n}\right| \leq W$ a.s., $n \in \mathbb{N}$ converging to $Z$ a.s. If $E(W \mid \mathcal{H})<\infty$ a.s. then $E\left(Z_{n} \mid \mathcal{H}\right) \rightarrow E(Z \mid \mathcal{H})$ a.s.

Proof. Let $A_{m} \in \mathcal{H}$ be a partition of $\Omega$ such that $E\left(W 1_{A_{m}}\right)<\infty$ for all $m$. Fixing $m$, the statement follows on $A_{m}$ by the ordinary conditional Lebesgue theorem. Since the $A_{m}$ form a partition, it holds a.s. on $\Omega$.

Corollary 6.5. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be convex and bounded from below. Let $E(Z \mid \mathcal{H})$ exist and be finite a.s. Then

$$
E(g(Z) \mid \mathcal{H}) \geq g(E(Z \mid \mathcal{H})) \text { a.s. }
$$

Proof. We may and will assume $g(0)=0$. Define $B:=\{E(g(Z) \mid \mathcal{H})<\infty\}$. The inequality is trivial on the complement of $B$.

As $E(|Z| \mid \mathcal{H})<\infty$ a.s. and $E\left(|g(Z)| 1_{B} \mid \mathcal{H}\right)<\infty$ a.s. (recall that $g$ is bounded from below), from Lemma 6.1, one can find a sequence $A_{m}$ such that $\cup_{m} A_{m}=\Omega$ and both $E\left(|Z| 1_{A_{m}}\right)<$ $\infty$ and $E\left(|g(Z)| 1_{A_{m}} 1_{B}\right)<\infty$ hold true for all $m$. From the ordinary (conditional) Jensen inequality we clearly have

$$
1_{B} E\left(g(Z) 1_{A_{m}} \mid \mathcal{H}\right)=E\left(g\left(Z 1_{A_{m}} 1_{B}\right) \mid \mathcal{H}\right) \geq g\left(E\left(Z 1_{A_{m}} 1_{B} \mid \mathcal{H}\right)\right)=g(E(Z \mid \mathcal{H})) 1_{A_{m}} 1_{B}, \text { a.s. }
$$

for all $m$, and the statement follows if we can apply Corollary 6.3, i.e. if $E\left(g(Z) 1_{A_{m}} \mid \mathcal{H}\right)$ exists and it is finite a.s. This holds true by the choice of $A_{m}$.

Lemma 6.6. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $U$ and $V$ from $\Omega \times \mathbb{R}$ to $\mathbb{R}$ such that for all $x \in \mathbb{R}, U(\cdot, x), V(\cdot, x)$ are $\mathcal{F}$-measurable. Assume that for a.e. $\omega, U(\omega, \cdot)$ and $V(\omega, \cdot)$ are either both right-continuous or both left-continuous.
(i) If $U(\cdot, q) \leq V(\cdot, q)$ holds simultaneously for all $q \in \mathbb{Q}$, a.s. then $U(\cdot, x) \leq V(\cdot, x)$ holds for all x, a.s.
(ii) If $U(\cdot, q)=V(\cdot, q)$ holds simultaneously for all $q \in \mathbb{Q}$, a.s. then $U(\cdot, x)=V(\cdot, x)$ holds for all $x$, a.s.

## Proof. Obvious.

Lemma 6.7. Let $(\Omega, \mathcal{H}, P)$ be a complete probability space. Let $\Xi^{d}$ be the set of $\mathcal{H}$-measurable $d$-dimensional random variables. Let $F: \Omega \times \mathbb{R}^{d} \rightarrow \mathbb{R}$ be a function such that for almost all $\omega \in \Omega, F(\omega, \cdot)$ is continuous and for each $y \in \mathbb{R}^{d}, F(\cdot, y)$ is $\mathcal{H}$-measurable. Let $K>0$ be an $\mathcal{H}$-measurable random variable.

Set $f(\omega)=$ ess. $\sup _{\xi \in \Xi^{d},|\xi| \leq K} F(\omega, \xi(\omega))$. Then, for almost all $\omega$,

$$
\begin{equation*}
f(\omega)=\sup _{y \in \mathbb{R}^{d},|y| \leq K(\omega)} F(\omega, y) . \tag{177}
\end{equation*}
$$

Proof. $F$ is easily seen to be $\mathcal{H} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$-measurable and so is

$$
\sup _{y \in \mathbb{R}^{d},|y| \leq K(\omega)} F(\omega, y)=\sup _{y \in \mathbb{Q}^{d},|y| \leq K(\omega)} F(\omega, y) \text {. }
$$

Hence $\sup _{y \in \mathbb{R}^{d},|y| \leq K(\omega)} F(\omega, y) \geq f(\omega)$ a.s. by the definition of essential supremum. Assume that the inequality is strict with positive probability. Then for some $\varepsilon>0$ the set

$$
A=\left\{(\omega, y) \in \Omega \times \mathbb{R}^{d}:|y| \leq K(\omega) ; F(\omega, y)-f(\omega) \geq \varepsilon\right\}
$$

has a projection $A^{\prime}$ on $\Omega$ with $P\left(A^{\prime}\right)>0$. Recall that $\omega \rightarrow F(\omega, \xi(\omega))$ is $\mathcal{H}$-measurable for $\xi \in \Xi^{d}$. By definition of the essential supremum, $f$ is $\mathcal{H}$-measurable and hence $A \in \mathcal{H} \otimes \mathcal{B}\left(\mathbb{R}^{d}\right)$. The measurable selection theorem (see Proposition III. 44 in [36]) applies and there exists $\eta \in \Xi^{d}$ such that $(\omega, \eta(\omega)) \in A$ for $\omega \in A^{\prime}$ (and $\eta(\omega)=0$ on the complement of $A^{\prime}$ ). This leads to a contradiction since for all $\omega \in A^{\prime}, f(\omega)<F(\omega, \eta(\omega))$ by the construction of $\eta$ and $f(\omega) \geq F(\omega, \eta(\omega))$ a.s. by the definition of $f$.

### 6.2 Compactness and integrability in Banach spaces

Lemma 6.8. Let $\mathcal{H} \subset \mathcal{F}$ be a sigma-algebra. Let $X_{n}$ be a sequence of $\mathcal{H}$-measurable $d$ dimensional random variables such that

$$
\begin{equation*}
\liminf _{n \rightarrow \infty}\left|X_{n}\right|<\infty \tag{178}
\end{equation*}
$$

almost surely. Then there exist $\mathcal{H}$-measurable random variables $n_{k}: \Omega \rightarrow \mathbb{N}, k \in \mathbb{N}$ with $n_{k}(\omega)<n_{k+1}(\omega)$ for all $\omega \in \Omega$ and $k \in \mathbb{N}$ and an $\mathcal{H}$-measurable random variable $X$ such that $X_{n_{k}} \rightarrow X$ a.s. In such a case we write that there exists an $\mathcal{H}$-measurable random subsequence.

Proof. This is Lemma 2 of [55]. Its proof also shows that when (178) holds for all $\omega \in \Omega$ then convergence takes place for all $\omega \in \Omega$.

Lemma 6.9. Let $Y_{n}, n=1, \ldots, M$ be random variables. Then there exists $Q \sim P$ with $d Q / d P \in$ $L^{\infty}$ such that, for all $n=1, \ldots, M, E_{Q}\left|Y_{n}\right|<\infty$.

Proof. Indeed, take $d Q / d P:=e^{-\sum_{n=1}^{M}\left|Y_{n}\right|} / E e^{-\sum_{n=1}^{M}\left|Y_{n}\right|}$. Interestingly, the result remains true for a countable sequence $Y_{n}, n \in \mathbb{N}$ as well, see page 266 of [37].

Lemma 6.10. Let $f_{n}$ be a sequence of $[0, \infty)$-valued random variables. Then there exist $M(n) \geq$ $n$ and $\alpha_{j}(n) \geq 0, j=n, \ldots, M(n)$ with $\sum_{j=n}^{M(n)} \alpha_{j}(n)=1$ such that $g_{n}:=\sum_{j=n}^{M(n)} \alpha_{j}(n) f_{j}, n \in \mathbb{N}$ satisfy $g_{n} \rightarrow g, n \rightarrow \infty$ a.s., where $g$ is $a[0, \infty]$-valued random variable.

If $f_{n}$ are $\mathbb{R}$-valued random variables with $\sup _{n} E\left|f_{n}\right|<\infty$ then the conclusion holds with $a$ real-valued $g$ satisfying $E|g|<\infty$.

Proof. See Lemma 9.8.1 in [35]. Clearly, the same statement is true also in the case where $(\Omega, \mathcal{F}, P)$ is a measure space with $P$ a finite measure.

Let $\mathbb{B}$ be a separable Banach space. The norm of $\mathbb{B}$ is denoted by $\|\cdot\|_{\mathbb{B}}$. We call a mapping $f: \Omega \rightarrow \mathbb{B}$ measurable if $g \circ f$ is measurable for all $g \in \mathbb{B}^{\prime}$ where $\mathbb{B}^{\prime}$ is the dual space of $\mathbb{B}$.

For a measurable $f$ satisfying

$$
\begin{equation*}
\|f\|_{\mathbb{L}}:=\int_{\Omega}\|f(\omega)\|_{\mathbb{B}} P(d \omega)<\infty \tag{179}
\end{equation*}
$$

one can simply define the $\mathbb{B}$-valued integral $\int_{\Omega} f(\omega) P(d \omega)$, see e.g. Chapter 5 of [67]. We denote by $\mathbb{L}:=L^{1}(\Omega, \mathcal{F}, P ; \mathbb{B})$ the space of (equivalence classes of) measurable functions $f$ satisfying (179), this is a Banach space with the norm $\|\cdot\|_{\mathbb{L}}$ defined above. One can construct such (Bochner) integral for non-separable $\mathbb{B}$ as well but we do not need this in the present dissertation.

Lemma 6.11. If $\mathbb{B}$ is a reflexive Banach space and $f_{n}, n \in \mathbb{N}$ is a bounded sequence in $\mathbb{L}$ then there are $M(n) \geq n$ and $\alpha_{j}(n) \geq 0, j=n, \ldots, M(n)$ with $\sum_{j=n}^{M(n)} \alpha_{j}(n)=1$ such that

$$
g_{n}:=\sum_{j=n}^{M(n)} \alpha_{j}(n) f_{j} \text { satisfies }\left\|g_{n}-g\right\|_{\mathbb{B}} \rightarrow 0, n \rightarrow \infty
$$

for a.e. $\omega$, with some $g \in \mathbb{L}$.
Proof. See Theorem 15.2.6 in [35].
Lemma 6.12. Let $\mathcal{H} \subset \mathcal{F}$ be a sigma-algebra. Define $K:=[-N, N]^{n}$. Let $L: \Omega \times K \rightarrow \mathbb{R}^{m}$ be such that for a.e. $\omega \in \Omega, L(\omega, \cdot)$ is continuous and for all $x \in K, L(\cdot, x)$ is measurable such that $\sup _{z \in K}|L(\omega, z)|$ is integrable. Then there is $l: \Omega \times K \rightarrow \mathbb{R}^{m}$ such that for a.e. $\omega \in \Omega, l(\omega, \cdot)$ is continuous and for all $k \in K, E(L(k) \mid \mathcal{H})=l(k)$ a.s.

Proof. Consider the separable Banach space $\mathbb{B}:=C\left([-N, N]^{n}\right)$ with the supremum norm. Clearly, $L: \Omega \rightarrow \mathbb{B}$ and for all $\mu \in \mathbb{B}^{\prime}$ (which can be represented as a Borel signed measure), $\mu(L)=\int_{K} L(\omega, x) \mu(d x)$ is a measurable function on $(\Omega, \mathcal{F})$ : indeed, this is clear for $\mu$ with finite support and then follows for general $\mu$ by approximation. Note that, for each $k \in K$, the linear functional $f_{k}(x):=x(k), x \in \mathbb{B}$ is continuous (w.r.t. the norm of $\mathbb{B}$ ) so $f_{k} \in \mathbb{B}^{\prime}$. Now it follows from Proposition V.2.5. of [67] that there is a measurable $l: \Omega \rightarrow \mathbb{B}$ such that

$$
l(k)=f_{k}(l)=E\left(f_{k}(L) \mid \mathcal{H}\right)=E(L(k) \mid \mathcal{H})
$$

for each $k \in K$, as claimed.

### 6.3 Asymptotic elasticity

When $u$ is concave and differentiable, the asymptotic elasticities of $u$ at $\pm \infty$ are defined as

$$
\begin{align*}
& A E_{+}(u)=\limsup _{x \rightarrow \infty} \frac{u^{\prime}(x) x}{u(x)}  \tag{180}\\
& A E_{-}(u)=\liminf _{x \rightarrow-\infty} \frac{u^{\prime}(x) x}{u(x)} \tag{181}
\end{align*}
$$

see [61], [91] and the references therein.
It is shown in Lemma 6.3 of [61] that, when $u(\infty)>0, A E_{+}(u)$ equals ${ }^{5}$ the infimum of those $\alpha$ for which there is $\bar{x}>0$ s.t.

$$
\begin{equation*}
u(\lambda x) \leq \lambda^{\alpha} u(x) \text { for } x \geq \bar{x}, \lambda \geq 1 \tag{182}
\end{equation*}
$$

Similarly, $A E_{-}(u)$ equals the supremum of those $\beta$ for which there is $\underline{x}>0$ such that

$$
\begin{equation*}
u(\lambda x) \leq \lambda^{\beta} u(x) \text { for } x \leq-\underline{x}, \lambda \geq 1 \tag{183}
\end{equation*}
$$

Note that the proof of the equivalence $(180) \Leftrightarrow(182)$ in Lemma 6.3 of [61] does not use the concavity of $u$. So if $u$ is nondecreasing and continuously differentiable then (180) makes sense and it is equivalent to (182) in the above sense. Similarly, (181) is equivalent to (183).

It thus seems reasonable to extend the definitions of $A E_{+}(u)$ (resp. $A E_{-}(u)$ ) to possibly non-concave and non-differentiable $u$ as the infimum (resp. supremum) of $\alpha$ (resp. $\beta$ ) such that (182) (resp. (183)) holds. Doing so we see (looking at Assumption 2.13) that in Chapter 2 we assert the existence of an optimal strategy whenever $A E_{+}(u)<A E_{-}(u)$.

[^3]
### 6.4 Continuously differentiable versions

We return to the setting and notation of Section 2.5. We assume that, for a.e. $\omega \in \Omega$, $x \rightarrow V(\omega, x)$ is continuously differentiable, concave, nondecreasing and, for each $N>0$,

$$
\begin{equation*}
\sup _{(x, y) \in[-N, N]^{d+1}}\left|V^{\prime}(x+y Y)\right|(1+|Y|) \tag{184}
\end{equation*}
$$

is integrable.
Proposition 6.13. The function

$$
(x, y) \rightarrow E(V(x+y Y) \mid \mathcal{H})
$$

has a version $G(x, y, \omega)$ which is continuously differentiable in $(x, y) \in \mathbb{R}^{d+1}$,

$$
\begin{equation*}
\partial_{i} G(x, y, \omega)=E\left(V^{\prime}(x+y Y) Y^{i} \mid \mathcal{H}\right), \quad 1 \leq i \leq d \tag{185}
\end{equation*}
$$

where $\partial_{i}$ is the derivative with respect to $y^{i}$,

$$
\begin{equation*}
\partial_{x} G(x, y, \omega)=E\left(V^{\prime}(x+y Y) \mid \mathcal{H}\right) \tag{186}
\end{equation*}
$$

Furthermore, for any $\xi \in \Xi^{d}$ and $X \in \Xi^{1}$,

$$
\begin{align*}
G(X, \xi, \omega) & =E(V(X+\xi Y) \mid \mathcal{H})  \tag{187}\\
\partial_{i} G(X, \xi, \omega) & =E\left(V^{\prime}(X+\xi Y) Y^{i} \mid \mathcal{H}\right), 1 \leq i \leq d  \tag{188}\\
\partial_{x} G(X, \xi, \omega) & =E\left(V^{\prime}(X+\xi Y) \mid \mathcal{H}\right) \tag{189}
\end{align*}
$$

The function

$$
x \rightarrow \text { ess. } \sup _{\xi \in \Xi} E(V(x+\xi Y) \mid \mathcal{H})
$$

has a version $A(\omega, x)$ which is almost surely concave, continuously differentiable and (a.s.)

$$
\begin{equation*}
A(X)=\text { ess. } \sup _{\xi \in \Xi} E(V(X+\xi Y) \mid \mathcal{H}) \tag{190}
\end{equation*}
$$

Also,

$$
E\left(V^{\prime}(X+\tilde{\xi}(X) Y) Y^{i} \mathcal{H}\right)=0 \text { a.s. }, 1 \leq i \leq d
$$

where $\tilde{\xi}$ is the function constructed in Proposition 2.38.
Proof. It is enough to construct $G$ for $(x, y) \in[-N, N]^{d+1}$, for all $N \in \mathbb{N}$ (we mean that $G$ is differentiable in $(x, y) \in(-N, N)^{d+1}$ and its derivative extends as a continuous function to $[-N, N]^{d+1}$ ). Fix $N$. By (184), as in Proposition 2.33, Lemma 6.12 implies the existence of $H$ : $\Omega \rightarrow C\left([-N, N]^{d+1} ; \mathbb{R}^{d+1}\right)$ such that, for each $x, y, H(\omega, x, y)^{i}$ is a version of $E\left(V^{\prime}(x+y Y) Y^{i} \mid \mathcal{H}\right)$ for $i=1, \ldots, d$ and $H(\omega, x, y)^{0}$ is a version of $E\left(V^{\prime}(x+y Y) \mid \mathcal{H}\right)$. Let $W$ be and arbitrary version of $E(V(0) \mid \mathcal{H})$.

Since

$$
V(x+y Y)=V(0)+\sum_{j=1}^{d} \int_{0}^{y^{j}} V^{\prime}\left(u_{j} Y^{j}+\sum_{l=1}^{j-1} y^{l} Y^{l}\right) d u_{j}+\int_{0}^{x} V^{\prime}(w+y Y) d w
$$

applications of Fubini's theorem show that the expression

$$
G(\omega, x, y):=W+\sum_{j=1}^{d} \int_{0}^{y^{j}} H^{j}\left(\omega, 0, y^{1}, \ldots, y^{j-1}, u_{j}, 0, \ldots\right) d u_{j}++\int_{0}^{x} H^{0}\left(\omega, w, y^{1}, \ldots, y^{d}\right) d w
$$

defines a version $G(\omega, x, y)$ of $E(V(x+y Y) \mid \mathcal{H})$, it is a.s. continuously differentiable and satisfies (185) and (186). This version can be used throughout the arguments of Section 2.5.

It is enough to show (187), (188) and (189) for $(X, \xi) \in[-N, N]^{d+1}$ for each $N$ fixed. It is clear for step functions and taking (uniformly bounded) step-function approximations $\left(X_{m}, \xi_{m}\right), m \in \mathbb{N}$ of $(X, \xi)$ we get

$$
H^{i}\left(X_{m}, \xi_{m}\right)=\partial_{i} G\left(X_{m}, \xi_{m}\right) \rightarrow \partial_{i} G(X, \xi)=H^{i}(X, \xi)
$$

by continuity of $H$ and

$$
E\left(V^{\prime}\left(X_{n}+\xi_{n} Y\right) Y^{i} \mid \mathcal{H}\right) \rightarrow E\left(V^{\prime}(X+\xi Y) Y^{i} \mid \mathcal{H}\right)
$$

by (184) and the (conditional) Lebesgue theorem, for $i=1, \ldots, d$. A similar argument works for $\partial_{x} G$.

Take $A$ as constructed in Proposition 2.35 (using the $G$ just obtained). Now we borrow a trick from Theorem 4.13 of [95]. Outside a null set, for all $x$ and for any $h \in \mathbb{N}$

$$
\begin{equation*}
h(A(x \pm 1 / h)-A(x)) \geq h(G(x \pm 1 / h, \tilde{\xi}(x))-G(x, \tilde{\xi}(x))) . \tag{191}
\end{equation*}
$$

The left- and right-handed derivatives of $A$ exist by concavity and $A^{\prime}(x+) \leq A^{\prime}(x-)$. Letting $h \rightarrow \infty$ in (191) we find that

$$
\partial_{x} G(x, \tilde{\xi}(x)) \geq A^{\prime}(x-) \geq A^{\prime}(x+) \geq \partial_{x} G(x, \tilde{\xi}(x))
$$

so $A$ is indeed smooth almost everywhere. (190) for general $X$ follows from Proposition 2.38. Outside a $P$-zero set, for all $x \in \mathbb{R}$,

$$
\begin{equation*}
G(\omega, x, \tilde{\xi}(x))=A(\omega, x) \tag{192}
\end{equation*}
$$

From the definition of $A$,

$$
\begin{equation*}
G(\omega, x, y) \leq A(\omega, x) \text { for all } x, y \tag{193}
\end{equation*}
$$

outside a $P$-zero-set. Now the last assertion follows from (192) and (193): as $G(\omega, x, \tilde{\xi}(x))$ is a local maximum of $y \rightarrow G(\omega, x, y)$ we get that $\partial_{i} G(\omega, x, \tilde{\xi}(x))=0, i=1, \ldots, d$ simultaneously for all $x$, a.s. This completes the proof, remembering (188).

### 6.5 Further auxiliary results

Lemma 6.14. If $Y \in \mathcal{W}$ then

$$
\int_{0}^{\infty} P^{\delta}(Y \geq y) d y<\infty
$$

for all $\delta>0$.
Proof. By Markov's inequality,

$$
P(Y \geq y) \leq M(N) y^{-N}, \quad y>0
$$

for all $N>0$, with a constant $M(N):=E Y^{N}<\infty$. We can now choose $N$ so large that $N \delta>1$, showing that the integral in question is finite.

Lemma 6.15 below is folklore and its proof is omitted.
Lemma 6.15. Let $X$ be a real-valued random variable with atomless law. Let $F(x):=P(X \leq$ $x)$ denote its cumulative distribution function. Then $F(X)$ has uniform law on $[0,1]$.

The following Lemmata should be fairly standard. We nonetheless included their proofs since we could not find an appropriate reference.

Lemma 6.16. Let $\mu(d y, d z)=\nu(y, d z) \delta(d y)$ be a probability on $\mathbb{R}^{N_{2}} \times \mathbb{R}^{N_{1}}$ such that $\delta(d y)$ is a probability on $\mathbb{R}^{N_{2}}$ and $\nu(y, d z)$ is a probabilistic kernel. Assume that $Y$ has law $\delta(d y)$ and $E$ is independent of $Y$ and uniformly distributed on $[0,1]$. Then there is a measurable function $G: \mathbb{R}^{N_{2}} \times[0,1] \rightarrow \mathbb{R}^{N_{1}}$ such that $(Y, G(Y, E))$ has law $\mu(d y, d z)$.

Proof. We first recall that if $\mathcal{Y}_{1}, \mathcal{Y}_{2}$ are uncountable Polish spaces then they are Borel isomorphic, i.e. there is a bijection $\psi: \mathcal{Y}_{1} \rightarrow \mathcal{Y}_{2}$ such that $\psi, \psi^{-1}$ are measurable (with respect to the respective Borel fields); see e.g. page 159 of [36].

Fix a Borel isomorphism $\psi: \mathbb{R} \rightarrow \mathbb{R}^{N_{1}}$. Consider the measure on $\mathbb{R}^{N_{2}} \times \mathbb{R}$ defined by $\tilde{\mu}(A \times B):=\int_{A} \nu(y, \psi(B)) \delta(d y), A \in \mathcal{B}\left(\mathbb{R}^{N_{2}}\right), B \in \mathcal{B}(\mathbb{R})$. For $\delta$-almost every $y, \nu(y, \psi(\cdot))$ is a probability measure on $\mathbb{R}$. Let $F(y, z):=\nu(y, \psi((-\infty, z])))$ denote its cumulative distribution function and define

$$
F^{\dagger}(y, u):=\inf \{q \in \mathbb{Q}: F(y, q) \geq u\}, u \in(0,1),
$$

this is easily seen to be $\mathcal{B}\left(\mathbb{R}^{N_{2}}\right) \otimes \mathcal{B}([0,1])$-measurable. Then, for $\delta$-almost every $y, F^{\dagger}(y, E)$ has law $\nu(y, \psi(\cdot))$. Hence $\left(Y, F^{\dagger}(Y, E)\right)$ has law $\tilde{\mu}$. Consequently, $\left(Y, \psi\left(F^{\dagger}(Y, E)\right)\right)$ has law $\mu$ and we may conclude setting $G(y, u):=\psi\left(F^{\dagger}(y, u)\right)$. The idea of this proof is well-known, see e.g. page 228 of [13].

Lemma 6.17. Let $(X, W)$ be an $(n+m)$-dimensional random variable such that the conditional law of $X$ w.r.t. $\sigma(W)$ is a.s. atomless. Then there is a measurable $G: \mathbb{R}^{n+m} \rightarrow \mathbb{R}$ such that $G(X, W)$ is independent of $W$ with uniform law on $[0,1]$.

Proof. Let us fix a Borel-isomorphism $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}$. Note that $\psi(X)$ also has an a.s. atomless conditional law w.r.t. $\sigma(W)$. Define (using a regular version of the conditional law),

$$
H(x, w):=P(\psi(X) \leq x \mid W=w),(x, w) \in \mathbb{R} \times \mathbb{R}^{m},
$$

this is $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}\left(\mathbb{R}^{m}\right)$-measurable (using the fact that $H$ is continuous in $x$ a.s. by hypothesis and measurable for each fixed $w$ since we took a regular version of the conditional law). It follows that the conditional law of $H(\psi(X), W)$ w.r.t. $\sigma(W)$ is a.s. uniform on $[0,1]$ (see Lemma 6.15) which means that it is independent of $W$. Hence we may define $G(x, w):=H(\psi(x), w)$, which is measurable since $H$ and $\psi$ are.

## 7 Present and future work

In Chapter 3 it was found that the parameter restrictions $\alpha<\beta$ and $\alpha / \gamma \leq \beta / \delta$ are necessary for well-posedness of the optimal investment problem of a CPT investor in a multistep discrete-time model. Are these restrictions also sufficient? Theorems 3.16 and 4.16 cover a substantial proportion of this parameter range, but not all of it. It would also be nice to unify the arguments of these two theorems and to simplify/generalize the proof of Theorem 3.4.

A desirable, but challenging research direction is to extend the results of Chapter 4 to relevant classes of incomplete continuous-time markets. [77] is a first step into this direction.

Let $u$ be concave (as in Chapter 5 and in Section 2.7) but let the market contain a countably infinite number of risky assets. Such "large financial markets" were proposed already in [87] and their systematic study began in [53]. Optimal investment in this context leads to an interesting infinite dimensional optimization problem, see [73].

Finally, [7] proposes a model for price impact which is fundamentally different from that of Chapter 5: the "market makers" create prices by seeking a microeconomic equilibrium and a large investor is moving these prices by his/her actions. This leads to a complicated, nonlinear dynamics and hence it is intriguing how to pose and solve optimal investment problems for the large investor in this setting.

## Epilógus

Két dolog fontos: a művészet és a szerelem. (Marton Éva, [63])
Sokan vannak, akiknek erőfeszítései hozzájárultak ahhoz, hogy ez a disszertáció elkészülhessen és akikre hálával gondolok: tanáraim, társszerzőim, munkatársaim, barátaim.

Most elsősorban Feleségemnek és Lányaimnak mondok köszönetet: öröm egy ennyire jó csapatban játszani. Köszönöm Szüleimnek, hogy felébresztették bennem a szellemi dolgok iránti érdeklődést és mindig segítettek tanulmányaimban. Köszönöm Keresztanyámnak, Nővéremnek és Családjának támogatását, Vancsó Imréné tanárnőnek pedig azt, hogy megszerettette velem a matematikát.

Itt köszönök el az eddig kitartó kedves olvasóktól.

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[^0]:    1 "Short-selling" means to sell a stock without actually posessing it at the given moment. When the stock needs to be delivered phisically, it can be bought in the market. Some of the financial markets allow short-selling. In theoretical works short-selling is usually permitted as it makes the mathematical models more tractable.

[^1]:    ${ }^{2}$ The predictable sigma-algebra $\mathfrak{P}$ on $\Omega \times[0, T]$ is the one defined by adapted left-continuous processes. A predictable process is one that is measurable w.r.t. $\mathfrak{P}$.

[^2]:    ${ }^{3}$ More precisely, the Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel.
    ${ }^{4}$ Unless $\Omega$ is a union of finitely many atoms, a discrete-time market model of Section 1.2 is always incomplete, see Theorem 1.40 of [44]. Actually, one needs a very specific structure in order to get a complete model in finite discrete time, see [51] for details.

[^3]:    ${ }^{5}$ More precisely, in that Lemma (182) is assumed with $<$ instead of $\leq$ and with $\lambda>1$ instead of $\lambda \geq 1$. A careful inspection of the proof reveals that this is equivalent to our formulation (182).

